Exercises involving Cauchy's integral formula

1. *The following were parts of question 2 on the May 2023* MA3614 *exam paper and was worth 9 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z.$$

   Use this result to determine I_1 and I_2 in the following where Γ is the circle with centre at 2 and radius 5.

   $$I_1 = \oint_\Gamma \frac{\mathrm{e}^{\pi z}}{(z - i)^3} \,\mathrm{d}z, \qquad I_2 = \oint_\Gamma \frac{\sin(z)}{(z - \pi/2)^2 (z + 2\pi)} \,\mathrm{d}z.$$

   (b) Let $f(z)$ be a function which is analytic in the disk $\{z : |z| < R\}$, let C denote the circle with centre 0 and radius $r < R$ traversed once in the anti-clockwise sense and further let $0 < h < r$.

   i. Determine the partial fraction representation of

   $$\frac{2z}{z^2 - h^2}$$

   and use this to show that

   $$\frac{f(h) + f(-h)}{2} = \frac{1}{2\pi i} \oint_C \left(\frac{z}{z^2 - h^2} \right) f(z) \,\mathrm{d}z.$$

   ii. Let $\omega = \mathrm{e}^{2\pi i/5}$. The 5 roots of unity are $1, \omega, \omega^2, \omega^3, \omega^4$. Determine the partial fraction representation of

   $$\frac{5z^4}{z^5 - h^5}$$

   and use this to show that

   $$\frac{f(h) + f(\omega h) + f(\omega^2 h) + f(\omega^3 h) + f(\omega^4 h)}{5} = \frac{1}{2\pi i} \oint_C \left(\frac{z^4}{z^5 - h^5} \right) f(z) \,\mathrm{d}z.$$

   You need to justify your steps.

Solution

(a)

$$\frac{e^{\pi z}}{(z-i)^3} = \frac{f(z)}{(z-z_0)^{n+1}}$$

with $f(z) = e^{\pi z}$, $z_0 = i$ and $n = 2$. $f(z)$ is analytic inside Γ.

$$I_1 = \frac{2\pi i}{2!} f''(z_0) = \pi i f''(i).$$

$$f''(z) = \pi^2 e^{\pi z}, \quad f''(i) = \pi^2 e^{i\pi} = -\pi^2.$$

$$I_1 = -i\pi^3.$$

$$\frac{\sin(z)}{(z-\pi/2)^2(z+2\pi)} = \frac{f(z)}{(z-z_0)^{n+1}}$$

with $f(z) = \sin(z)/(z+2\pi)$, $z_0 = \pi/2$ and $n = 1$. $f(z)$ is analytic inside Γ.

$$I_2 = 2\pi i f'(\pi/2).$$

$$f'(z) = \frac{(z+2\pi)\cos(z) - \sin(z)}{(z+2\pi)^2}, \quad f'(\pi/2) = -\frac{1}{(2\pi+\pi/2)^2}.$$

$$I_2 = -\frac{2\pi i}{(2\pi+\pi/2)^2} = -\frac{8\pi i}{(5\pi)^2} = -\frac{8i}{25\pi}.$$

(b) i. $z^2 - h^2 = (z+h)(z-h)$,

$$g(z) = \frac{2z}{z^2-h^2} = \frac{A}{z+h} + \frac{B}{z-h}.$$

By a property of the limit and L'Hopital's rule

$$A = \lim_{z\to -h}(z+h)g(z) = (2z)|_{z=-h} \left.\frac{1}{2z}\right|_{z=-h} = 1.$$

Similarly $B = 1$. The numerator in the expression for $g(z)$ is the derivative of the denominator.

$$g(z) = \frac{2z}{z^2-h^2} = \frac{1}{z+h} + \frac{1}{z-h}.$$

By the Cauchy integral formula applied twice

$$f(-h)+f(h) = \frac{1}{2\pi i}\oint_\Gamma \left(\frac{1}{z+h} + \frac{1}{z-h}\right) f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_\Gamma \frac{2z}{z^2-h^2} f(z)\,\mathrm{d}z.$$

Dividing by 2 gives the result.

ii.
$$z^5 - h^5 = (z-h)(z-\omega h)(z-\omega^2 h)(z-\omega^3 h)(z-\omega^4 h).$$
As $5z^4$ is the derivative of z^5 we have
$$\frac{1}{z-h} + \frac{1}{z-\omega h} + \frac{1}{z-\omega^2 h} + \frac{1}{z-\omega^3 h} + \frac{1}{z-\omega^4 h} = \frac{5z^4}{z^5-h^5}.$$
Multiplying by $f(z)$, integrating over Γ and using the Cauchy integral formula gives
$$f(h) + f(\omega h) + f(\omega^2 h) + f(\omega^3 h) + f(\omega^4 h) = \frac{1}{2\pi i}\oint_C \left(\frac{5z^4}{z^5-h^5}\right) f(z)\,\mathrm{d}z.$$
Dividing by 5 gives the result.

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2. *The following were parts of question 2 on the May 2022* `MA3614` *exam paper and was worth 10 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by
   $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$
   Use this result to determine I_1 and I_2 in the following where Γ is the circle with centre at -1 and radius 3. In each case you need to express the value in cartesian form and you need to justify your workings.
   $$I_1 = \oint_\Gamma \frac{2z^2+z^4}{(z+2)^4}\,\mathrm{d}z, \qquad I_2 = \oint_\Gamma \frac{\cosh(z)}{z^2(z-6)}\,\mathrm{d}z.$$

   (b) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. When $0 < h < 1$ use the generalised Cauchy integral formula to show that
   $$\frac{f(-h) - 2f(0) + f(h)}{h^2} - f''(0) = \frac{h^2}{\pi i}\oint_C \frac{f(z)}{z^3(z^2-h^2)}\,\mathrm{d}z.$$

Solution

(a)
$$\frac{f(z)}{(z-z_0)^{n+1}}=\frac{2z^2+z^4}{(z+2)^4},\quad \text{with } z_0=-2,\quad n=3,\quad f(z)=2z^2+z^4.$$

All the requirements of the generalised Cauchy integral formula are satisfied and thus
$$\frac{f'''(-2)}{3!}=\frac{1}{2\pi i}I_1,\quad I_1=\frac{2\pi i}{6}f'''(-2).$$
$$f'''(z)=24z,\quad f'''(-2)=-48,\quad I_1=-i(16\pi).$$

$$\frac{f(z)}{(z-z_0)^{n+1}}=\frac{\cosh(z)}{z^2(z-6)}\quad \text{with } z_0=0,\quad n=1,\quad f(z)=\frac{\cosh(z)}{z-6}.$$

All the requirements of the generalised Cauchy integral formula are satisfied as $f(z)$ is analytic inside the circle.

$$f'(0)=\frac{1}{2\pi i}I_2,\quad I_2=2\pi i f'(0).$$

$$f'(z)=\cosh(z)\frac{-1}{(z-6)^2}+\frac{\sinh(z)}{z-6},\quad f'(0)=-\frac{1}{36},\quad I_2=-i\frac{\pi}{18}.$$

(b) By using the Cauchy integral formula for each of $f(-h)$, $f(0)$ and $f(h)$ we have

$$f(-h)-2f(0)+f(h)=\frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z+h}-\frac{2}{z}+\frac{1}{z-h}\right)\,\mathrm{d}z.$$

$$\frac{1}{z+h}-\frac{2}{z}+\frac{1}{z-h}=\frac{z(z-h)-2(z^2-h^2)+z(z+h)}{(z^2-h^2)z}=\frac{2h^2}{(z^2-h^2)z}.$$

Thus

$$\frac{f(-h)-2f(0)+f(h)}{h^2}=\frac{1}{2\pi i}\oint_C\frac{2f(z)}{(z^2-h^2)z}\,\mathrm{d}z=\frac{1}{\pi i}\oint_C\frac{f(z)}{(z^2-h^2)z}\,\mathrm{d}z.$$

Next use the representation of $f''(0)$.

$$\frac{f(-h)-2f(0)+f(h)}{h^2}-f''(0)=\frac{1}{\pi i}\oint_C f(z)\left(\frac{1}{(z^2-h^2)z}-\frac{1}{z^3}\right)\,\mathrm{d}z.$$

Now
$$\frac{1}{(z^2-h^2)z}-\frac{1}{z^3}=\frac{z^2-(z^2-h^2)}{(z^2-h^2)z^3}=\frac{h^2}{(z^2-h^2)z^3}$$

and the result follows.

3. *The following was part of question 2 on the May 2021* MA3614 *exam paper and was worth 7 of the 20 marks of the entire question.*

Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

Use this result to evaluate the following where the problems you need to consider depends on the 6th digit of your 7-digit student id.. If the 6th digit is even then you do the loop integrals on the left hand side and if it is odd then you do the loop integrals on the right hand side. In each case you need to express the value in cartesian form and you need to justify your workings. In the case of I_2, Log means the principal value logarithm.

If the 6th digit is one of $0, 2, 4, 6, 8$ then you do these cases. In each case Γ is the circle with centre at $1+2i$ and radius 4.	If the 6th digit is one of $1, 3, 5, 7, 9$ then you do these cases. In each case Γ is the circle with centre at $-2+i$ and radius 5.
$\begin{aligned} I_1 &= \oint_\Gamma \frac{z^3 - 2z}{(z-1)^3}\,\mathrm{d}z, \\ I_2 &= \oint_\Gamma \frac{\mathrm{Log}(z - 7\sqrt{3} + 7i)}{z - \sqrt{3} + i}\,\mathrm{d}z, \\ I_3 &= \oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-6)}\,\mathrm{d}z. \end{aligned}$	$\begin{aligned} I_1 &= \oint_\Gamma \frac{z^3 + 3z}{(z+2)^3}\mathrm{d}z, \\ I_2 &= \oint_\Gamma \frac{\mathrm{Log}(z + 9 - i9\sqrt{3})}{z + 1 - i\sqrt{3}}\,\mathrm{d}z, \\ I_3 &= \oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-3)}\,\mathrm{d}z. \end{aligned}$

Solution

This is the version for a 6th digit of 0, 2, 4, 6, 8.

The generalised Cauchy integral formula can be written in the form

$$\oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = 2\pi i \frac{f^{(n)}(z_0)}{n!}.$$

In the I_1 case we take $f(z) = z^3 - 2z$, $n + 1 = 3$ and $z_0 = 1$. This $f(z)$ is an entire function and z_0 is inside Γ.

$$f''(z) = 6z, \quad f''(1) = 6.$$

Thus

$$I_1 = \pi i f''(1) = 6\pi i.$$

In the I_2 case we take $z_0 = \sqrt{3} - i$ which is inside Γ and $n + 1 = 1$. We also take $f(z) = \mathrm{Log}(z - 7z_0)$. The branch point of $f(z)$ is at $7z_0$ which is outside Γ and the function is analytic inside Γ.

$$|z_0|^2 = 4, \quad |z_0| = 2, \quad -z_0 = -\sqrt{3} + i, \quad \mathrm{Arg}(-z_0) = \frac{5\pi}{6}.$$

For $f(z_0) = \mathrm{Log}(-6z_0)$ we have

$$|-6z_0| = 12, \quad \mathrm{Arg}(-6z_0) = \frac{5\pi}{6}.$$

$$I_2 = 2\pi i \mathrm{Log}(-6z_0) = 2\pi i\left(\ln(12) + i\frac{5\pi}{6}\right) = -\frac{5\pi^2}{3} + 2\pi\ln(12)i.$$

In the I_3 case the integrand has singularities at 0 and 6 but as $|6-(1+2i)| = |5-2i| > 5$ only the singularity at 0 is inside Γ. Hence we take

$$f(z) = \frac{\mathrm{e}^z}{z-6}, \quad n+1 = 2, \quad z_0 = 0.$$

$$f'(z) = \frac{(z-6)\mathrm{e}^z - \mathrm{e}^z}{(z-6)^2}, \quad f'(0) = -\frac{7}{36}.$$

$$I_3 = 2\pi i f'(0) = -\frac{7\pi i}{18}.$$

This is the version for a 6th digit of 1, 3, 5, 7, 9.

The generalised Cauchy integral formula can be written in the form

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = 2\pi i\frac{f^{(n)}(z_0)}{n!}.$$

In the I_1 case we take $f(z) = z^3 + 3z$, $n+1 = 3$ and $z_0 = -2$. This $f(z)$ is an entire function and z_0 is inside Γ.

$$f''(z) = 6z, \quad f''(-2) = -12.$$

Thus

$$I_1 = \pi i f''(1) = -12\pi i.$$

In the I_2 case we take $z_0 = -1 + i\sqrt{3}$ which is inside Γ and $n+1 = 1$. We also take $f(z) = \mathrm{Log}(z - 9z_0)$. The branch point of $f(z)$ is at $9z_0$ which is outside Γ and the function is analytic inside Γ.

$$|z_0|^2 = 4, \quad |z_0| = 2, \quad -z_0 = 1 - i\sqrt{3}, \quad \mathrm{Arg}(-z_0) = -\frac{\pi}{3}.$$

For $f(z_0) = \mathrm{Log}(-8z_0)$ we have

$$|8z_0| = 16, \quad \mathrm{Arg}(-8z_0) = -\frac{\pi}{3}.$$

$$I_2 = 2\pi i \mathrm{Log}(-8z_0) = 2\pi i\left(\ln(16) - i\frac{\pi}{3}\right) = \frac{2\pi^2}{3} + 2\pi\ln(16)i.$$

In the I_3 case the integrand has singularities at 0 and 3 but as $|3-(-2+i)| = |5-i| > 5$ only the singularity at 0 is inside Γ. Hence we take

$$f(z) = \frac{e^z}{z-3}, \quad n+1 = 2, \quad z_0 = 0.$$

$$f'(z) = \frac{(z-3)e^z - e^z}{(z-3)^2}, \quad f'(0) = -\frac{4}{9}.$$

$$I_3 = 2\pi i f'(0) = -\frac{8\pi i}{9}.$$

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4. *The following was part of question 2 on the May 2021* `MA3614` *exam paper and was worth 5 of the 20 marks of the entire question.*

This part of the question is for all student numbers.

Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. Let $0 < h < 1$ and let $\omega = e^{2\pi i/3}$. By using the generalised Cauchy integral formula (which is stated in the previous part), or otherwise, show the following.

$$\frac{f(h) + \omega^2 f(\omega h) + \omega f(\omega^2 h)}{3h} - f'(0) = \frac{h^3}{2\pi i} \oint_C \frac{f(z)}{(z^3 - h^3)z^2} \, dz.$$

Solution

By using the Cauchy integral formula or the generalised version for each of the terms on the left hand side we get

$$\frac{1}{2\pi i} \oint_C f(z) \left(\frac{1}{3h} \left(\frac{1}{z-h} + \frac{\omega^2}{z-\omega h} + \frac{\omega}{z-\omega^2 h} \right) - \frac{1}{z^2} \right) dz.$$

We consider parts of the integrand. As 1, ω and ω^2 are the 3 roots of unity we have $z^3 - h^3 = (z-h)(z-\omega h)(z-\omega^2 h)$.

$$\frac{1}{z-h} + \frac{\omega^2}{z-\omega h} + \frac{\omega}{z-\omega^2 h} = \frac{\text{quadratic term}}{z^3 - h^3}.$$

where the quadratic term is

$$(z-\omega h)(z-\omega^2 h) + \omega^2 (z-h)(z-\omega^2 h) + \omega(z-h)(z-\omega h).$$

The coefficient of z^2 is $1 + \omega + \omega^2 = 0$.

The constant term is

$$h^2(\omega^3 + \omega^4 + \omega^2) = h^2(1 + \omega + \omega^2) = 0.$$

The coefficient of z is

$$-h(\omega + \omega^2 + \omega^2(1+\omega^2) + \omega(1+\omega)) = -3h(\omega + \omega^2) = 3h.$$

Thus

$$\frac{1}{3h}\left(\frac{1}{z-h}+\frac{\omega^2}{z-\omega h}+\frac{\omega}{z-\omega^2 h}\right)=\frac{z}{z^3-h^3}.$$

$$\frac{z}{z^3-h^3}-\frac{1}{z^2}=\frac{z^3-(z^3-h^3)}{(z^3-h^3)z^2}=\frac{h^3}{(z^3-h^3)z^2}$$

and the identity is shown.

5. *The following was part of question 2 on the Aug 2020* `MA3614` *exam paper and was worth 11 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

   i.
   $$\oint_\Gamma \frac{\mathrm{e}^{2z}}{(z-1)^3} \,\mathrm{d}z.$$

   ii.
   $$\oint_\Gamma \frac{z^3}{(z+i)^4} \,\mathrm{d}z.$$

   iii.
   $$\oint_\Gamma \frac{\mathrm{Log}(z+4)}{(z+i)^2} \,\mathrm{d}z,$$

   where Log denotes the principal valued logarithm.

   (b) With the same set-up as the previous part let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction.

   Consider the following two functions $g_1(z)$ and $g_2(z)$ where z is a point which is not on Γ.

   $$g_1(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0}, & \text{when } z \ne z_0, \\ f'(z_0), & z = z_0, \end{cases} \quad \text{and} \quad g_2(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta-z)(\zeta-z_0)} \,\mathrm{d}\zeta.$$

   When z is any point inside Γ explain why $g_1(z) = g_2(z)$.

   Give the value of $g_2(z)$ when z is outside of Γ.

Solution

(a) i. In this case $f(z) = \mathrm{e}^{2z}$, $n = 2$ and $z_0 = 1$. The value of the integral is

$$\frac{2\pi i}{2!} f''(1) = \pi i \left(4\mathrm{e}^2\right).$$

ii. In this case $f(z) = z^3$, $n = 3$ and $z_0 = -i$.

$$f'''(z) = 6.$$

The value of the integral is

$$\frac{2\pi i}{3!} f'''(i) = 2\pi i.$$

iii. In this case $f(z) = \mathrm{Log}(z + 4)$, $n = 1$ and $z_0 = -i$. The function $f(z)$ has a branch point at $z = -4$ and is analytic on and inside the circle being considered.

$$f'(z) = \frac{1}{z+4}, \quad f'(-i) = \frac{1}{-i+4} = \frac{4+i}{17}.$$

The value of the integral is

$$\frac{2\pi i}{1!} f'(-i) = (2\pi i)\left(\frac{4+i}{17}\right) = \frac{2\pi}{17}(-1 + 4i).$$

(b) When $z = z_0$ we have

$$g_2(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\, \mathrm{d}\zeta.$$

By the generalised Cauchy integral formula this is $f'(z_0)$ and hence $g_2(z_0) = g_1(z_0)$.

The Cauchy integral formula used twice gives

$$f(z) - f(z_0) = \frac{1}{2\pi i} \oint f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right) \mathrm{d}\zeta = \frac{1}{2\pi i} \oint f(\zeta) \left(\frac{-z_0 + z}{(\zeta - z)(\zeta - z_0)}\right) \mathrm{d}\zeta.$$

When $z \neq z_0$ dividing by $z - z_0$ hence shows that $g_1(z) = g_2(z)$.

When z is outside of the closed loop the function $\phi(\zeta) = f(\zeta)/(\zeta - z)$ is analytic as a function of ζ inside the closed loop and hence by the Cauchy integral formula applied to $\phi(\zeta)$ we have

$$g_2(z) = \phi(z_0) = \frac{f(z_0)}{z_0 - z} = -\frac{f(z_0)}{z - z_0}.$$

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6. *The following was part of question 2 on the May 2020* MA3614 *exam paper and was worth 12 of the 20 marks of the entire question.*

(a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

Use this result to evaluate the following when Γ is the circle $\{z : \ |z| = 4\}$. In your answer you must show intermediate workings.

i.
$$\oint_\Gamma \frac{z\mathrm{e}^z}{(z-1)^3}\,\mathrm{d}z.$$

ii.
$$\oint_\Gamma \frac{\sin(\pi z)}{(z-2)^2(z+6)}\,\mathrm{d}z.$$

(b) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. By using the generalised Cauchy integral formula (which is stated in the previous part), or otherwise, show that when $0 < h < 1$ we have the following.

$$\begin{aligned}
\frac{f(h)-f(-h)}{2h} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^2-h^2}\,\mathrm{d}z, \\
\frac{f(h)-f(-h)}{2h} - f'(0) &= \frac{h^2}{2\pi i}\oint_C \frac{f(z)}{z^2(z^2-h^2)}\,\mathrm{d}z, \\
\frac{f(h)-if(ih)-f(-h)+if(-ih)}{4h} &= \frac{1}{2\pi i}\oint_C \frac{z^2 f(z)}{z^4-h^4}\,\mathrm{d}z.
\end{aligned}$$

Solution

(a) i. To use the generalised Cauchy integral formula we take $f(z) = z\mathrm{e}^z$, $n = 2$ and $z_0 = 1$.

$$f''(z) = z\mathrm{e}^z + 2\mathrm{e}^z = (z+2)\mathrm{e}^z.$$

The value is

$$\frac{2\pi i}{2} f''(z_0) = i(3\pi\mathrm{e}).$$

ii. To use the generalised Cauchy integral formula we take $f(z) = \sin(\pi z)/(z+6)$, $n = 1$ and $z_0 = 2$. $f(z)$ is analytic inside the circle Γ.

$$f'(z) = \frac{-\sin(\pi z)}{(z+6)^2} + \frac{\pi\cos(\pi z)}{z+6}.$$

The value is

$$2\pi i f'(z_0) = 2\pi i \frac{\pi}{8} = i\frac{\pi^2}{4}.$$

(b) Using the Cauchy integral formula for $f(h)$ and $f(-h)$ we have

$$f(h) - f(-h) = \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-h} - \frac{1}{z+h}\right)\mathrm{d}z = \frac{1}{2\pi i}\oint_C f(z)\left(\frac{2h}{z^2-h^2}\right)\mathrm{d}z.$$

Thus

$$\frac{f(h)-f(-h)}{2h} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^2-h^2}\,\mathrm{d}z.$$

From considering the integrand in the expression for $f'(0)$ we have

$$\frac{1}{z^2-h^2} - \frac{1}{z^2} = \frac{h^2}{z^2(z^2-h^2)}.$$

Thus

$$\frac{f(h)-f(-h)}{2h} - f'(0) = \frac{h^2}{2\pi i}\oint_C \frac{f(z)}{z^2(z^2-h^2)}\,\mathrm{d}z.$$

Now

$$-if(ih) + if(-ih) = \frac{f(ih)-f(-ih)}{i}$$

and hence

$$\frac{f(h) - if(ih) - f(-h) + if(-ih)}{4h} = \frac{1}{2}\left(\frac{f(h)-f(-h)}{2h} + \frac{f(ih)-f(-ih)}{2ih}\right).$$

By replacing h by ih in the earlier result gives

$$\frac{f(ih)-f(-ih)}{2ih} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^2+h^2}\,\mathrm{d}z.$$

As

$$\frac{1}{z^2-h^2} + \frac{1}{z^2+h^2} = \frac{2z^2}{z^4-h^4}$$

we get

$$\frac{f(h) - if(ih) - f(-h) + if(-ih)}{4h} = \frac{1}{2\pi i}\oint_C \frac{z^2 f(z)}{z^4-h^4}\,\mathrm{d}z.$$

7. *The following was part of question 2 on the May 2019* `MA3614` *exam paper and was worth 6 of the 20 marks of the entire question.*

   Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle with centre at $1+i$ and radius 2.

   (a)
$$\oint_\Gamma \frac{\mathrm{Log}(3-z)}{(z-2)^3} \, \mathrm{d}z,$$
   where Log denotes the principal value logarithm.

   (b)
$$\oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-3)} \, \mathrm{d}z.$$

Solution

(a) The integrand has an isolated singularity at 2 which is inside the circle Γ. We can use the generalised formula with $n=2$, $z_0=2$ and $f(z)=\mathrm{Log}(3-z)$ and the value of the integral is
$$I = \frac{2\pi i}{2} f''(2).$$
$$f'(z) = \frac{-1}{3-z} = \frac{1}{z-3} \quad \text{and} \quad f''(z) = -\frac{1}{(z-3)^2}.$$
Hence $I=-\pi i$.

(b) The integrand has isolated singularities at 0 and at 3 but only $z=0$ is inside the loop. We can use the generalised formula with $n=1$, $z_0=0$ and $f(z)=\mathrm{e}^z/(z-3)$ and the value of the integral is
$$I = 2\pi i f'(0).$$
$$f'(z) = \mathrm{e}^z \left(\frac{1}{z-3} - \frac{1}{(z-3)^2} \right).$$
$$f'(0) = -\frac{1}{3} - \frac{1}{9} = -\frac{4}{9}.$$
$$I = -\frac{8\pi i}{9}.$$

8. *The following was part of question 2 on the May 2017* `MA3614` *exam paper and was worth 9 of the 20 marks of the entire question.*

(a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, \mathrm{d}z.$$

Use this result to evaluate the following when Γ is the circle $\{z : \ |z-1| = 2\}$.

i.
$$\oint_\Gamma \frac{\mathrm{e}^z}{z(4-z)}\, \mathrm{d}z.$$

ii.
$$\oint_\Gamma \frac{z^4}{(z-1)^3}\, \mathrm{d}z.$$

(b) By using the generalised Cauchy integral formula given in the previous part of the question, show that if Γ is a closed loop traversed once in the anti-clockwise direction, $z_0 \ne 0$ and 0 are inside Γ and f is a function analytic on Γ and inside Γ, then
$$\frac{z_0^2}{2\pi i} \oint_\Gamma \frac{f(z)}{z^2(z-z_0)}\, \mathrm{d}z = f(z_0) - f(0) - z_0 f'(0).$$

Solution

(a) i. The integrand only has a singularity inside the contour at $z_0 = 0$ and the conditions of the integral formula hold if we take $n = 0$ and

$$f(z) = \frac{\mathrm{e}^z}{4-z}.$$

The value of the integral is

$$2\pi i f(0) = \frac{2\pi i}{4} = \frac{\pi i}{2}.$$

ii. The integrand only has a singularity inside the contour at $z_0 = 1$ and the condition of the integral formula hold if we take $n = 2$ and

$$f(z) = z^4.$$

The value of the integral is

$$2\pi i \frac{f''(1)}{2} = 2\pi i \frac{12}{2} = 12\pi i.$$

(b) We consider the right hand side as we can represent each of the 3 terms using the generalised Cauchy integral formula.

$$2\pi i(f(z_0) - f(0) - z_0 f'(0)) = \oint_\Gamma f(z) \left(\frac{1}{z-z_0} - \frac{1}{z} - \frac{z_0}{z^2} \right) \mathrm{d}z.$$

Now

$$\frac{1}{z-z_0}-\frac{1}{z}-\frac{z_0}{z^2} = \frac{z^2-z(z-z_0)-z_0(z-z_0)}{z^2(z-z_0)} = \frac{z_0^2}{z^2(z-z_0)}.$$

We thus have

$$2\pi i(f(z_0)-f(0)-z_0 f'(0)) = \oint_\Gamma f(z)\frac{z_0^2}{z^2(z-z_0)}\,\mathrm{d}z$$

as required.

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9. *The following was part of question 2 on the May 2016* MA3614 *exam paper and was worth 5 of the 20 marks of the entire question.*

   Let C be the unit circle traversed once in the anti-clockwise direction. Let $g(z)$ denote a function which is continuous on C and let

   $$G(z) = \oint_C \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

   Given that for all z inside the unit circle the limit

   $$\lim_{h\to 0}\oint_C \frac{g(\zeta)\mathrm{d}\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

   exists explain why $G(z)$ is analytic in $\{z:\ |z|<1\}$ with derivative

   $$\oint_C \frac{g(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta.$$

Solution

The function $G(z)$ is analytic inside C if the following limit exists for all z inside C.

$$\lim_{h\to 0}\frac{G(z+h)-G(z)}{h}.$$

The integrand in the $G(z+h)-G(z)$ part involves

$$\frac{1}{\zeta-(z+h)}-\frac{1}{\zeta-z} = \frac{(\zeta-z)-(\zeta-(z+h))}{(\zeta-(z+h))(\zeta-z)} = \frac{h}{(\zeta-(z+h))(\zeta-z)}.$$

Thus

$$\begin{aligned} & \frac{G(z+h)-G(z)}{h} - \oint_C \frac{g(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta \\ = & \oint_C \frac{g(\zeta)}{\zeta-z}\left(\frac{1}{(\zeta-(z+h))}-\frac{1}{(\zeta-z)}\right)\mathrm{d}\zeta \\ = & \ h\oint_C \frac{g(\zeta)}{(\zeta-(z+h))(\zeta-z)^2}\,\mathrm{d}\zeta \to 0 \quad \text{as } h\to 0 \end{aligned}$$

using the condition in the question. Hence G is analytic inside C with the specified derivative.

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10. *The following was part of question 2 on the May 2016* `MA3614` *exam paper and was worth 6 of the 20 marks of the entire question.*

    Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

    Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

    (a)
$$\oint_\Gamma \frac{z^2}{(z-2)^3}\,\mathrm{d}z.$$

    (b)
$$\oint_\Gamma \frac{z\,\mathrm{e}^{3z}}{(z+2)^2}\,\mathrm{d}z.$$

Solution

(a) With $f(z) = z^2$, $z_0 = 2$ and $n = 2$ we have $f''(z) = 2$ and

$$f''(2) = \frac{2}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-2)^3}\,\mathrm{d}z.$$

The value is hence

$$\oint_\Gamma \frac{f(z)}{(z-2)^3}\,\mathrm{d}z = \pi i f''(2) = 2\pi i.$$

(b) Now take $n = 1$ and $f(z) = z\mathrm{e}^{3z}$. The value of the integral is

$$2\pi i f'(-2).$$

$$f'(z) = \mathrm{e}^{3z}(1+3z).$$

The value is hence

$$2(-5)\pi i\mathrm{e}^{-6} = -10\pi\mathrm{e}^{-6} i.$$

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11. Suppose that $f(z)$ is analytic in a domain which contains $\{z : \ |z| \le 1\}$ and suppose that $f(0) = 0$. Show that the function

$$F(z) = \begin{cases} \dfrac{f(z)}{z}, & z \ne 0, \\ f'(0), & z = 0 \end{cases}$$

    is analytic at $z = 0$. [Hint use the Cauchy integral formula and the result of question 9.]

Solution

Let Γ denote the unit circle. The function $f(z)/z$ is analytic on the unit circle and by question 1 the function

$$G(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)/\zeta}{\zeta - z} \, \mathrm{d}\zeta$$

is analytic for z inside Γ.

$$G(0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta^2} \, \mathrm{d}\zeta = f'(0)$$

by the generalised Cauchy integral formula.

For $z \neq 0$ we have

$$\frac{1}{\zeta(\zeta - z)} = \frac{A}{\zeta} + \frac{B}{\zeta - z}$$

and

$$1 = A(\zeta - z) + B\zeta, \quad A = -\frac{1}{z}, \quad B = \frac{1}{z}.$$

Thus for $z \neq 0$

$$\begin{aligned} G(z) &= \left(\frac{1}{2\pi i}\right)\left(\frac{1}{z}\right) \oint_\Gamma \left(-\frac{f(\zeta)}{\zeta} + \frac{f(\zeta)}{\zeta - z}\right) \mathrm{d}\zeta \\ &= \left(\frac{1}{z}\right)(-f(0) + f(z)) = \frac{f(z)}{z} \end{aligned}$$

by using the Cauchy integral formula for each part and that $f(0) = 0$.

The function $G(z)$ is hence analytic and it is the same as $F(z)$.

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12. Let C be the unit circle traversed once in the anti-clockwise direction, left $f(z)$ be a function analytic inside C and let h be such that $0 < |h| < 1/2$. Show the following involving approximations to $f'(0)$.

$$\begin{aligned} \frac{-f(2h) + 4f(h) - 3f(0)}{2h} &= \frac{1}{2\pi i} \oint_C \frac{(z - 3h) f(z)}{(z - 2h)(z - h)z} \, \mathrm{d}z, \\ \frac{-f(2h) + 4f(h) - 3f(0)}{2h} - f'(0) &= -\frac{2h^2}{2\pi i} \oint_C \frac{f(z)}{(z - 2h)(z - h)z^2} \, \mathrm{d}z. \end{aligned}$$

Solution

By using the Cauchy integral formula for each of $f(2h)$, $f(h)$ and $f(0)$ and combining the integrands we get

$$-f(2h) + 4f(h) - f(0) = \frac{1}{2\pi i} \oint_C \left(-\frac{1}{z - 2h} + \frac{4}{z - h} - \frac{3}{z}\right) \mathrm{d}z.$$

Now

$$\begin{aligned} -\frac{1}{z - 2h} + \frac{4}{z - h} - \frac{3}{z} &= \frac{-(z - h)z + 4(z - 2h)z - 3(z - 2h)(z - h)}{(z - 2h)(z - h)z} \\ &= \frac{2hz - 6h^2}{(z - 2h)(z - h)z}. \end{aligned}$$

Thus
$$\frac{-f(2h)+4f(h)-3f(0)}{2h} = \frac{1}{2\pi i}\oint_C \frac{(z-3h)f(z)}{(z-2h)(z-h)z}\,\mathrm{d}z.$$

For the second part we make use of
$$f'(0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^2}\,\mathrm{d}z$$

and note that
$$\frac{(z-3h)}{(z-2h)(z-h)z} - \frac{1}{z^2} = \frac{z(z-3h)-(z-2h)(z-h)}{(z-2h)(z-h)z^2} = -\frac{2h^2}{(z-2h)(z-h)z^2}$$

The result then follows.