Chap 3: some of the main points

Definitions: Complex differentiable, analytic ...

As was introduced in week 03.

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at** z_0 is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

  provided the limit exists.
- A function f is **analytic at** z_0 if f is differentiable at all points in some neighbourhood of z_0.
- A function f is **analytic in a domain** if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an **entire function** if it is analytic on the whole complex plane $\mathbb{C}$.

The Cauchy Riemann equations for $f(z) = u(x, y) + iv(x, y)$

When f is analytic at z_0 the following limit exists.

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

By considering the case when h is real and then purely imaginary we get

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If u and v have continuous first partial derivatives on a domain D and the Cauchy Riemann equations are satisfied then f is analytic on D.

Representations for $f'(z)$

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, && \text{(only involving derivatives with respect to } x), \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, && \text{(only involving derivatives with respect to } y), \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, && \text{(only involving } u), \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}, && \text{(only involving } v). \end{aligned}$$

The different versions are because of the CR equations.

$f'(z)$ is thus completely determined by ∇u.

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Expressing in terms of z only

In the polynomial cases we directly showed how to write in terms of z.

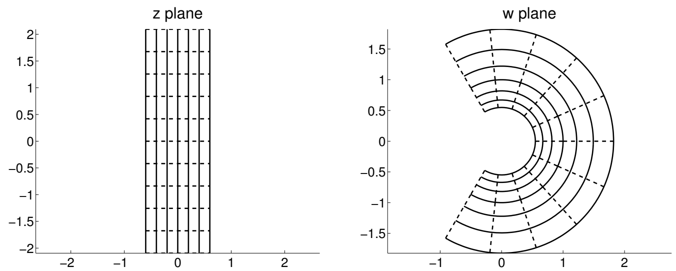
The level curves of u and v are orthogonal

By using the CR equations

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.$$

As a non-zero gradient vector is normal to a level curve this implies that the level curves of u and v are orthogonal.

Mapping of $w = \exp(z)$, level curves of $z = \mathrm{Log}(w)$



The circles and radial lines are level curves of $\mathrm{Log}(w)$, i.e. they are the curves where the real and imaginary parts are constant.

Harmonic functions and analytic function

- $\phi(x, y)$ is **harmonic** if

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

- If $f = u + iv$ is analytic then u and v are harmonic functions. v is said to be the **harmonic conjugate** of u.

If u is known then we can attempt to get v as follows.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Partially integrate w.r.t. x to get

$$\begin{aligned} v(x, y) &= \text{some function} + g(y) \\ \frac{\partial v}{\partial y} &= \text{deriv of some function} + g'(y) = \frac{\partial u}{\partial x} \end{aligned}$$

This gives $g'(y)$ and then we get $g(y)$.

Chap 4: Elementary functions of z

We consider the following.

1. Polynomials.
2. Rational functions.
3. $e^z = \exp(z)$.
4. $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$, $\tan(z)$, $\cot(z)$, $\tanh(z)$.
5. $\log(z)$, $\mathrm{Log}(z)$.
6. Complex powers, i.e. z^α.

With rational functions and the functions $\tan(z)$, $\cot(z)$ and $\tanh(z)$ we mention pole singularities and residues for the first time. They all have isolated singularities.

In the case of the logarithm and complex powers we mainly restrict to the "principal value case" which involves using the principal argument $\mathrm{Arg}(z)$.

Representation of polynomials and zeros

Polynomials are entire functions and can be represented in several ways.

$$\begin{aligned} p_n(z) &= \sum_{k=0}^{n} a_k z^k \\ &= \sum_{k=0}^{n} \frac{p_n^{(k)}(0)}{k!} z^k, \quad \text{(finite Maclaurin series)}, \\ &= \sum_{k=0}^{n} \frac{p_n^{(k)}(z_0)}{k!} (z - z_0)^k, \quad \text{(Taylor polynomial)}, \\ &= a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n), \quad \text{(in terms of the zeros)}, \\ &= a_n (z - z_1)^{r_1} (z - z_2)^{r_2} \cdots (z - z_m)^{r_m}, \end{aligned}$$

where $z_1, \ldots, z_m$ are the distinct zeros and $r_1 + \cdots + r_m = n$.

At the zero z_k of multiplicity r_k we have

$$p_n(z_k) = p_n'(z_k) = \cdots = p_n^{(r_k - 1)}(z_k) = 0, \quad p_n^{(r_k)}(z_k) \neq 0.$$

Rational functions

These are the ratio of two polynomials.

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

$z_1, \ldots, z_n$ are singular points.

If the limit exists as $z \to z_k$ then z_k is a **removable singularity**.

Otherwise $R(z)$ has a **pole singularity** at z_k. A **simple pole** is the case when $1/R(z)$ has a simple zero at z_k.

The order of the pole of $R(z)$ is the multiplicity of the zero of $1/R(z)$.

These terms will appear again in term 2 when we classify functions more generally which have isolated singularities. This will be in the Laurent series section.

Partial fractions representation – just simple poles case

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

When $\deg p(z) < \deg q(z)$ and the zeros of $q(z)$ are simple we have the partial fraction representation of the form

$$R(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

When $\deg p(z) \ge \deg q(z)$ and the zeros of $q(z)$ are simple we have a representation of the form

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

In either case A_k is the **residue** at z_k.

Getting the residues when we only have simple poles

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

To get A_k we have

$$\begin{aligned} A_k &= \lim_{z \to z_k} (z - z_k) R(z) = \lim_{z \to z_k} \frac{(z - z_k) p(z)}{q(z)} \\ &= p(z_k) \lim_{z \to z_k} \frac{(z - z_k)}{q(z)} = \frac{p(z_k)}{q'(z_k)}. \end{aligned}$$

Multiple poles case

When $q(z)$ has a zero at ζ of multiplicity $r \ge 1$ we need terms involving

$$\frac{1}{z - \zeta}, \quad \frac{1}{(z - \zeta)^2}, \quad \ldots, \quad \frac{1}{(z - \zeta)^r}.$$

Partial fraction examples

Note that $z^2 + 1 = (z + i)(z - i)$

$$f_1(z) = \frac{1}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i},$$

we need to determine A and B.

$$f_2(z) = \frac{z^3}{z^2 + 1} = (\text{deg 1 polynomial}) + \frac{A}{z + i} + \frac{B}{z - i},$$

we need to determine A and B,

$$f_3(z) = \frac{4}{(z^2 + 1)(z - 1)^2} = \frac{A}{z + i} + \frac{B}{z - i} + \frac{C_1}{z - 1} + \frac{C_2}{(z - 1)^2}$$

we need to determine A, B, C_1, C_2.

All the functions have pole singularities at $\pm i$ and $f_3(z)$ also has a pole at 1.

The residues are associated with the simple pole terms and are labelled as A and B in the case of f_1 and f_2 and are labelled as A_1 and B_1 in the case of f_3.

The representation when $q(z) = (z - z_0)^m$

Suppose the only singularity is at $z = z_0$ and we have

$$R(z) = \frac{p(z)}{(z - z_0)^m}, \quad p(z) \text{ is a polynomial of degree } n.$$

In this case we use the finite Taylor polynomial representation of $p(z)$ about z_0, i.e.

$$p(z) = p(z_0) + p'(z_0)(z - z_0) + \cdots + \frac{p^{(n)}(z_0)}{n!}(z - z_0)^n.$$

$$R(z) = \frac{p(z_0)}{(z - z_0)^m} + \frac{p'(z_0)}{(z - z_0)^{m-1}} + \cdots + \frac{p^{(n)}(z_0)/n!}{(z - z_0)^{m-n}}.$$

The residue is

$$\frac{p^{(m-1)}(z_0)}{(m - 1)!}.$$

Further comments about the $f_1(z)$ example

$$f_1(z) = \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}.$$

There is not too much to do by any of the methods in this case.

If we put the RHS on a common denominator (i.e. z^2+1) and equate the result numerators then we get

$$1 = A(z-i) + B(z+i).$$

As this is true for all z we can separately set $z=i$ or $z=-i$ to get B and then A. Thus $B = 1/(2i) = -i/2$.

If we just get B by the above then equating the coefficient of z gives

$$0 = A + B, \quad A = -B.$$

Further comments about the $f_1(z)$ example continued

We can use limits and L'Hopital's rule. This is because we have the following.

$$\begin{aligned}(z+i)f_1(z) &= A + (z+i)(\text{func analytic at } -i) \to A \quad \text{as } z \to -i,\\ (z-i)f_1(z) &= B + (z-i)(\text{func analytic at } i) \to B \quad \text{as } z \to i.\end{aligned}$$

$$\begin{aligned}A &= \lim_{z\to -i} \frac{z+i}{z^2+1} = \left.\frac{1}{2z}\right|_{z=-i} = -\frac{1}{2i} = \frac{i}{2},\\ B &= \lim_{z\to i} \frac{z-i}{z^2+1} = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i} = -\frac{i}{2}.\end{aligned}$$

Further comments about the $f_2(z)$ example

$$f_2(z) = \frac{z^3}{z^2+1} = (\text{deg 1 polynomial}) + \frac{A}{z+i} + \frac{B}{z-i}.$$

$$\begin{aligned}A &= \lim_{z\to -i} \frac{(z+i)z^3}{z^2+1},\\ B &= \lim_{z\to i} \frac{(z-i)z^3}{z^2+1}.\end{aligned}$$

We can use properties of limits before L'Hopital's rule is used. In the case of getting A we have

$$\begin{aligned}A &= \left(\lim_{z\to -i} z^3\right)\left(\lim_{z\to -i} \frac{z+i}{z^2+1}\right),\\ &= \left(z^3\big|_{z=-i}\right)\left(\left.\frac{1}{2z}\right|_{z=-i}\right)\\ &= \frac{(-i)^3}{2(-i)} = \frac{(-i)^2}{2} = -\frac{1}{2}.\end{aligned}$$

Is a partial fraction representation always possible?

Suppose $\deg(p(z)) < \deg(q(z))$ with

$$q(z) = (z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}$$

and let

$$R(z) = \frac{p(z)}{q(z)}.$$

Assuming a representation is possible, i.e.

$$\left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z-z_n} + \cdots + \frac{A_{r_n,n}}{(z-z_n)^{r_n}}\right)$$

we can get the coefficients as in the examples and see the next slides.

Getting the residue and the other coefficients

Re-label to concentrate on one of zeros of $q(z)$ at ζ and write

$$R(z) = \cdots + \frac{B_1}{z-\zeta} + \cdots + \frac{B_r}{(z-\zeta)^r} + \cdots$$

Then

$$\begin{aligned}(z-\zeta)^r R(z) &= B_r + B_{r-1}(z-\zeta) + \cdots + B_1 (z-\zeta)^{r-1} \\ &\qquad + (z-\zeta)^r (\text{a function analytic at } \zeta).\end{aligned}$$

To get B_j we have

$$(r-j)! B_j = \lim_{z\to\zeta} \frac{\mathrm{d}^{r-j}}{\mathrm{d}z^{r-j}} \left((z-\zeta)^r R(z)\right) \quad j = 1, 2, \ldots, r.$$

General case ...comments on the validity

Let

$$R(z) = \frac{p(z)}{(z-z_1)^{r_1}(z-z_2)^{r_2}\cdots(z-z_n)^{r_n}}$$

and let

$$\tilde{R}(z) = \left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z-z_n} + \cdots + \frac{A_{r_n,n}}{(z-z_n)^{r_n}}\right)$$

with the coefficients as given above and let

$$g(z) = R(z) - \tilde{R}(z).$$

g is a rational function and $z_1, \ldots, z_n$ are the only possible points where it might have poles.

General case ...comments on the validity continued

Without giving too many details the following are the main steps to show that $g(z) = R(z) - \tilde{R}(z) = 0$.

1. In case of z_1 consider

$$\begin{aligned}&R(z) - \left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) \\ &= \frac{\phi_1(z)}{(z-z_1)^{r_1} Q(z)}, \quad Q(z) = (z-z_2)^{r_2}\cdots(z-z_n)^{r_n},\end{aligned}$$

   where $\phi_1(z)$ is a polynomial. The detail is in showing that $\phi_1(z)$ has a zero at z_1 of multiplicity of at least r_1 which implies that there is no pole singularity at z_1.
   A similar argument applies to all the points $z_1, \ldots, z_n$.

General case ...comments on the validity ..last step

$$g(z) = R(z) - \tilde{R}(z).$$

2. As a consequence of the previous step the rational function $g(z)$ has no poles and hence it is a polynomial.
   As we have $R(z) \to 0$ and $\tilde{R}(z) \to 0$ as $|z| \to \infty$ we have $g(z) \to 0$ as $|z| \to \infty$. This implies that $g(z) = 0$ as non-constant polynomials are unbounded in the complex plane.

By considering the details in the complex case justifies the rules you are likely to have used earlier when constructing partial fraction representations.