Exercises involving analytic functions, harmonic functions and harmonic conjugates

Some of the questions have been taken from past May exams of MA3614 and some questions are from past class tests. The format of the past May exams was answer 3 from 4 in 3 hours with each question worth 20 marks. Hence if a question given here was worth 10 marks then as a percentage this was worth 16.7%. Up to December 2019 the length of the past class tests was 70 or 75 minutes. The class tests in January 2021, December 2021, December 2022 and December 2023 were 90 minutes. (I do not include any questions from the January 2021 or May 2021 exams as these were online at-home exams, due to covid, the format of the questions was a little different.) In all cases students had to answer all questions in the class test to get full marks and the sub-marks added to 100 marks. In some questions the term harmonic appears and the connection between analytic functions and harmonic functions is likely to be covered in about week 5. Techniques to express "in terms of z only" is likely also to be done in week 5 in the lectures or possibly slightly before.

1. Let $z_1, z_2, \dots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Prove by induction on n that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

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2. Let $z = x + iy$ and $f = u + iv$, where as usual x, y, u and v are real, If $f(z)$ is analytic in a domain D then show the following.

   (a) If $v(x, y) = 0$ in D then $f(z)$ is a real constant.

   (b) If $u(x, y) = 0$ in D then $f(z)$ is a pure imaginary constant.

   (c) If $|f(z)|$ is constant in D then $f(z)$ is a constant. Hint: First show that if

$$\phi(z) = \frac{1}{2}|f(z)|^2$$

   then

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \mathrm{Re}\left(f(z)\overline{f'(z)}\right), \\ \frac{\partial \phi}{\partial y} &= \mathrm{Im}\left(f(z)\overline{f'(z)}\right). \end{aligned}$$

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3. *This was in the class test in December 2023 and was worth 19 of the 100 marks on the paper.*

   Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in the complex plane $\mathbb{C}$ or if it is not analytic in $\mathbb{C}$.

   If a function is analytic in $\mathbb{C}$ then express it in terms of z alone. Full reasoning must be given to get all the marks.

$$
\begin{aligned}
f_1(x+iy) &= x^2 + y^2, \\
f_2(x+iy) &= (2x - 3y) + i(-3x - 2y), \\
f_3(x+iy) &= (-x - 2y + 3x^2 - 3y^2 + 2xy) + i(2x - y - x^2 + y^2 + 6xy).
\end{aligned}
$$

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4. *This was in the class test in December 2023 and was worth 19 of the 100 marks on the paper.*

   Let $x, y \in \mathbb{R}$ and let

$$
u(x,y) = x^3 - 3xy^2 + 3x^2y - y^3 + \sin(x)\left(\mathrm{e}^{y} + \mathrm{e}^{-y}\right).
$$

   Show that u is harmonic.

   Find the harmonic conjugate $v(x,y)$ satisfying $v(1,0) = 0$.

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5. *This was in the class test in December 2022 and was worth 28 of the 100 marks on the paper.*

   (a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in the complex plane $\mathbb{C}$ or if it is not analytic in $\mathbb{C}$.

   If a function is analytic in $\mathbb{C}$ then express it in terms of z alone. Full reasoning must be given to get all the marks.

$$
\begin{aligned}
f_1(x+iy) &= x + x^2 - y^2 + i(-y - 2xy), \\
f_2(x+iy) &= (2x + 3y + 5x^2 - 5y^2 + 2xy) + i(-3x + 2y - x^2 + y^2 + 10xy).
\end{aligned}
$$

   (b) Let $x, y \in \mathbb{R}$. If $\phi(x,y)$ is harmonic then explain why

$$
g(x,y) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}
$$

   is analytic.

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6. *This was in the class test in December 2021 and was worth 25 of the 100 marks on the paper.*

   In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 1, 2, 3, 4 then you do part (a) whilst if it is one of the digits 5, 6, 7, 8, 9 then you do part (b).

   (a) This is the version if the 6th digit is one of the digits of 0, 1, 2, 3, 4.
   Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (-2x^2 - 10xy + 6x + 2y^2 + 15y) + i(5x^2 - 4xy - 15x - 5y^2 + 6y), \\ f_2(x+iy) &= (x - 2y) + i(-2x - y). \end{aligned}$$

   (b) This is the version if the 6th digit is one of the digits of 5, 6, 7, 8, 9.
   Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of your functions you need to determine if it is analytic in $\mathbb{C}$ or it is not analytic in $\mathbb{C}$, and if a function is analytic express it in terms of z alone. Full reasoning must be given to get all the marks.

$$\begin{aligned} f_1(x+iy) &= (2x + y) + i(x - 2y), \\ f_2(x+iy) &= (-12x^2 - 18xy + 4x + 12y^2 + 3y) + i(9x^2 - 24xy - 3x - 9y^2 + 4y). \end{aligned}$$

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7. *This was in the class test in December 2022 and was worth 11 of the 100 marks on the paper.*

   Let $x, y \in \mathbb{R}$ and let

$$u(x,y) = -5x^4y + 10x^2y^3 - y^5.$$

   Show that u is harmonic and find the harmonic conjugate $v(x,y)$ satisfying $v(1,0) = 2$.

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8. *This was in the class test in December 2021 and was worth 15 of the 100 marks on the paper.*

   In this question the version that you do depends on the 6th digit of your 7-digit student id.. If the 6th digit is one of the digits 0, 2, 4, 6, 8 then

$$u(x,y) = -\mathrm{e}^y \sin(x) - 2\mathrm{e}^{-x} \sin(y)$$

   whilst if it is one of the digits 1, 3, 5, 7, 9 then

$$u(x,y) = \mathrm{e}^y \cos(x) + 2\mathrm{e}^{-x} \cos(y)$$

   with in all cases $x, y \in \mathbb{R}$. Show that your version of $u(x,y)$ is a harmonic function and determine the harmonic conjugate $v(x,y)$ satisfying $v(0,0) = 4$.

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9. *This was in the class test in December 2019 and was worth 25 of the 100 marks on the paper.*

   Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.
   State the Cauchy Riemann equations.

   Let
   $$u(x, y) = 2x + y + x^2 - y^2 - 2xy.$$

   Show that this function is harmonic and determine the harmonic conjugate v which satisfies $v(0, 0) = 1$.

   Express the function $f = u + iv$ in terms of z alone. You need to give reasoning for your answer.

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10. *This was in the class test in December 2018 and was worth 26 of the 100 marks on the paper.*

    Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ with $x, y, u, v \in \mathbb{R}$.

    State the Cauchy Riemann equations.

    By using the Cauchy Riemann equations, or otherwise, determine if the following functions are analytic in $\mathbb{C}$. If a function is analytic then express it in terms of z alone.

    (a)
    $$f(x + iy) = (x^3 - 3xy^2) + i(-3x^2y + y^3).$$

    (b)
    $$g(x + iy) = (y^3 - 3x^2y + 2xy + 2x^2 - 2y^2) + i(x^3 - 3xy^2 + 4xy - x^2 + y^2).$$

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11. *This was question 1 of the May 2024 exam paper.*

(a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions determine if they are analytic in the complex plane, giving reasons for your answer in each case.

i.
$$f_1(z) = (2x^3 + 3x^2y - 6xy^2 - y^3) + i(-x^3 + 6x^2y + 3xy^2 - 2y^3).$$

ii.
$$f_2(z) = \frac{\partial \phi}{\partial y} + i\frac{\partial \phi}{\partial x},$$
where $\phi(x, y)$ denotes any function which is harmonic at all points (x, y).

iii.
$$f_3(z) = \sinh(x)\cos(y) + i\cosh(x)\sin(y).$$

iv.
$$f_4(z) = \mathrm{e}^{2x}\left(\cos(2y) - i\sin(2y)\right).$$

(b) Let $D = \{z : |z| > 0\}$ and let $z = r\mathrm{e}^{i\theta}$ where $r \ge 0$ and $\theta \in \mathbb{R}$ denote the usual polar coordinates. Further let

$$u(r, \theta) = \frac{\cos(\theta)}{r}.$$

It can be shown that $u(r, \theta)$ is harmonic in D. It can also be shown that the following function is analytic in D.

$$g(z) = \frac{1}{\mathrm{e}^{i\theta}}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right).$$

Give an expression for the value $g(z)$ in terms of r and θ and also write it in terms of z alone.

(c) Let $z = x + iy$ with $x, y \in \mathbb{R}$ and let a_0, a_1, a_2, b_1 and b_2 denote non-zero complex numbers. Consider the following three functions $h_1(z)$, $h_2(z)$ and $h_3(z)$.

$$\begin{aligned}
h_1(x + iy) &= (3x + 2y + 2xy) + i(-2x + 3y - x^2 + y^2),\\
h_2(z) &= \overline{\psi_2(\overline{z})}, \quad \text{where } \psi_2(z) = a_0 + a_1 z + a_2 z^2,\\
h_3(z) &= \frac{1}{\overline{\psi_3(\overline{z})}}, \quad \text{where } \psi_3(z) = \frac{b_1 z + 1}{b_2 z + 1}.
\end{aligned}$$

i. Show that $h_1(x + iy)$ is analytic and express in terms of z alone.

ii. It can be shown that $h_2(z)$ and $h_3(z)$ are analytic. In each case express them in terms of z alone.

iii. Give the most general conditions on a_0, a_1 and a_2 such that $h_2(z) = \psi_2(z)$ for all z.

iv. Give the most general conditions on b_1 and b_2 such that $h_3(z) = \psi_3(z)$ for all z.

12. *This was question 1 of the May 2023 exam paper.*

(a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the real valued functions $p(x, y)$ and $q(x, y)$ are such that $p(x, y) + iq(x, y)$ is analytic in the entire complex plane.

i.
$$f_1(z) = (x - 2xy) + i(-y - x^2 + y^2).$$

ii.
$$f_2(z) = (-y + 2x^3 - 6xy^2) + i(x + 6x^2y - 2y^3).$$

iii.
$$f_3(z) = \mathrm{e}^{-x-3y}(\cos(3x - y) + i\sin(3x - y)).$$

iv.
$$f_4(z) = (xp(x, y) - yq(x, y)) + i(yp(x, y) + xq(x, y)).$$

(b) Let $u(x, y) = \cosh(x)\cos(y)$. The function u is harmonic. Find the harmonic conjugate $v(x, y)$ such that $v(0, 0) = 0$.

(c) Let $z = r\mathrm{e}^{i\theta}$ with $r > 0$ and $-\pi < \theta \le \pi$ and let

$$u(r, \theta) = r^{1/3}\cos(\theta/3), \quad v(r, \theta) = r^{1/3}\sin(\theta/3), \quad \text{and} \quad g(r\mathrm{e}^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

Give the first order partial derivatives

$$\frac{\partial u}{\partial r}, \quad \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}$$

in the part of the complex plane where the derivatives exist. The Cauchy Riemann equations in polar coordinates r and θ are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

In which part of the complex plane is $g(r\mathrm{e}^{i\theta})$ analytic?
Determine in terms of r and θ the simplest cartesian form of the following limit.

$$\lim_{h \to 0} \frac{g((r + h)\mathrm{e}^{i\theta}) - g(r\mathrm{e}^{i\theta})}{h\mathrm{e}^{i\theta}}.$$

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13. *This was question 1 of the May 2022 exam paper.*

(a) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the entire complex plane giving reasons for your answers in each case. In the case of $f_4(z)$ the function $\phi(x, y)$ is an infinitely continuously differentiable harmonic function.

i.
$$f_1(z) = (x - y) - i(x + y).$$

ii.
$$f_2(z) = (x^3 - 3xy^2 - 4xy) + i(3x^2y - y^3 + 2x^2 - 2y^2).$$

iii.
$$f_3(z) = \mathrm{e}^x(2\cos(y) - \sin(y)) + i\mathrm{e}^x(\cos(y) + 2\sin(y)).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} + i\frac{\partial^2 \phi}{\partial x^2}.$$

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$ and let

$$g(x + iy) = (x^4 - 6x^2y^2 + y^4 - 2xy) + i(4x^3y - 4xy^3 + x^2 - y^2).$$

The function $g(z)$ is analytic. Express $g(z)$ in terms of z alone. You must justify your answer.

(c) Let $x, y \in \mathbb{R}$ and let

$$u(x, y) = \cos(x)\cosh(y) + \sin(x)\sinh(y).$$

The function $u(x, y)$ is harmonic (you do not need to verify this). Determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 1$.

The analytic function $f(x + iy) = u(x, y) + iv(x, y)$ can be written as a linear combination of e^{iz} and e^{-iz}, i.e. as

$$c\mathrm{e}^{iz} + d\mathrm{e}^{-iz},$$

where c and d are complex constants. Determine the constants c and d.

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14. *This was most of question 1 of the May 2020* MA3614 *paper.*

(a) Let $z = x + iy$, with $x, y \in \mathbb{R}$. Let $f(z) = u(x, y) + iv(x, y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

The Cauchy Riemann equations in polar coordinates r and θ for an analytic function $f(re^{i\theta}) = \tilde{u}(r, \theta) + i\tilde{v}(r, \theta)$, with $\tilde{u}(r, \theta)$ and $\tilde{v}(r, \theta)$ being real, are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

In the case of $f(z) = 1/z$, $z \neq 0$, give $\tilde{u}$, $\tilde{v}$ and the first order partial derivatives

$$\frac{\partial \tilde{u}}{\partial r}, \quad \frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{\partial \tilde{u}}{\partial \theta} \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial r}.$$

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = 3x - iy.$$

ii.
$$f_2(z) = -3x^2y + y^3 + i(x^3 - 3xy^2).$$

iii.
$$f_3(z) = \sinh(x)\cos(y) - i\cosh(x)\sin(y).$$

iv.
$$f_4(z) = \frac{\partial^2 \phi}{\partial x \partial y} - i\frac{\partial^2 \phi}{\partial y^2}$$

where $\phi(x, y)$ is a harmonic function with partial derivatives of all orders being continuous.

(c) The function $u(x, y) = \cosh(2x)\cos(2y)$ is harmonic for all x and y. Determine the harmonic conjugate $v(x, y)$ such that $v(0, 0) = 0$ and indicate all the zeros of the analytic function $u(x, y) + iv(x, y)$.

15. *This was most of question 1 of the May 2019* MA3614 *paper and was worth 16 of the 20 marks.*

(a) Let $z = x + iy$, with x, $y \in \mathbb{R}$, and let $f(z) = u(x, y) + iv(x, y)$ denote a function defined in the complex plane $\mathbb{C}$, with u and v being real-valued functions which have continuous partial derivatives of all orders.

State the Cauchy Riemann equations for an analytic function in terms of partial derivatives of u and v with respect to x and y.

(b) Let $z = x + iy$ with $x, y \in \mathbb{R}$. For each of the following functions, determine whether or not it is analytic in the complex plane, giving reasons for your answers in each case.

i.
$$f_1(z) = y.$$

ii.
$$f_2(z) = (-x - 4xy) + i(2x^2 - 2y^2 - y).$$

iii.
$$f_3(z) = \mathrm{e}^x(x\cos(y) - y\sin(y)) + i\mathrm{e}^x(x\sin(y) + y\cos(y)).$$

iv.
$$f_4(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$$

where $\phi(x, y)$ is a harmonic function and the first partial derivatives are not constant.

(c) Let $u(x, y) = \cosh(x)\cos(y)$. Show that u is harmonic and determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 0$.