More exercises before the class test

There are a possibly slightly more questions than can be covered in the week 12 sessions and modification may be made depending on what is asked in the Monday and Tuesday sessions. Solutions to all questions will be given whether they are answered or not during the sessions.

1. *This question was in the class test in 2015/6 and was worth 8 marks.*

   Let $f(z)$ be defined by
   $$f(z) = \frac{1}{z-1}.$$

   (a) Give in cartesian form the following complex numbers: $f(-1)$, $f(i)$ and $f(-i)$.

   (b) Prove that the real part of $f(\mathrm{e}^{i\theta})$ is constant for $\theta \in (0, 2\pi)$.

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2. Let $f(z)$ be defined by
   $$f(z) = \frac{4z-1}{z-4}.$$
   Determine z in cartesian form such that $f(z) = -i$.

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3. *This question was in the class test in 2014/5 and was worth 6 marks.*
   $$f(z) = \frac{z-1}{z+1}.$$
   Show that for $-\pi < \theta < \pi$ we have
   $$f(\mathrm{e}^{i\theta}) = i \tan\left(\frac{\theta}{2}\right).$$

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4. Let $\omega = \mathrm{e}^{2\pi i/5}$, where as usual $i = \sqrt{-1}$ is the imaginary unit, and let $c = \cos(2\pi/5)$.

   (a) What are the following values in cartesian form?
   $$\omega^5 \quad \text{and} \quad 1 + \omega + \omega^2 + \omega^3 + \omega^4.$$

   (b) Explain the following.
   $$\begin{aligned} \omega + \omega^4 &= 2c, \\ \omega^2 + \omega^3 &= 2(2c^2 - 1), \\ 4c^2 + 2c - 1 &= 0, \\ c &= \frac{-1+\sqrt{5}}{4}, \\ \cos(4\pi/5) &= \frac{-1-\sqrt{5}}{4}. \end{aligned}$$

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5. *This question was in the class test in 2013/4 and was worth 25 marks.*

   Determine if the following functions are analytic in $\mathbb{C}$ and if a function is analytic express it in terms of z alone.

   (a)
   $$f(x+iy) = y.$$

   (b)
   $$f(x+iy) = 2 + y^3 - 3x^2y + 2x + i(x^3 - 3xy^2 + 2y).$$

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6. *This question was in the class test in 2014/5 and was worth 10 marks.*

   Determine the following in Cartesian form.

   (a) $\mathrm{Log}(1-i)$.

   (b) z^α where $z = 1 - i$ and $\alpha = 1 + i$ and where we mean the principal value of the complex power.

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7. The definition of $\tan(z)$ is
   $$\tan(z) = \frac{\sin(z)}{\cos(z)} = \left(\frac{1}{i}\right)\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}.$$

   Show that
   $$\tan(2z) = \frac{2\tan(z)}{1 - \tan^2(z)}.$$

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8. Give in polar form all values of z which satisfy
   $$z^{10} = -1 + i$$
   and which have negative real part.

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9. Show that the following function $f(z)$, $z = x + iy$, $(x, y \in \mathbb{R})$ is analytic and express it in terms of z alone
   $$f(x+iy) = (x^2 - y^2 - x + y - 2xy) + i(x^2 - y^2 - x + 2xy - y).$$

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10. Show that $u(x,y) = \sin(ax)\cosh(ay)$ is harmonic, where a is a real constant, and determine a harmonic conjugate of u.

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11. Determine the partial fraction representation of $f(z)$ given by
    $$f(z) = \frac{1}{z^2(z^2-1)}.$$

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12. Determine the residue at $z = i$ of
$$\frac{z^8}{z^8 - 1}.$$

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13. *This question was in the class test in 2015/6 and was worth 24 marks.*

    (a) Express the function $f_1(z)$ defined below in partial fraction form and state the residue at any pole.
    $$f_1(z) = \frac{3z}{(z-1)(z+2)}.$$

    (b) Determine the residue at 2 of the following rational function.
    $$f_2(z) = \frac{z^8}{z^2 - 4}.$$

    (c) Determine the residue at -1 of the following rational function.
    $$f_3(z) = \frac{z(z+3)}{(z+1)^3}.$$

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14. Determine the principal value of each of the following.
$$(-1)^{1/2}, \qquad 1^i, \qquad (-i)^{1/2}, \qquad (\sqrt{2}(1+i))^{1+i}.$$

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15. *This question was in the class test in 2015/6 and was worth 6 marks.*

    Determine in polar form and in cartesian form the 3 solutions of
    $$z^3 = i.$$

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16. Determine the partial fraction representation of
$$f(z) = \frac{1}{z(z-1)^3}$$
and state the residues at the poles.
