Revision: Key formula

Let f be a function which is analytic in a domain D and let Γ be a positively orientated loop in D and let z be a point inside D.

The generalised Cauchy integral formula giving $f^{(n)}(z_0)$

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z, \quad n = 0, 1, 2, \cdots$$

Taylor's series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

If $f(z)$ is analytic in $|z-z_0| < R$ then we have uniform convergence to $f(z)$ in $|z-z_0| \le R' < R$ for all $R' < R$.

Results with power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} f(re^{it})\,\mathrm{e}^{-int}\,\mathrm{d}t.$$

- Odd functions only involve odd powers. Even functions only involve even powers. Real valued functions have real coefficients.
- In the region where the series converges we can do the following.
  We can differentiate and integrate term-by-term.
  We can multiply two series, i.e.

$$c_0+c_1z+c_2z^2+\cdots = (a_0+a_1z+a_2z^2+\cdots)(b_0+b_1z+b_2z^2+\cdots),$$

$$\begin{aligned} c_0 &= a_0b_0, \\ c_1 &= a_1b_0 + a_0b_1, \\ c_2 &= a_2b_0 + a_1b_1 + a_0b_2. \end{aligned}$$

The formula for c_n is known as the Cauchy product.

Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

The terms a_0, a_1, ... are the coefficients of the power series.

The series always converges at $z = z_0$. When it converges at other points the region where it converges is a disk $\{z : \ |z-z_0| < R\}$ and it is analytic in the disk. A proof was given last week.

The largest R is the **radius of convergence**. When $R < \infty$ $\{z : \ |z-z_0| = R\}$ is the **circle of convergence**. In all cases

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

In our examples we obtain R using the ratio test or the root test.

$R = 0$ when we only have convergence at $z = z_0$.

$R = \infty$ when we have convergence for all z.

Some examples of power series

1. $$\sum_{n=0}^{\infty}(nz)^n, \qquad \sum_{n=0}^{\infty}\frac{2^n}{n!}z^n.$$

   The first series only converges at $z = 0$. The terms are not bounded when $z \ne 0$.
   By the ratio test the second series converges for all z.

2. $$\sum_{n=0}^{\infty}\frac{n+1}{n^2+2}(z-1)^n.$$

   By the ratio test the circle of convergence is $|z-1| = 1$.

3. $$\sum_{n=0}^{\infty}(2+\sin(n))z^n.$$

   With $a_n = 2 + \sin(n) \in [1, 3]$, $a_n^{1/n} \to 1$ as $n \to \infty$ and by the root test the circle of convergence is $|z| = 1$.

Comments about the "general case"

Suppose the sequence $(|a_n|^{1/n})$ does not converge and thus the root test cannot be used. If the sequence $(|a_n|^{1/n})$ is not bounded then for all $z \ne z_0$ we have for some sufficiently large n

$$|a_n|^{1/n} > \frac{1}{|z-z_0|} \quad \text{and hence } |a_n||z-z_0|^n > 1$$

and the terms $(a_n(z-z_0)^n)$ cannot tend to 0 as $n \to \infty$. Thus the series only converges at $z = z_0$.

If the sequence is bounded then we can define

$$b_n = \sup\{|a_m|^{1/m} : \ m \ge n\} \ge 0.$$

This is a decreasing sequence bounded below by 0 and converges by the monotone convergence theorem. We label the limit as $\alpha \ge 0$. There is a theorem known as the Cauchy-Hadamard theorem which is briefly that

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{has radius of convergence } R = \frac{1}{\alpha}.$$

Properties of a function defined by a power series

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad R = \frac{1}{\limsup |a_n|^{1/n}}.$$

When $R > 0$ this defines an analytic function in $|z-z_0| < R$.

One way to relate the coefficients a_n to the derivatives of $f(z)$ is to use the generalised Cauchy integral formula. We take a loop Γ in the disk with z_0 inside the loop.

$$\begin{aligned}\frac{f^{(m)}(z_0)}{m!} &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{m+1}}\,dz \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} a_n \oint_\Gamma (z-z_0)^{n-(m+1)}\,dz.\end{aligned}$$

The only integral in the last line which is non-zero is when $n-(m+1) = -1$, i.e. when $n = m$ and we get

$$\frac{f^{(m)}(z_0)}{m!} = a_m.$$

Laurent series

A Laurent series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$

When it converges the region is an annulus $\{z : \ r < |z-z_0| < R\}$.

$$\sum_{n=-\infty}^{-1} a_n(z-z_0)^n, \qquad \text{converges in } |z-z_0| > r.$$

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \text{converges in } |z-z_0| < R.$$

To be a function defined at some points we need the coefficients a_n to be such that $r < R$.

Example: construction of a Laurent series

$$f(z) = \frac{1}{(1-z)(2-z)} = \frac{A}{1-z} + \frac{B}{2-z}.$$

$\frac{1}{1-z}$	This has a geometric series representation in $\lvert z\rvert < 1$. It has a series representation in $\lvert z\rvert > 1$ involving powers of $1/z$.
$\frac{1}{2-z}$	This has a geometric series representation in $\lvert z\rvert < 2$ involving powers of $z/2$. It has a series representation in $\lvert z\rvert > 2$ involving powers of $2/z$.

Laurent series for $f(z)$ in different regions.

$\lvert z\rvert < 1$	Combine the geometric series.
$1 < \lvert z\rvert < 2$	Combine the power series for the $1/(2-z)$ term with the series with negative powers for the $1/(1-z)$ term.
$\lvert z\rvert > 2$	Combine the series involving only negative powers for both parts.

Some points about the manipulation

$$g(z) = \frac{1}{c - z}.$$

$$c - z = c\left(1 - \frac{z}{c}\right) = -z\left(1 - \frac{c}{z}\right).$$

When $|z/c| < 1$ we have the geometric series

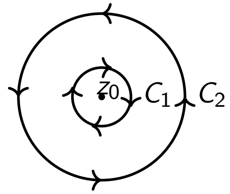
$$g(z) = \left(\frac{1}{c}\right)\left(1 + \left(\frac{z}{c}\right) + \left(\frac{z}{c}\right)^2 + \cdots\right)$$

When $|z/c| > 1$, $|c/z| < 1$ and we have

$$g(z) = -\left(\frac{1}{z}\right)\left(1 + \frac{c}{z} + \left(\frac{c}{z}\right)^2 + \cdots\right).$$

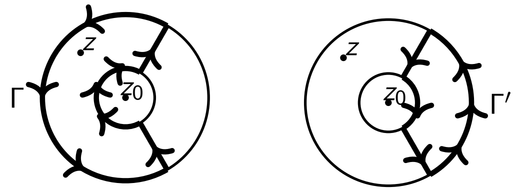
We get the representation involving negative powers.

Steps in proving the Laurent series representation



$C_1 \cup C_2$ is the boundary of an annulus where $f(z)$ is analytic in a slightly larger annulus. Note that C_1 is clockwise, C_2 is anti-clockwise.

The loop Γ is such that z is inside Γ.



Due to cancellation on the radial lines we have for any function g

$$\oint_\Gamma g(\zeta)\,\mathrm{d}\zeta + \oint_{\Gamma'} g(\zeta)\,\mathrm{d}\zeta = \oint_{C_1} g(\zeta)\,\mathrm{d}\zeta + \oint_{C_2} g(\zeta)\,\mathrm{d}\zeta.$$

Steps in proving $\cdots$ continued

Let z be inside Γ and outside Γ'. By the Cauchy integral formula

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\oint_{\Gamma\cup\Gamma'} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \end{aligned}$$

As in the Taylor series proof the non-negative powers part is

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,\mathrm{d}\zeta.$$

The negative powers come from re-writing the term

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k},$$

$$a_{-k} = -\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^{k-1}\,\mathrm{d}\zeta, \quad k = 1, 2, \ldots$$

Further effort enables C_2 and $-C_1$ to be replaced by a curve C.

Laurent series representation

Let $f(z)$ be analytic in an annulus $r < |z - z_0| < R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}.$$

The series converge uniformly in any closed sub-annulus $r < \rho_1 \le |z - z_0| \le \rho_2 < R$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z,$$

where C is any positively orientated simple closed curve lying in the annulus which has z_0 as an interior point.

This indicates that the representation is unique.

Also note that in none of the examples that have been done did we obtain a_n by evaluating this integral.

Laurent series: Classifying zeros and poles

When $f(z)$ has a **zero of multiplicity** $m \geq 1$ **at** z_0 we have

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots = (z-z_0)^m g(z)$$

with $g(z)$ being analytic at z_0 and $g(z_0) = a_m \neq 0$.

If $f(z)$ has **a removable singularity** at z_0 then it has a Laurent series valid in $0 < |z-z_0| < R$ with no negative powers, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \lim_{z\to z_0} f(z) = a_0.$$

If $f(z)$ has a **pole of order** m then in $0 < |z-z_0| < R$ we have

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = \frac{\phi(z)}{(z-z_0)^m}$$

with $\phi(z)$ being analytic at z_0 and $\phi(z_0) = a_{-m} \neq 0$.

An **essential singularity at** z_0 has infinitely many negative powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \quad 0 < |z-z_0| < R.$$

Isolated zeros of non-zero analytic functions

When $f(z)$ has a **zero of multiplicity** $m \geq 1$ **at** z_0 we have

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots = (z-z_0)^m g(z)$$

with $g(z)$ being analytic at z_0 and $g(z_0) = a_m \neq 0$. These properties of $g(z)$ imply that in a neighbourhood $\{z : |z-z_0| < \delta\}$, for some $\delta > 0$, $g(z)$ is non-zero and thus $f(z)$ is non-zero. The zeros of $f(z)$ are isolated.

As an example suppose that the Cauchy Riemann equations are used to show that the following is analytic.

$$f(x+iy) = (-2x^2-10xy+6x+2y^2+15y)+i(5x^2-4xy-15x-5y^2+6y).$$

$$f(x) = (-2x^2+6x) + i(5x^2-15x).$$

$$g(z) = (-2z^2+6z) + i(5z^2-15z).$$

$f(x+iy)$ and $g(z)$ are both analytic with $f(z) - g(z) = 0$ on the real line. Hence $f(z) = g(z)$ for all z.

Complex identity and the related real relation

The isolated zeros property of non-zero analytic functions is a way to quickly explain why many identities are also true in the complex plane. For example,

$$\begin{aligned} \cos^2(x) + \sin^2(x) &= 1, \\ \sin(2x) &= 2\sin(x)\cos(x), \end{aligned}$$

being true for all $x \in \mathbb{R}$ also hold for all $z \in \mathbb{C}$, i.e.

$$\begin{aligned} \cos^2(z) + \sin^2(z) &= 1, \\ \sin(2z) &= 2\sin(z)\cos(z). \end{aligned}$$

Integrating a Laurent Series

Let $f(z)$ be analytic in an annulus with the following Laurent series representation.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \quad 0 < |z-z_0| < R.$$

The coefficient a_{-1} is called the residue at z_0. We write $\mathrm{Res}(f, z_0)$.

Let Γ denote a loop traversed once in the anti-clockwise sense with z_0 inside Γ. Then term-by-term integration gives

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i a_{-1}.$$

This is one of properties we need to show residue theorem which is in chapter 8 of the main notes.