Exercises involving analytic functions, harmonic functions and harmonic conjugates

Some of the questions have been taken from past May exams of `MA3614` and some questions are from past class tests. The format of the past May exams was answer 3 from 4 in 3 hours with each question worth 20 marks. Hence if a question given here was worth 10 marks then as a percentage this was worth 16.7%. Up to December 2019 the length of the past class tests was 70 or 75 minutes. The class tests in January 2021, December 2021 and December 2022 were 90 minutes. In all cases students had to answer all questions in the class test to get full marks and the sub-marks added to 100 marks. In some questions the term harmonic appears and the connection between analytic functions and harmonic functions is likely to be covered in about week 5. Techniques to express "in terms of z only" is likely also to be done in week 5 in the lectures.

1. Let $z_1, z_2, \ldots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Prove by induction on n that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

Solution

To start the induction consider the case when $n = 1$ which gives $p_1'(z) = 1$. Thus

$$\frac{p_1'(z)}{p_1(z)} = \frac{1}{z - z_1}$$

and the result is true.

For the induction hypothesis suppose that it is true with m terms with $m \ge 1$. We now consider the case with $m + 1$ terms and note that by the product rule

$$p_{m+1}(z) = (z - z_{m+1})p_m(z) \quad \text{and} \quad p_{m+1}'(z) = (z - z_{m+1})p_m'(z) + p_m(z)$$

so that

$$\frac{p_{m+1}'(z)}{p_{m+1}(z)} = \frac{p_m'(z)}{p_m(z)} + \frac{1}{z - z_{m+1}}.$$

Now by the hypothesis we can replace the term $p_m'(z)/p_m(z)$ by the sum of m terms and hence

$$\frac{p_{m+1}'(z)}{p_{m+1}(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_{m+1}}.$$

This shows that the result is also true for $m + 1$ terms and by induction it is true for all $m = 1, 2, \ldots$.

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2. Let $z = x + iy$ and $f = u + iv$, where as usual x, y, u and v are real, If $f(z)$ is analytic in a domain D then show the following.

   (a) If $v(x, y) = 0$ in D then $f(z)$ is a real constant.

   (b) If $u(x, y) = 0$ in D then $f(z)$ is a pure imaginary constant.

(c) If $|f(z)|$ is constant in D then $f(z)$ is a constant. Hint: First show that if

$$\phi(z) = \frac{1}{2}|f(z)|^2$$

then

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \operatorname{Re}\left(f(z)\overline{f'(z)}\right), \\ \frac{\partial \phi}{\partial y} &= \operatorname{Im}\left(f(z)\overline{f'(z)}\right). \end{aligned}$$

Solution

(a) If $v(x,y)=0$ then since $f(z)$ is analytic the Cauchy Riemann equations imply that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0.$$

Both first partial derivatives of u being zero implies that $u(x,y)$ is a constant.

(b) If $u(x,y)=0$ then since $f(z)$ is analytic the Cauchy Riemann equations imply that

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=0 \quad \text{and} \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=0.$$

Both first partial derivatives of v being zero implies that $v(x,y)$ is a constant.

(c) Let

$$\phi(x,y):=\frac{1}{2}|f(z)|^2=\frac{1}{2}\left(u(x,y)^2+v(x,y)^2\right).$$

The first partial derivatives of $\phi(x,y)$ are as follows.

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x},\\ \frac{\partial \phi}{\partial y} &= u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\end{aligned}$$

where the last version follows by using the Cauchy Riemann equations. Both these first partial derivatives of ϕ are zero as ϕ is a constant.

Now $f'(z)$ can be expressed in the form

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \quad \text{and thus} \quad \overline{f'(z)}=\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}$$

and

$$\begin{aligned}f(z)\overline{f'(z)} &= (u+iv)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)\\ &= \left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)+i\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right).\end{aligned}$$

By comparing the expression for $f(z)\overline{f'(z)}$ and the earlier expressions for the partial derivatives of ϕ it follows that if $\phi(x,y)$ is constant then

$$f(z)\overline{f'(z)}=0.$$

As we are given that $|f(z)|$ is a constant in D it follows that if $f(z)=0$ at any point then $f(z)$ is the constant 0 throughout the domain D. If $f(z)\neq 0$ at all points then this implies that $f'(z)=0$ at all points and thus $f(z)$ is a constant. Hence in all cases $f(z)$ is constant.

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3. *This was in the class test in December 2022 and was worth 28 of the 100 marks on the paper.*

(a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in the complex plane $\mathbb{C}$ or if it is not analytic in $\mathbb{C}$.

If a function is analytic in $\mathbb{C}$ then express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= x + x^2 - y^2 + i(-y - 2xy), \\ f_2(x+iy) &= (2x + 3y + 5x^2 - 5y^2 + 2xy) + i(-3x + 2y - x^2 + y^2 + 10xy). \end{aligned}$$

(b) Let $x, y \in \mathbb{R}$. If $\phi(x, y)$ is harmonic then explain why

$$g(x, y) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

is analytic.

Solution

(a) Let $u = x + x^2 - y^2$ and $v = -y - 2xy$.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1 - 2x.$$

The Cauchy Riemann equation involving these two first partial derivatives is only satisfied when $1 = -1 - 2x$, i.e. when $x = -1$. The equation is not satisfied in the neighbourhood of any point on the line. Hence f_1 is not analytic.

Now let

$$u = 2x + 3y + 5x^2 - 5y^2 + 2xy \quad \text{and} \quad v = -3x + 2y - x^2 + y^2 + 10xy.$$

$$\frac{\partial u}{\partial x} = 2 + 10x + 2y, \quad \frac{\partial v}{\partial y} = 2 + 2y + 10x,$$

$$\frac{\partial u}{\partial y} = 3 - 10y + 2x, \quad \frac{\partial v}{\partial x} = -3 - 2x + 10y.$$

The Cauchy Riemann equations of

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied and hence the function f_2 is analytic.

f_2 is a polynomial of degree 2 and has the finite Maclaurin series representation

$$f_2(z) = f_2(0) + f_2'(0)z + \frac{f_2''(0)}{2}z^2.$$

$f_2(0) = 0$.

$$f_2'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2 + 10x + 2y + i(-3 - 2x + 10y), \quad f_2'(0) = 2 - 3i.$$

$$f_2''(z) = \frac{\partial}{\partial x} f_2'(z) = 10 - 2i.$$

Thus

$$f_2(z) = (2 - 3i)z + (5 - i)z^2.$$

As a check on the expression, letting $y = 0$ in the expression in the question gives

$$f_2(x) = (2x + 5x^2) + i(-3x - x^2) = (2 - 3i)x + (5 - i)x^2.$$

(b) $g = u + iv$ with

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \phi}{\partial y}.$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} - \left(-\frac{\partial^2 \phi}{\partial y^2}\right) = \nabla^2 \phi = 0$$

as ϕ is harmonic.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

as mixed partial derivatives can be done in any order. As both Cauchy Riemann equations are satisfied the function $g(z)$ is analytic.

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4. *This was in the class test in December 2021 and was worth 25 of the 100 marks on the paper.*

In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 1, 2, 3, 4 then you do part (a) whilst if it is one of the digits 5, 6, 7, 8, 9 then you do part (b).

(a) This is the version if the 6th digit is one of the digits of 0, 1, 2, 3, 4.

Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (-2x^2 - 10xy + 6x + 2y^2 + 15y) + i(5x^2 - 4xy - 15x - 5y^2 + 6y), \\ f_2(x+iy) &= (x - 2y) + i(-2x - y). \end{aligned}$$

(b) This is the version if the 6th digit is one of the digits of 5, 6, 7, 8, 9.

Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (2x + y) + i(x - 2y), \\ f_2(x+iy) &= (-12x^2 - 18xy + 4x + 12y^2 + 3y) + i(9x^2 - 24xy - 3x - 9y^2 + 4y). \end{aligned}$$

Solution

(a) This is the version if the 6th digit is one of the digits of 0, 1, 2, 3, 4.

Let u and v denote the real and imaginary parts of $f_1(z)$. The first partial derivatives are

$$\frac{\partial u}{\partial x} = -4x - 10y + 6, \quad \frac{\partial v}{\partial y} = -4x - 10y + 6,$$

$$\frac{\partial u}{\partial y} = -10x + 4y + 15, \quad \frac{\partial v}{\partial x} = 10x - 4y - 15.$$

Both Cauchy Riemann equations are satisfied and thus f_1 is analytic.

$f_1(z)$ is a polynomial in z of degree 2 and has the finite Maclaurin expansion

$$f_1(z) = f_1(0) + f_1'(0)z + \frac{f_1''(0)}{2}z^2.$$

$f_1(0) = 0$.

$$f_1'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad f_1'(0) = 6 - 15i.$$
$$f_1''(z) = -4 + 10i.$$

Hence

$$f_1(z) = (6 - 15i)z + (-2 + 5i)z^2 = (2 - 5i)(3z - z^2).$$

Now let u and v denote the real and imaginary parts of $f_2(z)$. The first partial derivatives are

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1, \quad \frac{\partial u}{\partial y} = -2, \quad \frac{\partial v}{\partial x} = -2.$$

The Cauchy Riemann equations do not hold at any point and this $f_2(z)$ is not analytic at any point.

(b) This is the version if the 6th digit is one of the digits of 5, 6, 7, 8, 9.
Let u and v denote the real and imaginary parts of $f_1(z)$. The first partial derivatives are

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial v}{\partial y} = -2, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 1.$$

The Cauchy Riemann equations do not hold at any point and this $f_1(z)$ is not analytic at any point.
Now let u and v denote the real and imaginary parts of $f_2(z)$. The first partial derivatives are

$$\frac{\partial u}{\partial x} = -24x - 18y + 4, \quad \frac{\partial v}{\partial y} = -24x - 18y + 4,$$

$$\frac{\partial u}{\partial y} = -18x + 24y + 3, \quad \frac{\partial v}{\partial x} = 18x - 24y - 3.$$

Both Cauchy Riemann equations are satisfied and thus f_2 is analytic.
$f_2(z)$ is a polynomial in z of degree 2 and has the finite Maclaurin expansion

$$f_2(z) = f_2(0) + f_2'(0)z + \frac{f_2''(0)}{2}z^2.$$

$f_2(0) = 0$.

$$f_2'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad f_2'(0) = 4 - 3i.$$
$$f_2''(z) = -24 + 18i.$$

Hence

$$f_2(z) = (4 - 3i)z + (-12 + 9i)z^2 = (4 - 3i)(z - 3z^2).$$

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5. *This was in the class test in January 2021 and was worth 20 of the 100 marks on the paper.*

   Let $z = x + iy$, $x, y \in \mathbb{R}$. In the following you have functions $f_1(z)$ and $f_2(z)$ to consider which are defined on $\mathbb{C}$ and the particular version in your case depends on the last digit of your 7-digit student id.. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

   You do exactly one of (a), (b) and (c) below. Please take care to do the correct version.

   (a) If your last digit is one of $0, 3, 6, 9$ then f_1 and f_2 are as follows.

$$\begin{aligned} f_1(x+iy) &= 3x^2 - 2xy + x - 3y^2 + 2y + i\left(-x^2 - 6xy - 2x + y^2 + y\right), \\ f_2(x+iy) &= 5x^2 + 2xy + x - 5y^2 + y + i\left(-x^2 + 10xy - x + y^2 + y\right). \end{aligned}$$

   (b) If your last digit is one of $1, 4, 7$ then f_1 and f_2 are as follows.

$$\begin{aligned} f_1(x+iy) &= 3x^2 + 2xy + x - 3y^2 + 2y + i\left(-x^2 + 6xy - 2x + y^2 + y\right), \\ f_2(x+iy) &= 5x^2 - 2xy + x - 5y^2 + y + i\left(-x^2 - 10xy - x + y^2 + y\right). \end{aligned}$$

   (c) If your last digit is one of $2, 5, 8$ then f_1 and f_2 are as follows.

$$\begin{aligned} f_1(x+iy) &= 4x^2 - 2xy + x - 4y^2 - y + i\left(x^2 + 8xy + x - y^2 + y\right), \\ f_2(x+iy) &= 5x^2 + 2xy + x - 5y^2 - y + i\left(x^2 - 10xy + x - y^2 + y\right). \end{aligned}$$

Solution

(a) This is the version for a last digit of $0, 3, 6, 9$.
For f_1 we have

$$u = 3x^2 - 2xy + x - 3y^2 + 2y, \quad v = -x^2 - 6xy - 2x + y^2 + y.$$

$$\frac{\partial u}{\partial x} = 6x - 2y + 1, \quad \frac{\partial v}{\partial y} = -6x + 2y + 1.$$

$$\frac{\partial u}{\partial y} = -2x - 6y + 2, \quad \frac{\partial v}{\partial x} = -2x - 6y - 2.$$

The Cauchy Riemann equations only hold when $6x - 2y = 0$ and $-2x - 6y = 0$ and this is just the point $x = y = 0$. The equations do not hold in the neighbourhood of any point and thus $f_1(z)$ is not analytic at any point.
For f_2 we have

$$u = 5x^2 + 2xy + x - 5y^2 + y, \quad v = -x^2 + 10xy - x + y^2 + y.$$

$$\frac{\partial u}{\partial x} = 10x + 2y + 1, \quad \frac{\partial v}{\partial y} = 10x + 2y + 1.$$

$$\frac{\partial u}{\partial y} = 2x - 10y + 1, \quad \frac{\partial v}{\partial x} = -2x + 10y - 1.$$

Both Cauchy Riemann equations hold at all points and thus $f_2(z)$ is analytic in $\mathbb{C}$.
The analytic function $f_2(z)$ is a polynomial of degree 2 and we can express in terms of z by considering a finite Maclaurin series representation. We have $f_2(0) = 0$.

$$\begin{aligned} f_2'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (10x + 2y + 1) + i(-2x + 10y - 1), \quad f_2'(0) = 1 - i, \\ f_2''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = 10 - 2i. \end{aligned}$$

Hence

$$f_2(z) = f_2(0) + f_2'(0)z + \frac{f_2''(0)}{2}z^2 = (1 - i)z + (5 - i)z^2.$$

(b) This is the version for a last digit of $1, 4, 7$.
For f_1 we have

$$u = 3x^2 + 2xy + x - 3y^2 + 2y, \quad v = -x^2 + 6xy - 2x + y^2 + y.$$

$$\frac{\partial u}{\partial x} = 6x + 2y + 1, \quad \frac{\partial v}{\partial y} = 6x + 2y + 1.$$

$$\frac{\partial u}{\partial y} = 2x - 6y + 2, \quad \frac{\partial v}{\partial x} = -2x + 6y - 2.$$

Both Cauchy Riemann equations hold at all points and thus $f_1(z)$ is analytic in $\mathbb{C}$.

The analytic function $f_1(z)$ is a polynomial of degree 2 and we can express in terms of z by considering a finite Maclaurin series representation. We have $f_1(0) = 0$.

$$\begin{aligned} f_1'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (6x + 2y + 1) + i(-2x + 6y - 2), \quad f'(0) = 1 - 2i, \\ f_1''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = 6 - 2i. \end{aligned}$$

Hence

$$f_1(z) = f_1(0) + f_1'(0)z + \frac{f_1''(0)}{2}z^2 = (1 - 2i)z + (3 - i)z^2.$$

For f_2 we have

$$u = 5x^2 - 2xy + x - 5y^2 + y, \quad v = -x^2 - 10xy - x + y^2 + y.$$

$$\frac{\partial u}{\partial x} = 10x - 2y + 1, \quad \frac{\partial v}{\partial y} = -10x + 2y + 1.$$

$$\frac{\partial u}{\partial y} = -2x - 10y + 1, \quad \frac{\partial v}{\partial x} = -2x - 10y - 1.$$

The Cauchy Riemann equations only hold when $10x - 2y = 0$ and $-2x - 10y = 0$ and this is just the point $x = y = 0$. The equations do not hold in the neighbourhood of any point and thus $f_2(z)$ is not analytic at any point.

(c) This is the version for a last digit of $2, 5, 8$.

For f_1 we have

$$u = 4x^2 - 2xy + x - 4y^2 - y, \quad v = x^2 + 8xy + x - y^2 + y.$$

$$\frac{\partial u}{\partial x} = 8x - 2y + 1, \quad \frac{\partial v}{\partial y} = 8x - 2y + 1.$$

$$\frac{\partial u}{\partial y} = -2x - 8y - 1, \quad \frac{\partial v}{\partial x} = 2x + 8y + 1.$$

Both Cauchy Riemann equations hold at all points and thus $f_1(z)$ is analytic in $\mathbb{C}$. The analytic function $f_1(z)$ is a polynomial of degree 2 and we can express in terms of z by considering a finite Maclaurin series representation. We have $f_1(0) = 0$.

$$\begin{aligned} f_1'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 8x - 2y + 1 + i(2x + 8y + 1), \quad f_1'(0) = 1 + i, \\ &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = 8 + 2i. \end{aligned}$$

Hence

$$f_1(z) = f_1(0) + f_1'(0)z + \frac{f_1''(0)}{2}z^2 = (1 + i)z + (4 + i)z^2.$$

For f_2 we have

$$u = 5x^2 + 2xy + x - 5y^2 - y, \quad v = x^2 - 10xy + x - y^2 + y.$$

$$\frac{\partial u}{\partial x} = 10x + 2y + 1, \quad \frac{\partial v}{\partial y} = -10x - 2y + 1.$$
$$\frac{\partial u}{\partial y} = 2x - 10y - 1, \quad \frac{\partial v}{\partial x} = 2x - 10y + 1.$$

Both Cauchy Riemann equations only hold when $10x+2y = 0$ and $2x-10y = 0$ and this is just the point $x = y = 0$. The equations do not hold in the neighbourhood of any point and thus $f_2(z)$ is not analytic at any point.

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6. *This was in the class test in December 2022 and was worth 11 of the 100 marks on the paper.*

   Let $x, y \in \mathbb{R}$ and let
   $$u(x, y) = -5x^4y + 10x^2y^3 - y^5.$$
   Show that u is harmonic and find the harmonic conjugate $v(x, y)$ satisfying $v(1, 0) = 2$.

Solution

$$\frac{\partial u}{\partial x} = -20x^3y + 20xy^3, \quad \frac{\partial^2 u}{\partial x^2} = -60x^2y + 20y^3.$$
$$\frac{\partial u}{\partial y} = -5x^4 + 30x^2y^2 - 5y^4, \quad \frac{\partial^2 u}{\partial y^2} = 60x^2y - 20y^3.$$

Thus $\nabla^2 u = 0$.

v is related to u by the Cauchy Riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 5x^4 - 30x^2y^2 + 5y^4.$$

Partially integrating with respect to x gives

$$v = x^5 - 10x^3y^2 + 5xy^4 + g(y)$$

for any differentiable function $g(y)$. Next partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = -20x^3y + 20xy^3 + g'(y) = \frac{\partial u}{\partial x} = -20x^3y + 20xy^3$$

and thus $g'(y) = 0$ and $g(y) = C$, where C is a constant.

$$v = x^5 - 10x^3y^2 + 5xy^4 + C, \quad v(1, 0) = 1 + C = 2, \quad C = 1.$$

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7. *This was in the class test in December 2021 and was worth 15 of the 100 marks on the paper.*

   In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 2, 4, 6, 8 then
   $$u(x, y) = -e^y \sin(x) - 2e^{-x} \sin(y)$$

whilst if it is one of the digits 1, 3, 5, 7, 9 then

$$u(x,y) = \mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\cos(y)$$

with in all cases $x, y \in \mathbb{R}$. Show that your version of $u(x,y)$ is a harmonic function and determine the harmonic conjugate $v(x,y)$ satisfying $v(0,0) = 4$.

Solution

This is the version if the 6th digit is one of the digits of 0, 2, 4, 6, 8.

$$\frac{\partial u}{\partial x} = -\mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\sin(y), \quad \frac{\partial u}{\partial y} = -\mathrm{e}^{y}\sin(x) - 2\mathrm{e}^{-x}\cos(y),$$

$$\frac{\partial^2 u}{\partial x^2} = \mathrm{e}^{y}\sin(x) - 2\mathrm{e}^{-x}\sin(y) \quad \frac{\partial^2 u}{\partial y^2} = -\mathrm{e}^{y}\sin(x) + 2\mathrm{e}^{-x}\sin(y), \quad \nabla^2 u = 0.$$

The harmonic conjugate v is related to u by the Cauchy Riemann equations and by using one of these we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \mathrm{e}^{y}\sin(x) + 2\mathrm{e}^{-x}\cos(y).$$

Partially integrating with respect to x gives

$$v(x,y) = -\mathrm{e}^{y}\cos(x) - 2\mathrm{e}^{-x}\cos(y) + g(y), \quad \text{for any differentiable function } g(y).$$

Partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = -\mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\sin(y) + g'(y) = \frac{\partial u}{\partial x} = -\mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\sin(y),$$

Thus $g'(y) = 0$ and $g(y) = C$, where C is a constant. $v(0,0) = -3 + C = 4$ if $C = 7$. The harmonic conjugate is

$$v(x,y) = -\mathrm{e}^{y}\cos(x) - 2\mathrm{e}^{-x}\cos(y) + 7.$$

This is the version if the 6th digit is one of the digits of 1, 3, 5, 7, 9.

$$\frac{\partial u}{\partial x} = -\mathrm{e}^{y}\sin(x) - 2\mathrm{e}^{-x}\cos(y), \quad \frac{\partial u}{\partial y} = \mathrm{e}^{y}\cos(x) - 2\mathrm{e}^{-x}\sin(y),$$

$$\frac{\partial^2 u}{\partial x^2} = -\mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\cos(y) \quad \frac{\partial^2 u}{\partial y^2} = \mathrm{e}^{y}\cos(x) - 2\mathrm{e}^{-x}\cos(y), \quad \nabla^2 u = 0.$$

The harmonic conjugate v is related to u by the Cauchy Riemann equations and by using one of these we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\mathrm{e}^{y}\cos(x) + 2\mathrm{e}^{-x}\sin(y).$$

Partially integrating with respect to x gives

$$v(x,y) = -e^y \sin(x) - 2e^{-x} \sin(y) + g(y), \quad \text{for any differentiable function } g(y).$$

Partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = -e^y \sin(x) - 2e^{-x} \cos(y) + g'(y) = \frac{\partial u}{\partial x} = -e^y \sin(x) - 2e^{-x} \cos(y),$$

Thus $g'(y) = 0$ and $g(y) = C$, where C is a constant. $v(0,0) = 0 + C = 4$ if $C = 4$. The harmonic conjugate is

$$v(x,y) = -e^y \sin(x) - 2e^{-x} \sin(y) + 4.$$

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8. *This was in the class test in December 2019 and was worth 25 of the 100 marks on the paper.*

   Let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.
   State the Cauchy Riemann equations.

   Let
   $$u(x,y) = 2x + y + x^2 - y^2 - 2xy.$$

   Show that this function is harmonic and determine the harmonic conjugate v which satisfies $v(0,0) = 1$.

   Express the function $f = u + iv$ in terms of z alone. You need to give reasoning for your answer.

Solution

The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

For the given function u we have

$$\frac{\partial u}{\partial x} = 2 + 2x - 2y, \quad \frac{\partial u}{\partial y} = 1 - 2y - 2x.$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2 \quad \text{and} \quad \nabla^2 u = 0.$$

The function u is harmonic.

By using one of the Cauchy Riemann equations we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1 + 2y + 2x$$

and partially integrating with respect to y gives

$$v = -x + 2xy + x^2 + g(y)$$

for any function $g(y)$. Partially differentiating this expression and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = 2x + g'(y) = \frac{\partial u}{\partial x} = 2 + 2x - 2y, \quad \text{which implies that } g'(y) = 2 - 2y.$$

Hence

$$g(y) = 2y - y^2 + C,$$

where C is a constant. To satisfy $v(0,0) = 1$ we need $C = 1$ and this gives

$$f = u + iv = (2x + y + x^2 - y^2 - 2xy) + i(-x + 2xy + x^2 + 2y - y^2 + 1).$$

$f(z)$ is a polynomial of degree 2.

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x} = (2 + 2x - 2y) + i(-1 + 2y + 2x), \\ f''(z) &= \frac{\partial^2 f}{\partial x^2} = 2 + 2i. \end{aligned}$$

Now $f(0) = i$, $f'(0) = 2-i$ and $f''(0) = 2+2i$. The finite Maclaurin series representation gives

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2} z^2 = i + (2 - i)z + (1 + i)z^2.$$

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9. *This was in the class test in December 2018 and was worth 26 of the 100 marks on the paper.*

Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.

State the Cauchy Riemann equations.

By using the Cauchy Riemann equations, or otherwise, determine if the following functions are analytic in $\mathbb{C}$. If a function is analytic then express it in terms of z alone.

(a)
$$f(x + iy) = (x^3 - 3xy^2) + i(-3x^2y + y^3).$$

(b)
$$g(x + iy) = (y^3 - 3x^2y + 2xy + 2x^2 - 2y^2) + i(x^3 - 3xy^2 + 4xy - x^2 + y^2).$$

Solution

The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

(a) $f = u + iv$ with $u = x^3 - 3xy^2$ and $v = -3x^2y + y^3$.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial y} = -3x^2 + 3y^2.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{only when } x^2 = y^2.$$

Points on the lines $x^2 = y^2$ do not have neighbourhoods which are also all on these lines and hence $f(z)$ is not analytic at any point.

(b) $g = u + iv$ with $u = y^3 - 3x^2y + 2xy + 2x^2 - 2y^2$ and $v = x^3 - 3xy^2 + 4xy - x^2 + y^2$.

$$\frac{\partial u}{\partial x} = -6xy + 2y + 4x = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 + 2x - 4y, \quad \frac{\partial v}{\partial x} = 3x^2 - 3y^2 + 4y - 2x.$$

Hence

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Both Cauchy Riemann equations hold and hence $g(z)$ is analytic.
$g(z)$ is a polynomial of degree 3 and is given by the finite Maclaurin representation

$$g(z) = g(0) + g'(0)z + \frac{g''(0)}{2}z^2 + \frac{g'''(0)}{6}z^3.$$

$g(0) = 0$.

$$\begin{aligned} g'(z) &= \frac{\partial g}{\partial x} = -6xy + 2y + 4x + i(3x^2 - 3y^2 + 4y - 2x), \\ g''(z) &= \frac{\partial^2 g}{\partial x^2} = -6y + 4 + i(6x - 2), \\ g'''(z) &= 6i. \end{aligned}$$

$g'(0) = 0$, $g''(0) = 4 - 2i$, $g'''(0) = 6i$.

$$g(z) = (2 - i)z^2 + iz^3.$$

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10. *This was question 1 of the May 2023 exam paper.*

(a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the real valued functions $p(x, y)$ and $q(x, y)$ are such that $p(x, y) + iq(x, y)$ is analytic in the entire complex plane.

i.
$$f_1(z) = (x - 2xy) + i(-y - x^2 + y^2).$$

ii.
$$f_2(z) = (-y + 2x^3 - 6xy^2) + i(x + 6x^2y - 2y^3).$$

iii.
$$f_3(z) = \mathrm{e}^{-x-3y}(\cos(3x - y) + i\sin(3x - y)).$$

iv.
$$f_4(z) = (xp(x, y) - yq(x, y)) + i(yp(x, y) + xq(x, y)).$$

(b) Let $u(x, y) = \cosh(x)\cos(y)$. The function u is harmonic. Find the harmonic conjugate $v(x, y)$ such that $v(0, 0) = 0$.

(c) Let $z = r\mathrm{e}^{i\theta}$ with $r > 0$ and $-\pi < \theta \le \pi$ and let

$$u(r,\theta) = r^{1/3}\cos(\theta/3), \quad v(r,\theta) = r^{1/3}\sin(\theta/3), \quad \text{and} \quad g(r\mathrm{e}^{i\theta}) = u(r,\theta) + iv(r,\theta).$$

Give the first order partial derivatives

$$\frac{\partial u}{\partial r}, \quad \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}$$

in the part of the complex plane where the derivatives exist. The Cauchy Riemann equations in polar coordinates r and θ are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

In which part of the complex plane is $g(r\mathrm{e}^{i\theta})$ analytic?
Determine in terms of r and θ the simplest cartesian form of the following limit.

$$\lim_{h\to 0} \frac{g((r+h)\mathrm{e}^{i\theta}) - g(r\mathrm{e}^{i\theta})}{h\mathrm{e}^{i\theta}}.$$

Solution

(a) i. Let $f_1 = u + iv$ with

$$u = x - 2xy \quad \text{and} \quad v = -y - x^2 + y^2.$$

$$\frac{\partial u}{\partial x} = 1 - 2y, \quad \frac{\partial v}{\partial y} = -1 + 2y, \quad \frac{\partial u}{\partial y} = -2x, \quad \frac{\partial v}{\partial x} = -2x.$$

The Cauchy Riemann equations only hold when $x = 0$ and $4y = 2$. As the equations do not hold in a neighbourhood of the point the function f_1 is not analytic at any point.

ii. Let $f_2 = u + iv$ with

$$u = -y + 2x^3 - 6xy^2 \quad \text{and} \quad v = x + 6x^2y - 2y^3.$$

$$\frac{\partial u}{\partial x} = 6x^2 - 6y^2, \quad \text{and} \quad \frac{\partial v}{\partial y} = 6x^2 - 6y^2.$$

Also

$$\frac{\partial u}{\partial y} = -1 - 12xy \quad \text{and} \quad \frac{\partial v}{\partial x} = 1 + 12xy.$$

Both Cauchy Riemann equations are satisfied and the function f_2 is analytic at all points.

iii. Let $f_3 = u + iv$ with

$$u = \mathrm{e}^{-x-3y}\cos(3x - y) \quad \text{and} \quad v = \mathrm{e}^{-x-3y}\sin(3x - y).$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \mathrm{e}^{-x-3y}(-3\sin(3x - y)) - \mathrm{e}^{x-3y}\cos(3x - y), \\
\frac{\partial v}{\partial y} &= \mathrm{e}^{-x-3y}(-\cos(3x - y)) + (-3\mathrm{e}^{-x-3y})\sin(3x - y), \\
\frac{\partial u}{\partial y} &= \mathrm{e}^{-x-3y}\sin(3x - y) + (-3\mathrm{e}^{-x-3y})\cos(3x - y), \\
\frac{\partial v}{\partial x} &= \mathrm{e}^{-x-3y}(3\cos(3x - y)) + (-\mathrm{e}^{-x-3y})\sin(3x - y).
\end{aligned}$$

Both Cauchy Riemann equations are satisfied and the function f_3 is analytic at all points.

iv. Let $f_4 = u + iv$ with

$$u = xp(x, y) - yq(x, y) \quad \text{and} \quad v = yp(x, y) + xq(x, y).$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \left(x\frac{\partial p}{\partial x} + p - y\frac{\partial q}{\partial x} \right) - \left(y\frac{\partial p}{\partial y} + p + x\frac{\partial q}{\partial y} \right) = 0$$

as $p + iq$ being analytic implies that

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}.$$

Also

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(x\frac{\partial p}{\partial y} - y\frac{\partial q}{\partial y} - q \right) + \left(y\frac{\partial p}{\partial x} + x\frac{\partial q}{\partial x} + q \right) = 0$$

as p and q satisfy the Cauchy Riemann equations.

(b) v is related to u by the Cauchy Riemann equations and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \cosh(x)\sin(y).$$

Partially integrating with respect to x gives

$$v = \sinh(x)\sin(y) + g(y)$$

for any differentiable function $g(y)$. Partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = \sinh(x)\cos(y) + g'(y) = \frac{\partial u}{\partial x} = \sinh(x)\cos(y).$$

Thus $g'(y) = 0$ and $g(y) = C$, a constant. The condition $v(0,0) = 0$ gives $C = 0$ and thus

$$v = \sinh(x)\sin(y).$$

(c)

$$\frac{\partial u}{\partial r} = \frac{1}{3}r^{-2/3}\cos(\theta/3), \quad \frac{\partial v}{\partial \theta} = \frac{1}{3}r^{1/3}\cos(\theta/3).$$
$$\frac{\partial u}{\partial \theta} = -\frac{1}{3}r^{1/3}\sin(\theta/3), \quad \frac{\partial v}{\partial r} = \frac{1}{3}r^{-2/3}\sin(\theta/3).$$

g is analytic where the partial derivatives exist and are continuous and the Cauchy Riemann equations hold. In this case this is

$$\left\{ r\mathrm{e}^{i\theta} : \ r > 0, \quad -\pi < \theta < \pi \right\}.$$

It is not analytic on the radial line $\theta = \pi$ as there is a jump discontinuity.

$$\lim_{h\to 0} \frac{g((r+h)\mathrm{e}^{i\theta}) - g(r\mathrm{e}^{i\theta})}{h\mathrm{e}^{i\theta}} = \frac{1}{\mathrm{e}^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$
$$= \frac{1}{3}r^{-2/3}\mathrm{e}^{-i\theta}\left(\cos(\theta/3) + i\sin(\theta/3)\right) = \frac{1}{3}r^{-2/3}\mathrm{e}^{-i\theta}\mathrm{e}^{i\theta/3} = \frac{1}{3}r^{-2/3}\mathrm{e}^{i(-2\theta/3)}$$
$$= \frac{1}{3}r^{-2/3}\left(\cos(2\theta/3) - i\sin(2\theta/3)\right).$$

11. *This was question 1 of the May 2022 exam paper.*

(a) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the function $\phi(x,y)$ is an infinitely continuously differentiable harmonic function.

i.
$$f_1(z) = (x-y) - i(x+y).$$

ii.
$$f_2(z) = (x^3 - 3xy^2 - 4xy) + i(3x^2y - y^3 + 2x^2 - 2y^2).$$

iii.
$$f_3(z) = \mathrm{e}^x(2\cos(y) - \sin(y)) + i\mathrm{e}^x(\cos(y) + 2\sin(y)).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} + i\frac{\partial^2 \phi}{\partial x^2}.$$

(b) Let $z = x+iy$ with $x, y \in \mathbb{R}$ and let

$$g(x+iy) = (x^4 - 6x^2y^2 + y^4 - 2xy) + i(4x^3y - 4xy^3 + x^2 - y^2).$$

The function $g(z)$ is analytic. Express $g(z)$ in terms of z alone. You must justify your answer.

(c) Let $x, y \in \mathbb{R}$ and let

$$u(x,y) = \cos(x)\cosh(y) + \sin(x)\sinh(y).$$

The function $u(x,y)$ is harmonic (you do not need to verify this). Determine the harmonic conjugate $v(x,y)$ satisfying $v(0,0) = 1$.

The analytic function $f(x+iy) = u(x,y) + iv(x,y)$ can be written as a linear combination of e^{iz} and e^{-iz}, i.e. as

$$c\mathrm{e}^{iz} + d\mathrm{e}^{-iz},$$

where c and d are complex constants. Determine the constants c and d.

Solution

(a) i. Let $f_1 = u + iv$ with

$$u = x - y, \quad v = -x - y.$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1.$$

One of the Cauchy Riemann equations is not satisfied and hence $f_1(z)$ is not analytic at any point.

ii. Let $f_2 = u + iv$ with

$$u = x^3 - 3xy^2 - 4xy, \quad v = 3x^2y - y^3 + 2x^2 - 2y^2.$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 4y, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 - 4y.$$

$$\frac{\partial u}{\partial y} = -6xy - 4x, \quad \frac{\partial v}{\partial x} = 6xy + 4x.$$

Both Cauchy Riemann equations are satisfied at all points and thus f_2 is analytic at all points.

iii. Let $f_3 = u + iv$ with

$$u = \mathrm{e}^x(2\cos(y) - \sin(y)), \quad v = \mathrm{e}^x(\cos(y) + 2\sin(y)).$$

$$\frac{\partial u}{\partial x} = u = \mathrm{e}^x(2\cos(y) - \sin(y)), \quad \frac{\partial v}{\partial y} = \mathrm{e}^x(-\sin(y) + 2\cos(y)).$$

$$\frac{\partial u}{\partial y} = \mathrm{e}^x(-2\sin(y) - \cos(y)), \quad \frac{\partial v}{\partial x} = v = \mathrm{e}^x(\cos(y) + 2\sin(y)).$$

Both Cauchy Riemann equations are satisfied at all points and thus f_3 is analytic at all points.

iv. Let $f_4 = u + iv$ with

$$u = \frac{\partial^2 \phi}{\partial x \partial y}, \quad v = \frac{\partial^2 \phi}{\partial x^2}.$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^3 \phi}{\partial^2 x \partial y} - \frac{\partial^3 \phi}{\partial y \partial x^2} = 0,$$

as mixed partial differentiation can be done in any order.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^3 \phi}{\partial y \partial x \partial y} + \frac{\partial^3 \phi}{\partial^3 x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0,$$

as mixed partial differentiation can be done in any order and ϕ is harmonic.

(b) As $g(z)$ is analytic we can get the derivatives by partially differentiating in the x direction. Hence

$$g'(z) = (4x^3 - 12xy^2 - 2y) + i(12x^2y - 4y^3 + 2x), \quad g'(0) = 0.$$

$$g''(z) = (12x^2 - 12y^2) + i(24xy + 2), \quad g''(0) = 2i.$$

$$g'''(z) = 24x + i(24y), \quad g'''(0) = 0.$$

$$g''''(z) = 24.$$

Now $g(0) = 0$ and the finite Maclaurin series representation is

$$g(z) = g(0) + g'(0)z + \frac{g''(0)}{2}z^2 + \frac{g'''(0)}{6}z^3 + \frac{g''''(0)}{24}z^4 = iz^2 + z^4.$$

As an added check, letting $y = 0$ in the expression for g gives $g(x) = x^4 + ix^2$.

(c) v is related to u by the Cauchy Riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\cos(x)\sinh(y) - \sin(x)\cosh(y).$$

Partially integrating with respect to x gives

$$v = -\sin(x)\sinh(y) + \cos(x)\cosh(y) + g(y)$$

for any differentiable function $g(y)$. Partially differentiating this version of v with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = -\sin(x)\cosh(y) + \cos(x)\sinh(y) + g'(y) = \frac{\partial u}{\partial x} = -\sin(x)\cosh(y) + \cos(x)\sinh(y)$$

and hence $g'(y) = 0$ and $g(y) = c$ is a constant.

$$v(0,0) = 1 + c = 1, \quad c = 0.$$

$$v(x,y) = -\sin(x)\sinh(y) + \cos(x)\cosh(y)$$

and

$$f(z) = \cos(x)\cosh(y) + \sin(x)\sinh(y) + i(-\sin(x)\sinh(y) + \cos(x)\cosh(y)).$$

Setting $y = 0$ gives

$$f(x) = \cos(x) + i\cos(x).$$

Let

$$h(z) = (1+i)\cos(z).$$

As $f(z)$ and $h(z)$ are both entire and agree on the real axis they must be the same for all z as the zeros of a non-zero analytic functions are isolated. The zeros of $f(z) - h(z)$ are not isolated.

$$h(z) = (1+i)\left(\frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}\right)$$

and thus

$$c = d = \frac{1+i}{2}.$$

12. *This was most of question 1 of the May 2021* MA3614 *paper.*

(a) In this part of the question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 1, 2, 3, 4 then you consider the first four functions and if the last digit is one of the digits 5, 6, 7, 8, 9 then you consider the second four functions.

Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

The functions for a last digit of 0, 1, 2, 3, 4 follow.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 + iy.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = \left(-6x^2y - 3x^2 + 2y^3 + 3y^2\right) + i\left(2x^3 - 6xy^2 - 6xy\right).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \sinh(x)\cos(y) - i\cosh(x)\sin(y).$$

iv.
$$f_4 : \mathbb{C}\backslash\{-i\} \to \mathbb{C}, \quad f_4(z) = \frac{(y+1) + ix}{x^2 + (y+1)^2}.$$

The functions for a last digit of 5, 6, 7, 8, 9 follow.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = y^2 + ix.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = \left(3x^3 - 9xy^2 + 4xy\right) + i\left(9x^2y - 2x^2 - 3y^3 + 2y^2\right).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \cosh(x)\cos(y) - i\sinh(x)\sin(y).$$

iv.
$$f_4 : \mathbb{C}\backslash\{-1\} \to \mathbb{C}, \quad f_4(z) = \frac{(x+1) - iy}{(x+1)^2 + y^2}.$$

(b) Let $\phi(x, y)$ denote a function defined for all x, $y \in \mathbb{R}$ which has continuous partial derivatives of all orders and which is harmonic. Also let ψ be defined for all x, $y \in \mathbb{R}$ by $\psi(x, y) = \phi(x, -y)$.

In this part of the question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 2, 4, 6, 8 then you consider the first two functions and if the last digit is one of the digits 1, 3, 5, 7, 9 then you consider the second two functions.

For your version of g_1 and g_2 you need to determine if it is analytic or not analytic at all points in the complex plane giving reasons for your answers in each case.

The functions for a last digit of 0, 2, 4, 6, 8 follow.

$$g_1(x+iy) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}, \quad g_2(x+iy) = \frac{\partial \psi}{\partial x} - i\frac{\partial \psi}{\partial y}.$$

The functions for a last digit of 1, 3, 5, 7, 9 follow.

$$g_1(x+iy) = \frac{\partial \phi}{\partial y} + i\frac{\partial \phi}{\partial x}, \quad g_2(x+iy) = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}.$$

(c) This part of the question is for all student numbers.
Let $f(z) = z^{1/2}$, where the principal value complex power is used. With $z = re^{i\theta}$, $r \ge 0$, $-\pi < \theta \le \pi$ give the real valued functions $u(r,\theta)$ and $v(r,\theta)$ in

$$f(z) = u(r,\theta) + iv(r,\theta).$$

State the domain of z where this function is analytic and give the functions

$$\frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial \theta} \quad \text{and} \quad f'(z).$$

Solution

(a) This is the version for a last digit of 0, 1, 2, 3, 4.

i. The real and imaginary parts of f_1 are $u = z^2$ and $v = y$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0.$$

The Cauchy Riemann equations are only satisfied at points on the line $x = 1/2$ but not in the neighbourhood of any point on the line. $f_1(z)$ is not analytic at any point in the complex plane.

ii. The real and imaginary parts of f_2 are

$$u = -6x^2y - 3x^2 + 2y^3 + 3y^2, \quad v = 2x^3 - 6xy^2 - 6xy.$$

$$\frac{\partial u}{\partial x} = -12xy - 6x = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -6x^2 + 6y^2 + 6y, \quad \frac{\partial v}{\partial x} = 6x^2 - 6y^2 - 6y = -\frac{\partial u}{\partial y}.$$

Both Cauchy Riemann equations are satisfied at all points and thus $f_2(z)$ is analytic at all points.

iii. The real and imaginary parts of f_3 are

$$u = \sinh(x)\cos(y), \quad v = -\cosh(x)\sin(y).$$

$$\frac{\partial u}{\partial x} = \cosh(x)\cos(y) = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\sinh(x)\sin(y) = \frac{\partial v}{\partial x}.$$

The Cauchy Riemann equations only hold at points for which all the first partial derivatives are 0.

$$\frac{\partial u}{\partial x} = 0 \quad \text{only when } \cos(y) = 0.$$

If $\cos(y) = 0$ then $|\sin(y)| = 1$ and the other equation is only satisfied as well if $x = 0$. The Cauchy Riemann equations are only satisfied at points $x = 0$, $y = \pi/2 + k\pi$, $k \in \mathbb{Z}$ but not in the neighbourhood of any of these points. $f_3(z)$ is not analytic at any point in the complex plane.

iv. The real and imaginary parts of f_4 are

$$u = \frac{y+1}{x^2 + (y+1)^2}, \quad v = \frac{x}{x^2 + (y+1)^2}.$$

$$\frac{\partial u}{\partial x} = \frac{-2x(y+1)}{\left(x^2 + (y+1)^2\right)^2} = \frac{\partial v}{\partial y}.$$

By the quotient rule

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\left(x^2 + (y+1)^2\right) - 2(y+1)^2}{\left(x^2 + (y+1)^2\right)^2} = \frac{x^2 - (y+1)^2}{\left(x^2 + (y+1)^2\right)^2}, \\
\frac{\partial v}{\partial x} &= \frac{\left(x^2 + (y+1)^2\right) - 2x^2}{\left(x^2 + (y+1)^2\right)^2} = \frac{-x^2 + (y+1)^2}{\left(x^2 + (y+1)^2\right)^2} = -\frac{\partial u}{\partial y}.
\end{aligned}$$

Both Cauchy Riemann equations are satisfied at all points in the domain and thus $f_4(z)$ is analytic in the domain.

This is the version for a last digit of 5, 6, 7, 8, 9.

i. The real and imaginary parts of f_1 are $u = y^2$ and $v = x$.

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 1.$$

The Cauchy Riemann equations are only satisfied at points on the line $y = 1/2$ but not in the neighbourhood of any point on the line. $f_1(z)$ is not analytic at any point in the complex plane.

ii. The real and imaginary parts of f_2 are

$$u = 3x^3 - 9xy^2 + 4xy, \quad v = 9x^2y - 2x^2 - 3y^3 + 2y^2.$$

$$\frac{\partial u}{\partial x} = 9x^2 - 9y^2 + 4y = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -18xy + 4x, \quad \frac{\partial v}{\partial x} = 18xy - 4x = -\frac{\partial u}{\partial y}.$$

Both Cauchy Riemann equations are satisfied at all points and thus $f_2(z)$ is analytic at all points.

iii. The real and imaginary parts of f_3 are

$$u = \cosh(x)\cos(y), \quad v = -\sinh(x)\sin(y).$$

$$\frac{\partial u}{\partial x} = \sinh(x)\cos(y) = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\cosh(x)\sin(y) = \frac{\partial v}{\partial x}.$$

The Cauchy Riemann equations only hold at points for which all the first partial derivatives are 0.

$$\frac{\partial u}{\partial y} = 0 \quad \text{only when } \sin(y) = 0.$$

If $\sin(y) = 0$ then $|\cos(y)| = 1$ and the other equation is only satisfied as well if $x = 0$. The Cauchy Riemann equations are only satisfied at points $x = 0$, $y = k\pi$, $k \in \mathbb{Z}$ but not in the neighbourhood of any of these points. $f_3(z)$ is not analytic at any point in the complex plane.

iv. The real and imaginary parts of f_4 are

$$u = \frac{x+1}{(x+1)^2 + y^2}, \quad v = \frac{-y}{(x+1)^2 + y^2}.$$

$$\frac{\partial u}{\partial y} = \frac{-2(x+1)y}{\left((x+1)^2 + y^2\right)^2} = -\frac{\partial v}{\partial x}.$$

By the quotient rule

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\left((x+1)^2+y^2\right)-2(x+1)^2}{\left((x+1)^2+y^2\right)^2} = \frac{-(x+1)^2+y^2}{\left((x+1)^2+y^2\right)^2}, \\
\frac{\partial v}{\partial y} &= \frac{\left((x+1)^2+y^2\right)(-1)-(-y)(2y)}{\left((x+1)^2+y^2\right)^2} = \frac{-(x+1)^2+y^2}{\left((x+1)^2+y^2\right)^2} = \frac{\partial u}{\partial x}.
\end{aligned}$$

Both Cauchy Riemann equations are satisfied at all points in the domain and thus $f_4(z)$ is analytic in the domain.

(b) This is the version for a last digit of 0, 2, 4, 6, 8.

The real and imaginary parts of g_1 are

$$u = \frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}.$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{as } \phi \text{ is harmonic.}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

as mixed partial derivatives can be done in any order. Both Cauchy Riemann equations are satisfied and thus g_1 is analytic.

With g_2 the real and imaginary parts are

$$u = \frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial y}.$$

By the chain rule

$$u = \frac{\partial \psi}{\partial x}(x,\, y) = \frac{\partial \phi}{\partial x}(x,\, -y), \quad v = -\frac{\partial \psi}{\partial y}(x,\, y) = +\frac{\partial \phi}{\partial y}(x,\, -y).$$

Thus

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} - \left(-\frac{\partial^2 \phi}{\partial y^2}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2},$$

with partial derivatives of ϕ evaluated at $(x, -y)$. As ϕ is harmonic at all points in the complex plane this is 0.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x \partial y} = 0,$$

again with partial derivatives of ϕ evaluated at $(x, -y)$. Both Cauchy Riemann equations are satisfied and thus g_2 is analytic. As an observation, $g_2(z) = \overline{g_1(\overline{z})}$.

This is the version for a last digit of 1, 3, 5, 7, 9.

The real and imaginary parts of g_1 are

$$u = \frac{\partial \phi}{\partial y}, \quad v = \frac{\partial \phi}{\partial x}.$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

as mixed partial derivatives can be done in any order.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \quad \text{as } \phi \text{ is harmonic.}$$

Both Cauchy Riemann equations are satisfied and thus g_1 is analytic.
With g_2 the real and imaginary parts are

$$u = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial y}.$$

By the chain rule

$$u = \frac{\partial \psi}{\partial y}(x,\, y) = -\frac{\partial \phi}{\partial y}(x,\, -y), \quad v = \frac{\partial \psi}{\partial x}(x,\, y) = \frac{\partial \phi}{\partial x}(x,\, -y).$$

Thus

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\frac{\partial^2 \phi}{\partial x \partial y} - \left(-\frac{\partial^2 \phi}{\partial y \partial x} \right),$$

with partial derivatives of ϕ evaluated at $(x, -y)$. This is 0 as mixed partial derivatives can be done in any order.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\left(-\frac{\partial^2 \phi}{\partial y^2} \right) + \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2},$$

again with partial derivatives of ϕ evaluated at $(x, -y)$. As ϕ is harmonic at all points in the complex plane this is 0. Both Cauchy Riemann equations are satisfied and thus g_2 is analytic. As an observation, $g_2(z) = \overline{g_1(\overline{z})}$.

(c) The definition of the principal value of z^α is

$$z^\alpha = \exp(\alpha \mathrm{Log}(z)) = \exp(\alpha(\ln r + i\theta)).$$

When $\alpha = 1/2$ we thus have

$$z^{1/2} = \exp((\ln r)/2) \exp(+i\theta/2)) = r^{1/2} \mathrm{e}^{i\theta/2} = r^{1/2} \left(\cos(\theta/2) + i \sin(\theta/2) \right).$$

The domain where this is analytic is

$$\left\{ z = r\mathrm{e}^{i\theta} \,:\, r > 0,\, -\pi < \theta < \pi \right\}.$$

Thus

$$u = r^{1/2} \cos(\theta/2), \quad v = r^{1/2} \sin(\theta/2).$$

The derivatives are

$$\frac{\partial u}{\partial r} = \frac{1}{2} r^{-1/2} \cos(\theta/2), \quad \frac{\partial u}{\partial \theta} = -\frac{1}{2} r^{1/2} \sin(\theta/2)$$

and

$$f'(z) = \frac{1}{2} z^{-1/2} = \frac{1}{2} r^{-1/2} \mathrm{e}^{-i\theta/2} = \mathrm{e}^{-i\theta} \left(\frac{\partial u}{\partial r} - \frac{i}{r} \frac{\partial u}{\partial \theta} \right).$$

13. *This was most of question 1 of the May 2020* MA3614 *paper.*

(a) Let $z = x + iy$, with $x,\, y \in \mathbb{R}$. Let $f(z) = u(x,\, y) + iv(x,\, y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

The Cauchy Riemann equations in polar coordinates r and θ for an analytic function $f(r\mathrm{e}^{i\theta}) = \tilde{u}(r,\, \theta) + i\tilde{v}(r,\, \theta)$, with $\tilde{u}(r,\, \theta)$ and $\tilde{v}(r,\, \theta)$ being real, are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

In the case of $f(z) = 1/z$, $z \neq 0$, give $\tilde{u}$, $\tilde{v}$ and the first order partial derivatives

$$\frac{\partial \tilde{u}}{\partial r}, \quad \frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{\partial \tilde{u}}{\partial \theta} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial r}.$$

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = 3x - iy.$$

ii.
$$f_2(z) = -3x^2y + y^3 + i(x^3 - 3xy^2).$$

iii.
$$f_3(z) = \sinh(x)\cos(y) - i\cosh(x)\sin(y).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} - i\frac{\partial^2 \phi}{\partial y^2}$$

where $\phi(x,\, y)$ is a harmonic function with partial derivatives of all orders being continuous.

(c) The function $u(x,\, y) = \cosh(2x)\cos(2y)$ is harmonic for all x and y. Determine the harmonic conjugate $v(x,\, y)$ such that $v(0,\, 0) = 0$ and indicate all the zeros of the analytic function $u(x,\, y) + iv(x,\, y)$.

Solution

(a) The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$f(z) = \frac{1}{z} = \frac{1}{r}\mathrm{e}^{-i\theta} = \frac{1}{r}(\cos(\theta) - i\sin(\theta)).$$

Thus

$$\tilde{u} = \frac{\cos(\theta)}{r}, \quad \tilde{v} = -\frac{\sin(\theta)}{r}$$

and

$$\frac{\partial \tilde{u}}{\partial r} = -\frac{\cos(\theta)}{r^2}, \quad \frac{\partial \tilde{v}}{\partial r} = \frac{\sin(\theta)}{r^2}, \quad \frac{\partial \tilde{u}}{\partial \theta} = -\frac{\sin(\theta)}{r} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial \theta} = -\frac{\cos(\theta)}{r}.$$

(b) i. For f_1 let

$$u = 3x \quad \text{and} \quad v = -y.$$

$$\frac{\partial u}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = -1.$$

One of the Cauchy Riemann equations is not satisfied and hence f_1 is not analytic.

ii. For f_2 let

$$u = -3x^2y + y^3 \quad \text{and} \quad v = x^3 - 3xy^2.$$

$$\frac{\partial u}{\partial x} = -6xy = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -3x^2 + 3y^2, \quad \frac{\partial v}{\partial x} = 3x^2 - 3y^2.$$

Both Cauchy Riemann equations are satisfied and thus f_2 is analytic.

iii. For f_3 let

$$u = \sinh(x)\cos(y) \quad \text{and} \quad v = -\cosh(x)\sin(y).$$

$$\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= \cosh(x)\cos(y) + \cosh(x)\cos(y) = 2\cosh(x)\cos(y), \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= -\sinh(x)\sin(y) - \sinh(x)\sin(y) = -2\sinh(x)\sin(y).
\end{aligned}$$

The first equation is only satisfied when $\cos(y) = 0$. When this is the case $|\sin(y)| = 1$ and the second equation is only satisfied as well when $x = 0$. Both equations are only satisfied at isolated points and as they are not satisfied in the neighbourhood of any point the function f_3 is not analytic.

iv. For f_4 let

$$u = \frac{\partial^2 \phi}{\partial x \partial y} \quad \text{and} \quad v = -\frac{\partial^2 \phi}{\partial y^2}.$$

$$\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= \frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial^3 \phi}{\partial y^3} \\
&= \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0
\end{aligned}$$

as mixed partial derivatives can be done in any order and because ϕ is harmonic.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$$

as mixed partial derivatives can be done in any order. Both Cauchy Riemann equations are satisfied and thus f_4 is analytic.

(c) Using the Cauchy Riemann equations the harmonic conjugate v satisfies

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2\cosh(2x)\sin(2y).$$

Partially integrating with respect to x gives

$$v = \sinh(2x)\sin(2y) + g(y)$$

for some function $g(y)$. Partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = 2\sinh(2x)\cos(2y) + g'(y) = \frac{\partial u}{\partial x} = 2\sinh(2x)\cos(2y)$$

and thus $g'(y) = 0$ and $g(y)$ is a constant. To satisfy $v(0,\, 0) = 0$ we have $g(y) = 0$ and

$$v = \sinh(2x)\sin(2y).$$

As $\cosh(2x) \ge 1$ we have $u(x\ y) = 0$ only when $\cos(2y) = 0$. When $\cos(2y) = 0$ we have $|v(x,\, y)| = |\sinh(2x)|$ and $v(x,\, y) = 0$ is only satisfied in this case when $x = 0$. The set of points where $u + iv = 0$ is thus

$$\left\{ i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) \,:\, k \in \mathbb{Z} \right\}.$$

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14. *This was most of question 1 of the May 2019* `MA3614` *paper and was worth 16 of the 20 marks.*

(a) Let $z = x + iy$, with $x,\, y \in \mathbb{R}$, and let $f(z) = u(x,y) + iv(x,y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = y.$$

ii.
$$f_2(z) = (-x - 4xy) + i(2x^2 - 2y^2 - y).$$

iii.

$$f_3(z) = \mathrm{e}^x(x\cos(y) - y\sin(y)) + i\mathrm{e}^x(x\sin(y) + y\cos(y)).$$

iv.

$$f_4(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$$

where $\phi(x, y)$ is a harmonic function and the first partial derivatives are not constant.

(c) Let $u(x, y) = \cosh(x)\cos(y)$. Show that u is harmonic and determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 0$.

Solution

(a) The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(b) i. With $u = y$ and $v = 0$ we get

$$\frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = 0.$$

The Cauchy Riemann equations are not satisfied and hence f_1 is not analytic.

ii. Let

$$u = -x - 4xy \quad \text{and} \quad v = 2x^2 - 2y^2 - y.$$

$$\frac{\partial u}{\partial x} = -1 - 4y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -4x, \quad \frac{\partial v}{\partial x} = 4x.$$

Both Cauchy Riemann equations are satisfied and hence f_2 is analytic.

iii. Let

$$u = \mathrm{e}^x(x\cos(y) - y\sin(y)) \quad \text{and} \quad v = \mathrm{e}^x(x\sin(y) + y\cos(y)).$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \mathrm{e}^x\cos(y) + \mathrm{e}^x(x\cos(y) - y\sin(y)), \\
\frac{\partial v}{\partial y} &= \mathrm{e}^x(x\cos(y) + \cos(y) - y\sin(y)), \\
\frac{\partial u}{\partial y} &= \mathrm{e}^x(-x\sin(y) - \sin(y) - y\cos(y)), \\
\frac{\partial v}{\partial x} &= \mathrm{e}^x\sin(y) + \mathrm{e}^x(x\sin(y) + y\cos(y)).
\end{aligned}$$

Both Cauchy Riemann equations are satisfied and hence f_3 is analytic.

iv. Let

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}.$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 2\frac{\partial^2 \phi}{\partial x^2} = -2\frac{\partial^2 \phi}{\partial y^2}$$

as ϕ is harmonic.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x \partial y} = 2\frac{\partial^2 \phi}{\partial x \partial y}$$

as mixed partial derivatives can be done in any order. If both right hand sides are 0 then the first partial derivatives of ϕ are constant. As we are told that this is not the case the Cauchy Riemann equations are not satisfied and hence f_4 is not analytic.

(c) The partial derivatives of u are

$$\frac{\partial u}{\partial x} = \sinh(x)\cos(y), \quad \frac{\partial u}{\partial y} = -\cosh(x)\sin(y), \quad \frac{\partial^2 u}{\partial x^2} = u, \quad \frac{\partial^2 u}{\partial y^2} = -u.$$

Hence u is harmonic.
The harmonic conjugate v is related to u by the Cauchy Riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \cosh(x)\sin(y).$$

Partially integrating with respect to x gives

$$v = \sinh(x)\sin(y) + g(y)$$

for any function $g(y)$.

$$\frac{\partial v}{\partial y} = \sinh(x)\cos(y) + g'(y) = \frac{\partial u}{\partial x}$$

which implies that $g'(y) = 0$ and $g(y) = C$ where C is a constant. $v(0,0) = C = 0$.
Thus

$$v = \sinh(x)\sin(y).$$

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15. *This was question 1 of the May 2018* `MA3614` *paper.*

(a) Let $z = x + iy$, with x, $y \in \mathbb{R}$, and let $f(z) = u(x, y) + iv(x, y)$ denote a function defined in the complex plane $\mathbb{C}$ with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

If $f(z)$ is analytic then express $f'(z)$ in terms of only partial derivatives of u and v with respect to x and also express $f'(z)$ in terms of partial derivatives of only the function u.

(b) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 - y^2 - i2xy.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = x - 4xy + i(y + 2x^2 - 2y^2).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \cos x \cosh y + i \sin x \sinh y.$$

iv.
$$f_4 : \mathbb{C} \to \mathbb{C}, \quad f_4(z) = \frac{\partial^2 \phi}{\partial x^2} - i \frac{\partial^2 \phi}{\partial x \partial y}$$

where ϕ is a harmonic function with continuous partial derivatives of all orders.

(c) Show that the function

$$u(x, y) = 5x^4 y - 10x^2 y^3 + y^5$$

is a harmonic function and determine the harmonic conjugate $v(x, y)$ which satisfies $v(0, 0) = 2$. For this function $v(x, y)$ express $u + iv$ in terms of z only, where as usual $z = x + iy$.

Solution

(a) The Cauchy Riemann equations in Cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In terms of partial derivatives with respect to x only

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

In terms of partial derivatives of u only

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

(b) i. $f_1(z) = x^2 - y^2 - i2xy$ gives $u = x^2 - y^2$ and $v = -2xy$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = -2y.$$

The Cauchy Riemann equations only hold at $x = y = 0$ but as they do not hold in the neighbourhood of the point the function f_1 is not analytic at any point.

ii. $f_2(z) = x - 4xy + i(y + 2x^2 - 2y^2)$ gives $u = x - 4xy$ and $v = y + 2x^2 - 2y^2$.

$$\frac{\partial u}{\partial x} = 1 - 4y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -4x, \quad \frac{\partial v}{\partial x} = 4x.$$

The Cauchy Riemann equations hold at all points and thus the function f_2 is analytic everywhere.

iii. $f_3(z) = \cos x \cosh y + i \sin x \sinh y$ gives $u = \cos x \cosh y$ and $v = \sin x \sinh y$.

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial v}{\partial y} = \sin x \cosh y,$$

and

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial v}{\partial x} = \cos x \sinh y.$$

The first Cauchy Riemann equation only holds when $\sin x = 0$. When this is the case $|\cos x| = 1$ and the second equation will only be satisfied as well when $y = 0$. Both equations only hold at $x = y = 0$ but as they do not hold in a neighbourhood of the point the function f_3 is not analytic at any point.

iv. $f_4(z) = \dfrac{\partial^2 \phi}{\partial x^2} - i\dfrac{\partial^2 \phi}{\partial x \partial y}$ gives $u = \dfrac{\partial^2 \phi}{\partial x^2}$ and $v = -\dfrac{\partial^2 \phi}{\partial x \partial y}$.

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} = \frac{\partial}{\partial x}\nabla^2 \phi = 0$$

as ϕ is harmonic.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^3 \phi}{\partial y \partial x^2} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$$

as mixed partial derivatives do not depend on the order. The Cauchy Riemann equations hold at all points and thus the function f_4 is analytic everywhere.

(c) The partial derivatives of u are

$$\frac{\partial u}{\partial x} = 20x^3y - 20xy^3, \quad \frac{\partial^2 u}{\partial x^2} = 60x^2y - 20y^3,$$

and

$$\frac{\partial u}{\partial y} = 5x^4 - 30x^2y^2 + 5y^4, \quad \frac{\partial^2 u}{\partial y^2} = -60x^2y + 20y^3.$$

$\nabla^2 u = 0$ and thus u is harmonic.

We use the Cauchy Riemann equations to attempt to get v.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -5x^4 + 30x^2y^2 - 5y^4.$$

Partially integrating with respect to x gives

$$v = -x^5 + 10x^3y^2 - 5y^4x + g(y)$$

where $g(y)$ is a differentiable function of y. Partially differentiating with respect to y and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = 20x^3y - 20y^3x + g'(y) = \frac{\partial u}{\partial x}.$$

Hence $g'(y) = 0$ and $g(y)$ is a constant. $v(0,0) = 2$ implies that $g(y) = 2$. Thus

$$v = -x^5 + 10x^3y^2 - 5y^4x + 2.$$

With $f = u+iv$ we have $f(0) = 2i$. As $f(z)$ is analytic we can partially differentiate in the x-direction to get the derivatives. It is a polynomial of degree 5 and we can express in terms of z by forming the Taylor polynomial.

$$\begin{aligned}
f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= 20x^3y - 20xy^3 + i(-5x^4 + 30x^2y^2 - 5y^4), \quad f'(0) = 0, \\
f''(z) &= 60x^2y - 20y^3 + i(-20x^3 + 60xy^2), \quad f''(0) = 0, \\
f'''(z) &= 120xy + i(-60x^2 + 60y^2), \quad f'''(0) = 0, \\
f''''(z) &= 120y + i(-120x), \quad f''''(0) = 0, \\
f^{(5)}(z) &= i(-120), \quad f^{(5)}(0) = -120i.
\end{aligned}$$

Hence

$$f(z) = 2i + \frac{f^{(5)}(0)}{5!}z^5 = 2i - iz^5.$$
