More exercises before the class test

There are a possibly slightly more questions than can be covered in the week 12 sessions and modification may be made depending on what is asked in the Monday and Tuesday sessions. Solutions to all questions will be given whether they are answered or not during the sessions.

1. *This question was in the class test in 2015/6 and was worth 8 marks.*

   Let $f(z)$ be defined by
$$f(z) = \frac{1}{z-1}.$$

   (a) Give in cartesian form the following complex numbers: $f(-1)$, $f(i)$ and $f(-i)$.

   (b) Prove that the real part of $f(\mathrm{e}^{i\theta})$ is constant for $\theta \in (0, 2\pi)$.

Solution

(a)

$$\begin{aligned} f(-1) &= \frac{1}{-1-1} = -\frac{1}{2}, \\ f(i) &= \frac{1}{i-1} = \frac{-i-1}{2}, \\ f(-i) &= \frac{1}{-i-1} = \frac{i-1}{2}. \end{aligned}$$

(b)

$$\frac{1}{\mathrm{e}^{i\theta}-1} = \frac{\mathrm{e}^{-i\theta}-1}{|\mathrm{e}^{\theta}-1|^2}.$$

Now for the denominator

$$|\mathrm{e}^{\theta}-1|^2 = (\mathrm{e}^{i\theta}-1)(\mathrm{e}^{-i\theta}-1) = 2 - (\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}) = 2(1-\cos\theta).$$

The real part of the numerator is $\cos\theta - 1$. Hence the real part of the expression is $-1/2$ for all $\theta \in (0, 2\pi)$.

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2. Let $f(z)$ be defined by
$$f(z) = \frac{4z-1}{z-4}.$$
   Determine z in cartesian form such that $f(z) = -i$.

Solution

$$\frac{4z-1}{z-4} = -i \quad \text{gives} \quad 4z-1 = -i(z-4).$$

Thus

$$(4+i)z = 1+4i, \quad \text{and} \quad z = \frac{1+4i}{4+i} = \frac{(1+4i)(4-i)}{17} = \frac{8+15i}{17}.$$

Just to note that $|1+4i|^2 = |4+i|^2 = 17$, the number has magnitude 1.

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3. *This question was in the class test in 2014/5 and was worth 6 marks.*

$$f(z) = \frac{z-1}{z+1}.$$

Show that for $-\pi < \theta < \pi$ we have

$$f(\mathrm{e}^{i\theta}) = i \tan\left(\frac{\theta}{2}\right).$$

Solution

$$\begin{aligned} f(\mathrm{e}^{i\theta}) &= \frac{\mathrm{e}^{i\theta}-1}{\mathrm{e}^{i\theta}+1} = \frac{\mathrm{e}^{i\theta/2}-\mathrm{e}^{-i\theta/2}}{\mathrm{e}^{i\theta/2}+\mathrm{e}^{-i\theta/2}} \\ &= \frac{2i\sin(\theta/2)}{2\cos(\theta/2)} = i\tan\left(\frac{\theta}{2}\right). \end{aligned}$$

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4. Let $\omega = \mathrm{e}^{2\pi i/5}$, where as usual $i = \sqrt{-1}$ is the imaginary unit, and let $c = \cos(2\pi/5)$.

   (a) What are the following values in cartesian form?

   $$\omega^5 \quad \text{and} \quad 1 + \omega + \omega^2 + \omega^3 + \omega^4.$$

   (b) Explain the following.

   $$\begin{aligned} \omega + \omega^4 &= 2c, \\ \omega^2 + \omega^3 &= 2(2c^2 - 1), \\ 4c^2 + 2c - 1 &= 0, \\ c &= \frac{-1+\sqrt{5}}{4}, \\ \cos(4\pi/5) &= \frac{-1-\sqrt{5}}{4}. \end{aligned}$$

Solution

(a) From the addition property of the exponential

$$\omega^5 = \left(\mathrm{e}^{2\pi i/5}\right)^5 = \mathrm{e}^{2\pi i} = 1.$$

ω is one of the roots of unity.
By the finite geometric series

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = \frac{1-\omega^5}{1-\omega} = 0$$

as $\omega^5 = 1$.

(b)

$$\begin{aligned}\omega+\omega^4 &= \mathrm{e}^{2\pi i/5}+\mathrm{e}^{8\pi i/5},\\ &= \mathrm{e}^{2\pi i/5}+\mathrm{e}^{-2\pi i/5},\\ &= 2\cos(2\pi/5)=2c\end{aligned}$$

using the definition of the cosine.
Similarly

$$\begin{aligned}\omega^2+\omega^3 &= \mathrm{e}^{4\pi i/5}+\mathrm{e}^{6\pi i/5},\\ &= \mathrm{e}^{4\pi i/5}+\mathrm{e}^{-4\pi i/5},\\ &= 2\cos(4\pi/5)=2(2\cos^2(2\pi/5)-1)=2(2c^2-1)\end{aligned}$$

using the cosine double angle formula in the last step.
From the previous results

$$1+(\omega+\omega^4)+(\omega^2+\omega^3)=1+2c+2(2c^2-1)=0.$$

This rearranges to

$$4c^2+2c-1=0.$$

$2\pi/5$ is in the first quadrant and $c>0$. The positive root of the quadratic is

$$c=\frac{-2+\sqrt{4-(-16)}}{8}=\frac{-1+\sqrt{5}}{4}.$$

By the double angle formula for the cosine

$$\cos(4\pi/5)=2\cos^2(2\pi/5)-1=2c^2-1.$$

$$\begin{aligned}\cos(4\pi/5) &= 2\left(\frac{1-2\sqrt{5}+5}{16}\right)-1\\ &= \frac{6-2\sqrt{5}}{8}-1=\frac{3-\sqrt{5}}{4}-1\\ &= \frac{-1-\sqrt{5}}{4}.\end{aligned}$$

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5. *This question was in the class test in 2013/4 and was worth 25 marks.*

Determine if the following functions are analytic in $\mathbb{C}$ and if a function is analytic express it in terms of z alone.

(a)

$$f(x+iy)=y.$$

(b)
$$f(x+iy) = 2 + y^3 - 3x^2y + 2x + i(x^3 - 3xy^2 + 2y).$$

Solution

(a)
$$u = y, \quad v = 0.$$
$$\frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 0.$$

The Cauchy Riemann equation involving these two partial derivatives is not satisfied and hence the function is not analytic.

(b)
$$u = 2 + y^3 - 3x^2y + 2x \quad \text{and} \quad v = x^3 - 3xy^2 + 2y$$
$$\frac{\partial u}{\partial x} = -6xy + 2, \quad \frac{\partial v}{\partial y} = -6xy + 2, \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = 3x^2 - 3y^2.$$

The Cauchy Riemann equations are satisfied at all points and thus the function is analytic everywhere.

To express in terms of z note that $f(0) = 2$ and

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x} = (-6xy + 2) + i(3x^2 - 3y^2), \\ f''(z) &= \frac{\partial^2 f}{\partial x^2} = (-6y) + i(6x), \\ f'''(z) &= \frac{\partial^3 f}{\partial x^3} = 6i. \end{aligned}$$

Thus
$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \frac{f'''(0)}{6}z^3 = 2 + 2z + iz^3.$$

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6. *This question was in the class test in 2014/5 and was worth 10 marks.*

Determine the following in Cartesian form.

(a) $\mathrm{Log}(1-i)$.

(b) z^α where $z = 1 - i$ and $\alpha = 1 + i$ and where we mean the principal value of the complex power.

Solution

(a) In polars
$$1 - i = re^{i\theta}, \quad r = \sqrt{2}, \quad \theta = -\frac{\pi}{4}.$$

Thus
$$\mathrm{Log}(1-i) = \frac{1}{2}\mathrm{Log}(2) - i\frac{\pi}{4}.$$

(b) With $\alpha = 1+i$,

$$\alpha \mathrm{Log}(1-i) = (1+i)\left(\frac{1}{2}\mathrm{Log}(2) - i\frac{\pi}{4}\right) = \left(\frac{1}{2}\mathrm{Log}(2) + \frac{\pi}{4}\right) + i\left(\frac{1}{2}\mathrm{Log}(2) - \frac{\pi}{4}\right).$$

$$z^\alpha = \exp\left(\frac{1}{2}\mathrm{Log}(2) + \frac{\pi}{4}\right)(\cos(\beta) + i\sin(\beta)), \quad \beta = \left(\frac{1}{2}\mathrm{Log}(2) - \frac{\pi}{4}\right).$$

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7. The definition of $\tan(z)$ is

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \left(\frac{1}{i}\right)\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}.$$

Show that

$$\tan(2z) = \frac{2\tan(z)}{1 - \tan^2(z)}.$$

Solution

First note that by replacing z by $2z$ gives

$$\tan(2z) = \left(\frac{1}{i}\right)\left(\frac{\mathrm{e}^{2iz} - \mathrm{e}^{-2iz}}{\mathrm{e}^{2iz} + \mathrm{e}^{-2iz}}\right).$$

The right hand side expression is

$$\frac{2\tan(z)}{1 - \tan^2(z)} = \frac{\left(\frac{2}{i}\right)\left(\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}\right)}{1 + \left(\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}\right)^2}$$

We multiply numerator and denominator by $(\mathrm{e}^{iz} + \mathrm{e}^{-iz})^2$ to give

$$\frac{\left(\frac{2}{i}\right)(\mathrm{e}^{iz} - \mathrm{e}^{-iz})(\mathrm{e}^{iz} + \mathrm{e}^{-iz})}{(\mathrm{e}^{iz} + \mathrm{e}^{-iz})^2 + (\mathrm{e}^{iz} - \mathrm{e}^{-iz})^2}$$

$$= \frac{\left(\frac{2}{i}\right)(\mathrm{e}^{2iz} - \mathrm{e}^{-2iz})}{2(\mathrm{e}^{2iz} + \mathrm{e}^{-2iz})} = \left(\frac{1}{i}\right)\left(\frac{\mathrm{e}^{2iz} - \mathrm{e}^{-2iz}}{\mathrm{e}^{2iz} + \mathrm{e}^{-2iz}}\right).$$

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8. Give in polar form all values of z which satisfy

$$z^{10} = -1 + i$$

and which have negative real part.

Solution

The magnitude of $-1+i$ is $\sqrt{2}$ and the principal argument is a value in $(\pi/2, \pi)$. Specifically a polar form of $-1+i$ is

$$2^{1/2}\exp(i3\pi/4).$$

One of the 10 solutions of $z^{10} = -1+\sqrt{3}i$ is

$$2^{1/20}\exp\left(i\frac{3\pi}{40}\right).$$

This is the principal solution. All 10 solutions are

$$2^{1/20}\exp\left(i\frac{3\pi}{40} + i\frac{2k\pi}{10}\right) = 2^{1/20}\exp\left(i\frac{3\pi}{40} + i\frac{k\pi}{5}\right) \quad k = 0, 1, \ldots, 9.$$

In the way the angles are described above the points with negative real part have an angle in $(\pi/2, 3\pi/2)$. As the points are equally spaced there will be about half the points which satisfy this.

When $k = 2$ the angle is

$$\pi\left(\frac{3}{40} + \frac{2}{5}\right) = \pi\left(\frac{3}{40} + \frac{16}{40}\right) = \frac{19\pi}{40} < \frac{\pi}{2}.$$

This point has positive real part.

When $k = 3$ the angle is

$$\frac{27\pi}{15} > \frac{\pi}{2}.$$

Hence z_3 has negative real part.

Now $z_7 = -z_2$ amd $z_8 = -z_3$. Hence z_7 has negative real part and z_8 has positive real part.

The 5 values corresponding to $k = 3, 4, 5, 6, 7$ have negative real part.

Just to note that the other 5 points have positive real part.

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9. Show that the following function $f(z)$, $z = x+iy$, $(x, y \in \mathbb{R})$ is analytic and express it in terms of z alone

$$f(x+iy) = (x^2 - y^2 - x + y - 2xy) + i(x^2 - y^2 - x + 2xy - y).$$

Solution

Let the real and imaginary parts be denoted by

$$u = x^2 - y^2 - x + y - 2xy \quad v = x^2 - y^2 - x + 2xy - y.$$

$$\frac{\partial u}{\partial x} = 2x - 1 - 2y, \quad \frac{\partial v}{\partial y} = -2y + 2x - 1 \quad \text{and hence } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

Also,

$$\frac{\partial u}{\partial y} = -2y + 1 - 2x, \quad \frac{\partial v}{\partial x} = 2x - 1 + 2y \quad \text{and hence } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

Both Cauchy Riemann equations are satisfied and hence f is analytic. $f(z)$ is a polynomial in z of degree 2 and one way to express in terms of z is to construct the Taylor polynomial. We have $f(0) = 0$.

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x} = (2x - 1 - 2y) + i(2x - 1 + 2y), \quad f'(0) = -1 - i, \\ f''(z) &= \frac{\partial^2 f}{\partial x^2} = 2 + 2i. \end{aligned}$$

The Taylor polynomial is

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2} z^2 = -(1+i)z + (1+i)z^2 = (1+i)(z^2 - z).$$

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10. Show that $u(x, y) = \sin(ax)\cosh(ay)$ is harmonic, where a is a real constant, and determine a harmonic conjugate of u.

Solution

$$\begin{aligned} \frac{\partial u}{\partial x} &= a\cos(ax)\cosh(ay), \quad \frac{\partial u}{\partial y} = a\sin(ax)\sinh(ay), \\ \frac{\partial^2 u}{\partial x^2} &= -a^2 u, \quad \frac{\partial^2 u}{\partial y^2} = a^2 u. \end{aligned}$$

Hence u is harmonic as

$$\nabla^2 u = 0.$$

Let v denote the harmonic conjugate. By one of the Cauchy Riemann equations it satisfies

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -a\sin(ax)\sinh(ay).$$

Partially integrating with respect to x gives

$$v(x, y) = \cos(ax)\sinh(ay) + g(y)$$

for any function $g(y)$. Using the other Cauchy Riemann equation we have

$$\frac{\partial v}{\partial y} = a\cos(ax)\cosh(ay) + g'(y) = \frac{\partial u}{\partial x} \quad \text{giving } g'(y) = 0.$$

Thus

$$v = \cos(ax)\sinh(ay) + \text{const}.$$

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11. Determine the partial fraction representation of $f(z)$ given by

$$f(z) = \frac{1}{z^2(z^2-1)}.$$

Solution

$z^2(z^2-1) = z^2(z-1)(z+1)$ has a pole at 0 or order 2 and simple poles at ± 1. The partial fraction representation is

$$f(z) = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C_1}{z} + \frac{C_2}{z^2}$$

As the denominator is already factored we have

$$\begin{aligned} A &= \lim_{z\to 1}(z-1)f(z) = \left.\frac{1}{z^4(z+1)}\right|_{z=1} = \frac{1}{2}, \\ B &= \lim_{z\to -1}(z+1)f(z) = \left.\frac{1}{z^4(z-1)}\right|_{z=-1} = -\frac{1}{2}. \end{aligned}$$

$$z^2 f(z) = \frac{1}{z^2-1} = C_2 + C_1 z + z^2(\text{function analytic at } z=0) \to C_2 \quad \text{as } z \to 0.$$

Thus $C_2 = -1$. With only C_1 still to find one way to determine it is to just let $z = 2$ in the expression giving

$$f(2) = \frac{1}{12} = 1/2 + \frac{-1/2}{3} + \frac{C_1}{2} + \frac{-1}{4} = \frac{1}{12} + \frac{C_1}{2}.$$

Thus $C_1 = 0$. This could have also be deduced by noting that $f_1(z)$ is even. A further may is to note that

$$zf(z) = \frac{1}{z(z^2-1)} \to 0 \quad \text{as } z \to \infty$$

and

$$z\left(\frac{A}{z-1} + \frac{B}{z+1} + \frac{C_1}{z} + \frac{C_2}{z^2}\right) \to A + B + C_1 \quad \text{as } z \to \infty.$$

Thus $A + B + C_1 = 0$ which gives $C_1 = 0$.

To summarize,

$$f(z) = \frac{1}{z^2(z^2-1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) - \frac{1}{z^2}.$$

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12. Determine the residue at $z = i$ of

$$\frac{z^8}{z^8-1}.$$

Solution

The residue at $z = i$ is given by

$$\lim_{z\to i}(z-i)\frac{z^8}{z^8-1} = \left(\lim_{z\to i} z^8\right)\lim_{z\to i}\frac{z-i}{z^8-1} = \frac{1}{8i^7} = \frac{i}{8}.$$

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13. *This question was in the class test in 2015/6 and was worth 24 marks.*

(a) Express the function $f_1(z)$ defined below in partial fraction form and state the residue at any pole.

$$f_1(z) = \frac{3z}{(z-1)(z+2)}.$$

(b) Determine the residue at 2 of the following rational function.

$$f_2(z) = \frac{z^8}{z^2-4}.$$

(c) Determine the residue at -1 of the following rational function.

$$f_3(z) = \frac{z(z+3)}{(z+1)^3}.$$

Solution

(a)

$$f_1(z) = \frac{3z}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}.$$

A is the residue at $z = 1$ and B is the residue at $z = -2$.

$$A = \lim_{z\to 1}(z-1)f_1(z) = \left.\frac{3z}{z+2}\right|_{z=1} = 1$$

and

$$B = \lim_{z\to -2}(z+2)f_1(z) = \left.\frac{3z}{z-1}\right|_{z=-2} = \frac{-6}{-3} = 2.$$

(b) $z^2 - 4 = (z-2)(z+2)$.

$$f_2(z) = \frac{z^8}{z^2-4} = (\text{polynomial of degree 6}) + \frac{A}{z-2} + \frac{B}{z+2}.$$

A is the residue at $z = 2$ and B is the residue at $z = -2$.

$$A = \lim_{z\to 2}(z-2)f_2(z) = \left.\frac{z^8}{z+2}\right|_{z=2} = \frac{256}{4} = 64.$$

(c) Let

$$g(z) = z(z+3) = z^2 + 3z.$$

The derivatives are

$$g'(z) = 2z + 3, \quad g''(z) = 2.$$

Hence

$$g(z) = g(-1) + g'(-1)(z+1) + \frac{g''(-1)}{2}(z+1)^2 = -2 + (z+1) + (z+1)^2.$$

Thus

$$f_3(z) = \frac{z(z+3)}{(z+1)^3} = \frac{-2}{(z+1)^3} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)}$$

and the residue at -1 is 1.

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14. Determine the principal value of each of the following.

$$(-1)^{1/2}, \qquad 1^i, \qquad (-i)^{1/2}, \qquad (\sqrt{2}(1+i))^{1+i}.$$

Solution

The definitions of the principal valued logarithm and the principal valued complex power are

$$\begin{aligned} \operatorname{Log} z &= \operatorname{Log}|z| + i\operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z \le \pi, \\ z^\alpha &= \exp(\alpha \operatorname{Log} z). \end{aligned}$$

In all cases $|z| = 1$ and thus $\operatorname{Log}|z| = 0$.

-1 has a principal argument of π and thus $(-1)^{1/2} = \exp(i\pi/2) = i$.

The principal argument of 1 is 0 and thus $1^i = 1$.

The principal argument of $-i$ is $-\pi/2$ and thus $\operatorname{Log}(-i) = -i\pi/2$ and

$$i^{1/2} = \exp(-i\pi/4) = \frac{1-i}{\sqrt{2}}.$$

The polar form of $\sqrt{2}(1+i)$ is

$$\sqrt{2}(1+i) = 2\exp(i\pi/4)$$

and thus

$$\operatorname{Log}(\sqrt{2}(1+i)) = \operatorname{Log}(2) + i\frac{\pi}{4}.$$

To complete the answer we need

$$(1+i)\operatorname{Log}(\sqrt{2}(1+i)) = \left(\operatorname{Log}(2) - \frac{\pi}{4}\right) + i\left(\operatorname{Log}(2) + \frac{\pi}{4}\right)$$

and

$$(\sqrt{2}(1+i))^{1+i} = \exp((1+i)\mathrm{Log}(\sqrt{2}(1+i))) = re^{i\theta}$$

with

$$r = \exp\left(\mathrm{Log}(2) - \frac{\pi}{4}\right), \quad \theta = \left(\mathrm{Log}(2) + \frac{\pi}{4}\right).$$

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15. *This question was in the class test in 2015/6 and was worth 6 marks.*

 Determine in polar form and in cartesian form the 3 solutions of

$$z^3 = i.$$

Solution

The polar form of i is

$$i = \cos(\pi/2) + i\sin(\pi/2).$$

The polar form of the 3 solutions is thus

$$z_k = \cos(\pi/6 + 2k\pi/3) + i\sin(\pi/6 + 2k\pi/3), \quad k = 0, 1, 2.$$

In cartesian form

$$\begin{aligned} z_0 &= \cos(\pi/6) + i\sin(\pi/6) = \frac{\sqrt{3}+i}{2}, \\ z_1 &= \cos(5\pi/6) + i\sin(5\pi/6) = \frac{-\sqrt{3}+i}{2}, \\ z_2 &= \cos(3\pi/2) + i\sin(3\pi/2) = -i. \end{aligned}$$

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16. Determine the partial fraction representation of

$$f(z) = \frac{1}{z(z-1)^3}$$

 and state the residues at the poles.

Solution

The partial fraction representation has the form

$$f(z) = \frac{A}{z} + \frac{B_1}{z-1} + \frac{B_2}{(z-1)^2} + \frac{B_3}{(z-1)^3}$$

with A being the residue at $z = 0$ and with B_1 being the residue at $z = 1$.

Note that

$$zf(z) = \frac{1}{(z-1)^3} \quad \text{and} \quad (z-1)^3 f(z) = \frac{1}{z}.$$

$$A = \lim_{z=0} z f(z) = \left. \frac{1}{(z-1)^3} \right|_{z=0} = -1.$$

$$(z-1)^3 f(z) = B_3 + B_2(z-1) + B_1(z-1)^2 + (z-1)^3(\text{function analytic at } 1).$$

$$B_3 = \left. \frac{1}{z} \right|_{z=1} = 1.$$

$$B_2 = \left. \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{z} \right|_{z=1} = \left. \frac{-1}{z^2} \right|_{z=1} = -1.$$

$$2B_1 = \left. \frac{\mathrm{d}}{\mathrm{d}z} \frac{-1}{z^2} \right|_{z=1} = \left. \frac{2}{z^3} \right|_{z=1} = 2.$$

Thus $B_1 = 1$.

To summarize the answer we have

$$f(z) = \frac{1}{z(z-1)^3} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}.$$

As a comment on the values, the sum of the residues is $A + B_1 = 0$. That the sum is zero can be deduced without explicitly determining A and B_1 as the magnitude of the function is order $1/|z|^4$ when $|z|$ is large. For this to be the case the contribution of the simple pole terms must be such that the magnitude decays like $1/|z|^2$ or faster.