Exercises involving analytic functions, harmonic functions and harmonic conjugates

Some of the questions have been taken from past May exams of MA3614 and some questions are from past class tests. The format of the past May exams was answer 3 from 4 in 3 hours with each question worth 20 marks. Hence if a question given here was worth 10 marks then as a percentage this was worth 16.7%. Up to December 2019 the length of the past class tests was 70 or 75 minutes. The class tests in January 2021, December 2021 and December 2022 were 90 minutes. In all cases students had to answer all questions in the class test to get full marks and the sub-marks added to 100 marks. In some questions the term harmonic appears and the connection between analytic functions and harmonic functions is likely to be covered in about week 5. Techniques to express "in terms of z only" is likely also to be done in week 5 in the lectures.

1. Let $z_1, z_2, \ldots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Prove by induction on n that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

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2. Let $z = x + iy$ and $f = u + iv$, where as usual x, y, u and v are real, If $f(z)$ is analytic in a domain D then show the following.

(a) If $v(x, y) = 0$ in D then $f(z)$ is a real constant.

(b) If $u(x, y) = 0$ in D then $f(z)$ is a pure imaginary constant.

(c) If $|f(z)|$ is constant in D then $f(z)$ is a constant. Hint: First show that if

$$\phi(z) = \frac{1}{2}|f(z)|^2$$

then

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \mathrm{Re}\left(f(z)\overline{f'(z)}\right), \\ \frac{\partial \phi}{\partial y} &= \mathrm{Im}\left(f(z)\overline{f'(z)}\right). \end{aligned}$$

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3. *This was in the class test in December 2022 and was worth 28 of the 100 marks on the paper.*

   (a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in the complex plane $\mathbb{C}$ or if it is not analytic in $\mathbb{C}$.

   If a function is analytic in $\mathbb{C}$ then express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= x + x^2 - y^2 + i(-y - 2xy), \\ f_2(x+iy) &= (2x + 3y + 5x^2 - 5y^2 + 2xy) + i(-3x + 2y - x^2 + y^2 + 10xy). \end{aligned}$$

   (b) Let $x, y \in \mathbb{R}$. If $\phi(x, y)$ is harmonic then explain why

$$g(x, y) = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y}$$

   is analytic.

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4. *This was in the class test in December 2021 and was worth 25 of the 100 marks on the paper.*

   In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 1, 2, 3, 4 then you do part (a) whilst if it is one of the digits 5, 6, 7, 8, 9 then you do part (b).

   (a) This is the version if the 6th digit is one of the digits of 0, 1, 2, 3, 4.

   Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (-2x^2 - 10xy + 6x + 2y^2 + 15y) + i(5x^2 - 4xy - 15x - 5y^2 + 6y), \\ f_2(x+iy) &= (x - 2y) + i(-2x - y). \end{aligned}$$

   (b) This is the version if the 6th digit is one of the digits of 5, 6, 7, 8, 9.

   Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (2x + y) + i(x - 2y), \\ f_2(x+iy) &= (-12x^2 - 18xy + 4x + 12y^2 + 3y) + i(9x^2 - 24xy - 3x - 9y^2 + 4y). \end{aligned}$$

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5. *This was in the class test in January 2021 and was worth 20 of the 100 marks on the paper.*

   Let $z = x + iy$, $x, y \in \mathbb{R}$. In the following you have functions $f_1(z)$ and $f_2(z)$ to consider which are defined on $\mathbb{C}$ and the particular version in your case depends on the last digit of your 7-digit student id.. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

   You do exactly one of (a), (b) and (c) below. Please take care to do the correct version.

   (a) If your last digit is one of $0, 3, 6, 9$ then f_1 and f_2 are as follows.

$$
\begin{aligned}
f_1(x+iy) &= 3x^2 - 2xy + x - 3y^2 + 2y + i\left(-x^2 - 6xy - 2x + y^2 + y\right), \\
f_2(x+iy) &= 5x^2 + 2xy + x - 5y^2 + y + i\left(-x^2 + 10xy - x + y^2 + y\right).
\end{aligned}
$$

   (b) If your last digit is one of $1, 4, 7$ then f_1 and f_2 are as follows.

$$
\begin{aligned}
f_1(x+iy) &= 3x^2 + 2xy + x - 3y^2 + 2y + i\left(-x^2 + 6xy - 2x + y^2 + y\right), \\
f_2(x+iy) &= 5x^2 - 2xy + x - 5y^2 + y + i\left(-x^2 - 10xy - x + y^2 + y\right).
\end{aligned}
$$

   (c) If your last digit is one of $2, 5, 8$ then f_1 and f_2 are as follows.

$$
\begin{aligned}
f_1(x+iy) &= 4x^2 - 2xy + x - 4y^2 - y + i\left(x^2 + 8xy + x - y^2 + y\right), \\
f_2(x+iy) &= 5x^2 + 2xy + x - 5y^2 - y + i\left(x^2 - 10xy + x - y^2 + y\right).
\end{aligned}
$$

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6. *This was in the class test in December 2022 and was worth 11 of the 100 marks on the paper.*

   Let $x, y \in \mathbb{R}$ and let
   $$u(x,y) = -5x^4y + 10x^2y^3 - y^5.$$
   Show that u is harmonic and find the harmonic conjugate $v(x,y)$ satisfying $v(1,0) = 2$.

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7. *This was in the class test in December 2021 and was worth 15 of the 100 marks on the paper.*

   In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 2, 4, 6, 8 then
   $$u(x,y) = -\mathrm{e}^y \sin(x) - 2\mathrm{e}^{-x} \sin(y)$$
   whilst if it is one of the digits 1, 3, 5, 7, 9 then
   $$u(x,y) = \mathrm{e}^y \cos(x) + 2\mathrm{e}^{-x} \cos(y)$$
   with in all cases $x, y \in \mathbb{R}$. Show that your version of $u(x,y)$ is a harmonic function and determine the harmonic conjugate $v(x,y)$ satisfying $v(0,0) = 4$.

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8. *This was in the class test in December 2019 and was worth 25 of the 100 marks on the paper.*

   Let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.
   State the Cauchy Riemann equations.

   Let
   $$u(x,y) = 2x + y + x^2 - y^2 - 2xy.$$
   Show that this function is harmonic and determine the harmonic conjugate v which satisfies $v(0,0) = 1$.

   Express the function $f = u + iv$ in terms of z alone. You need to give reasoning for your answer.

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9. *This was in the class test in December 2018 and was worth 26 of the 100 marks on the paper.*

   Let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.

   State the Cauchy Riemann equations.

   By using the Cauchy Riemann equations, or otherwise, determine if the following functions are analytic in $\mathbb{C}$. If a function is analytic then express it in terms of z alone.

   (a)
   $$f(x+iy) = (x^3 - 3xy^2) + i(-3x^2y + y^3).$$

   (b)
   $$g(x+iy) = (y^3 - 3x^2y + 2xy + 2x^2 - 2y^2) + i(x^3 - 3xy^2 + 4xy - x^2 + y^2).$$

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10. *This was question 1 of the May 2023 exam paper.*

   (a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the real valued functions $p(x, y)$ and $q(x, y)$ are such that $p(x, y) + iq(x, y)$ is analytic in the entire complex plane.

      i.
$$f_1(z) = (x - 2xy) + i(-y - x^2 + y^2).$$

      ii.
$$f_2(z) = (-y + 2x^3 - 6xy^2) + i(x + 6x^2y - 2y^3).$$

      iii.
$$f_3(z) = \mathrm{e}^{-x-3y}(\cos(3x - y) + i\sin(3x - y)).$$

      iv.
$$f_4(z) = (xp(x, y) - yq(x, y)) + i(yp(x, y) + xq(x, y)).$$

   (b) Let $u(x, y) = \cosh(x)\cos(y)$. The function u is harmonic. Find the harmonic conjugate $v(x, y)$ such that $v(0, 0) = 0$.

   (c) Let $z = r\mathrm{e}^{i\theta}$ with $r > 0$ and $-\pi < \theta \le \pi$ and let

$$u(r, \theta) = r^{1/3}\cos(\theta/3), \quad v(r, \theta) = r^{1/3}\sin(\theta/3), \quad \text{and} \quad g(r\mathrm{e}^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

   Give the first order partial derivatives

$$\frac{\partial u}{\partial r}, \quad \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}$$

   in the part of the complex plane where the derivatives exist. The Cauchy Riemann equations in polar coordinates r and θ are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

   In which part of the complex plane is $g(r\mathrm{e}^{i\theta})$ analytic?
   Determine in terms of r and θ the simplest cartesian form of the following limit.

$$\lim_{h \to 0} \frac{g((r + h)\mathrm{e}^{i\theta}) - g(r\mathrm{e}^{i\theta})}{h\mathrm{e}^{i\theta}}.$$

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11. *This was question 1 of the May 2022 exam paper.*

(a) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the function $\phi(x, y)$ is an infinitely continuously differentiable harmonic function.

i.
$$f_1(z) = (x-y) - i(x+y).$$

ii.
$$f_2(z) = (x^3 - 3xy^2 - 4xy) + i(3x^2y - y^3 + 2x^2 - 2y^2).$$

iii.
$$f_3(z) = \mathrm{e}^x(2\cos(y) - \sin(y)) + i\mathrm{e}^x(\cos(y) + 2\sin(y)).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} + i\frac{\partial^2 \phi}{\partial x^2}.$$

(b) Let $z = x+iy$ with $x, y \in \mathbb{R}$ and let

$$g(x+iy) = (x^4 - 6x^2y^2 + y^4 - 2xy) + i(4x^3y - 4xy^3 + x^2 - y^2).$$

The function $g(z)$ is analytic. Express $g(z)$ in terms of z alone. You must justify your answer.

(c) Let $x, y \in \mathbb{R}$ and let

$$u(x, y) = \cos(x)\cosh(y) + \sin(x)\sinh(y).$$

The function $u(x, y)$ is harmonic (you do not need to verify this). Determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 1$.

The analytic function $f(x+iy) = u(x, y) + iv(x, y)$ can be written as a linear combination of e^{iz} and e^{-iz}, i.e. as

$$c\mathrm{e}^{iz} + d\mathrm{e}^{-iz},$$

where c and d are complex constants. Determine the constants c and d.

12. *This was most of question 1 of the May 2021* MA3614 *paper.*

(a) In this part of the question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 1, 2, 3, 4 then you consider the first four functions and if the last digit is one of the digits 5, 6, 7, 8, 9 then you consider the second four functions.

Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

The functions for a last digit of 0, 1, 2, 3, 4 follow.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 + iy.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = \left(-6x^2y - 3x^2 + 2y^3 + 3y^2\right) + i\left(2x^3 - 6xy^2 - 6xy\right).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \sinh(x)\cos(y) - i\cosh(x)\sin(y).$$

iv.
$$f_4 : \mathbb{C}\backslash\{-i\} \to \mathbb{C}, \quad f_4(z) = \frac{(y+1)+ix}{x^2+(y+1)^2}.$$

The functions for a last digit of 5, 6, 7, 8, 9 follow.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = y^2 + ix.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = \left(3x^3 - 9xy^2 + 4xy\right) + i\left(9x^2y - 2x^2 - 3y^3 + 2y^2\right).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \cosh(x)\cos(y) - i\sinh(x)\sin(y).$$

iv.
$$f_4 : \mathbb{C}\backslash\{-1\} \to \mathbb{C}, \quad f_4(z) = \frac{(x+1)-iy}{(x+1)^2+y^2}.$$

(b) Let $\phi(x, y)$ denote a function defined for all x, $y \in \mathbb{R}$ which has continuous partial derivatives of all orders and which is harmonic. Also let ψ be defined for all x, $y \in \mathbb{R}$ by $\psi(x, y) = \phi(x, -y)$.

In this part of the question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 2, 4, 6, 8 then you consider the first two functions and if the last digit is one of the digits 1, 3, 5, 7, 9 then you consider the second two functions.

For your version of g_1 and g_2 you need to determine if it is analytic or not analytic at all points in the complex plane giving reasons for your answers in each case.

The functions for a last digit of 0, 2, 4, 6, 8 follow.

$$g_1(x+iy) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}, \quad g_2(x+iy) = \frac{\partial \psi}{\partial x} - i\frac{\partial \psi}{\partial y}.$$

The functions for a last digit of 1, 3, 5, 7, 9 follow.

$$g_1(x+iy) = \frac{\partial \phi}{\partial y} + i\frac{\partial \phi}{\partial x}, \quad g_2(x+iy) = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}.$$

(c) This part of the question is for all student numbers.
Let $f(z) = z^{1/2}$, where the principal value complex power is used. With $z = r\mathrm{e}^{i\theta}$, $r \ge 0$, $-\pi < \theta \le \pi$ give the real valued functions $u(r,\theta)$ and $v(r,\theta)$ in

$$f(z) = u(r,\theta) + iv(r,\theta).$$

State the domain of z where this function is analytic and give the functions

$$\frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial \theta} \quad \text{and} \quad f'(z).$$

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13. *This was most of question 1 of the May 2020* MA3614 *paper.*

(a) Let $z = x + iy$, with $x,\, y \in \mathbb{R}$. Let $f(z) = u(x,\, y) + iv(x,\, y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

The Cauchy Riemann equations in polar coordinates r and θ for an analytic function $f(re^{i\theta}) = \tilde{u}(r,\, \theta) + i\tilde{v}(r,\, \theta)$, with $\tilde{u}(r,\, \theta)$ and $\tilde{v}(r,\, \theta)$ being real, are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

In the case of $f(z) = 1/z$, $z \neq 0$, give $\tilde{u}$, $\tilde{v}$ and the first order partial derivatives

$$\frac{\partial \tilde{u}}{\partial r}, \quad \frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{\partial \tilde{u}}{\partial \theta} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial r}.$$

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = 3x - iy.$$

ii.
$$f_2(z) = -3x^2y + y^3 + i(x^3 - 3xy^2).$$

iii.
$$f_3(z) = \sinh(x)\cos(y) - i\cosh(x)\sin(y).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} - i\frac{\partial^2 \phi}{\partial y^2}$$

where $\phi(x,\, y)$ is a harmonic function with partial derivatives of all orders being continuous.

(c) The function $u(x,\, y) = \cosh(2x)\cos(2y)$ is harmonic for all x and y. Determine the harmonic conjugate $v(x,\, y)$ such that $v(0,\, 0) = 0$ and indicate all the zeros of the analytic function $u(x,\, y) + iv(x,\, y)$.

14. *This was most of question 1 of the May 2019* `MA3614` *paper and was worth 16 of the 20 marks.*

(a) Let $z = x + iy$, with $x,\ y \in \mathbb{R}$, and let $f(z) = u(x, y) + iv(x, y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = y.$$

ii.
$$f_2(z) = (-x - 4xy) + i(2x^2 - 2y^2 - y).$$

iii.
$$f_3(z) = \mathrm{e}^x(x\cos(y) - y\sin(y)) + i\mathrm{e}^x(x\sin(y) + y\cos(y)).$$

iv.
$$f_4(z) = \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}$$

where $\phi(x, y)$ is a harmonic function and the first partial derivatives are not constant.

(c) Let $u(x, y) = \cosh(x)\cos(y)$. Show that u is harmonic and determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 0$.

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15. *This was question 1 of the May 2018* `MA3614` *paper.*

(a) Let $z = x + iy$, with x, $y \in \mathbb{R}$, and let $f(z) = u(x, y) + iv(x, y)$ denote a function defined in the complex plane $\mathbb{C}$ with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

If $f(z)$ is analytic then express $f'(z)$ in terms of only partial derivatives of u and v with respect to x and also express $f'(z)$ in terms of partial derivatives of only the function u.

(b) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

i.
$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 - y^2 - i2xy.$$

ii.
$$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = x - 4xy + i(y + 2x^2 - 2y^2).$$

iii.
$$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \cos x \cosh y + i \sin x \sinh y.$$

iv.
$$f_4 : \mathbb{C} \to \mathbb{C}, \quad f_4(z) = \frac{\partial^2 \phi}{\partial x^2} - i \frac{\partial^2 \phi}{\partial x \partial y}$$

where ϕ is a harmonic function with continuous partial derivatives of all orders.

(c) Show that the function

$$u(x, y) = 5x^4 y - 10x^2 y^3 + y^5$$

is a harmonic function and determine the harmonic conjugate $v(x, y)$ which satisfies $v(0, 0) = 2$. For this function $v(x, y)$ express $u + iv$ in terms of z only, where as usual $z = x + iy$.
