Organisation and assessment

`MA3614` Complex variable methods and applications

Email: `Mike.Warby@brunel.ac.uk`

Handouts:
`http://people.brunel.ac.uk/~icstmkw/ma3614/`

Teaching times in term 1: Mon 15–16 and Tue 15–17.

From about week 2 one of the hours will be exercises.

In the week before the class test all sessions will be revision sessions.

Assessment:

Class test planned for the winter exam period (20%).

3 hour exam in April/May 2024 (80%).

Outline of some of the topics

Chap 3: Complex differentiable, analytic functions, Cauchy Riemann equations ...

Complex numbers.

$$z = x + iy = re^{i\theta}, \qquad x, y, r, \theta \in \mathbb{R}, \quad r \ge 0.$$

Functions of a complex variable:

$$w = f(z) = u(x, y) + iv(x, y), \quad u, v \in \mathbb{R}.$$

This reduces to the real valued case when $v(x, 0) = 0$.

When f is analytic we have the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This chapter will probably start in about week 3.

Outline of some of the topics continued

Chap 4: Elementary functions of a complex variable.

Polynomials, rational functions, $\exp(z)$, $\log(z)$, $\mathrm{Log}(z)$, z^α, $\sin(z)$, $\cos(z)$, etc. ...

A rational function can be re-expressed using partial fractions, e.g.

$$\frac{z^3}{z^2+1} = z - \frac{z}{z^2+1} = z - \frac{1}{2}\left(\frac{1}{z+i} + \frac{1}{z-i}\right).$$

This function has simple poles at $z = \pm i$.

$$\exp(x + iy) = e^{x+iy} := e^x(\cos y + i \sin y) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

This is periodic with period $2\pi i$. See chap 7 for the series part.

$\mathrm{Log}(z)$ is the principal valued logarithm.

With these we can give a meaning to i^i.

This chapter will probably start in about week 6.

Outline of some of the topics continued

Chap 5: Integration in the complex plane.

Consider a curve Γ in the complex plane described by

$$\Gamma = \{z(t) : \ a \le t \le b\}.$$

The contour integral is given by

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

If f has an anti-derivative F (i.e. $f = F'$) then this can be evaluated easily.

In many cases we consider curves which are closed loops and functions which are analytic except for known singular points.

This chapter will probably start in about week 9.

Outline of some of the topics continued

Chap 6 (in term 2): Cauchy's integral formula and consequences

When f is analytic inside a closed curve Γ, which we traverse once in an anti-clockwise direction, and z is inside Γ we have

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

This is a key result from which we deduce that one derivative existing actually implies that all derivatives exist with

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

The fundamental theorem of algebra, which is about polynomials having roots, can also be proved without too many steps from this result.

Outline of some of the topics continued

Chap 7: Taylor series and Laurent series

If f is analytic in a disk with centre z_0 then it has a Taylor series in the disk of the form.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If f is analytic in an annulus

$$\{z : \ r < |z - z_0| < R\}$$

then it has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

How do we determine r and R? What determines r and R?

Outline of some of the topics continued

Chap 8: Residue theory

When

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is valid in $\{z : \ 0 < |z - z_0| < R\}$ the coefficient a_{-1} is the residue at z_0. Determining the residues of a function inside a closed curve will be one of the steps in computing integrals.

Examples:

1. $$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, \mathrm{d}\theta$$

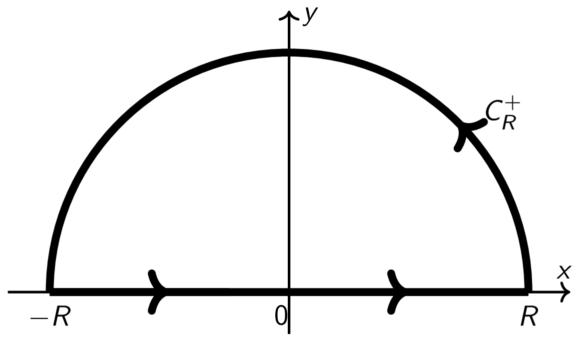
   With the substitution $z = \mathrm{e}^{i\theta}$ we get a problem involving integration around the unit circle. As an example we can show that

   $$I = \int_{-\pi}^{\pi} \frac{4\mathrm{d}\theta}{5 + 2\cos\theta} = \frac{8\pi}{\sqrt{21}}.$$

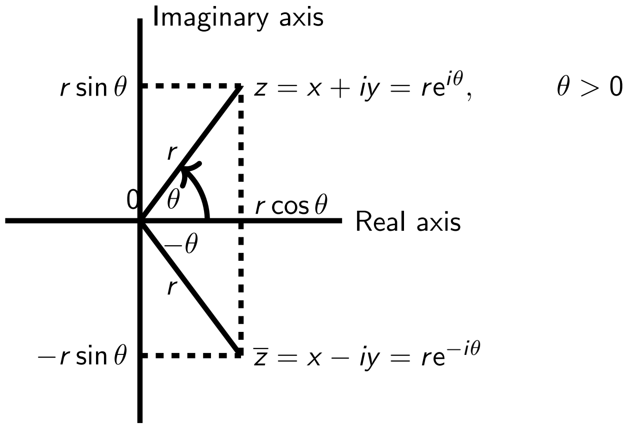
2. We can show that

   $$I = \int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2}\, \mathrm{d}x = \frac{\pi}{e}.$$

   This will involve considering a loop involving a half circle which will help to get the integral along the real line.



Week 1 key points: representations of z and $\overline{z}$



$$e^{i\theta} = \cos\theta + i\sin\theta$$

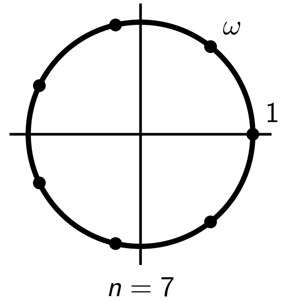
$\operatorname{Arg} z \in (-\pi, \pi]$ =principal argument. (arg z is multi-valued.)
Note $|z|^2 = z\,\overline{z} = x^2 + y^2$. $|z|$ =absolute value (i.e. modulus or magnitude).

Multiplication, powers, roots of unity

Suppose $z = re^{i\theta}$, $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$.
Multiplication: $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
Powers: $z^n = r^n e^{in\theta}$, $n = 0, \pm 1, \pm 2, \ldots$.
Observe $e^{2\pi i} = \exp(2\pi i) = \cos(2\pi i) + i\sin(2\pi i) = 1$.
Roots of unity: Let $\omega = \exp(2\pi i/n)$.

$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$

all satisfy $z^n - 1 = 0$ and are uniformly spaced on the unit circle.



Triangle inequality in $\mathbb{C}$

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Convergence of a sequence in $\mathbb{C}$

A sequence $z_0, z_1, z_2, \ldots$ converges to z if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$|z_n - z| < \epsilon \quad \text{for all } n \ge N.$$

In all these cases $|.|$ now means the absolute value of a complex number.

Power series

1. A power series

   $$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_k(z-a)^k + \cdots$$

   has a radius of convergence R This will be done in term 2. The ratio test and/or root test can often determine R.
2. Functions such as

   $$\frac{1}{1-z}, \quad \frac{1}{1+z^2}$$

   have pole singularities on the unit circle and power series representations inside the unit circle.
3. A function $f(z)$ has a power series representation in a neighbourhood of a point if the function is analytic at the point. Being analytic is a stronger requirement than being infinitely differentiable in the real sense.

We start to consider analytic functions in about 2 weeks.

When did people start using complex numbers?

The quadratic equations formula

The quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

can be solved by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This has been known for a long time and it was also recognised that in some cases when $a, b, c \in \mathbb{R}$ we have $b^2 - 4ac < 0$ but such cases did not involve real solutions.

Methods to solve cubics are usually considered as the starting point of the interest in complex numbers.

Cardano's method for solving cubic equations

With a change of variable the general cubic case can be changed to

$$x^3 + cx + d = 0.$$

Cardano had a method which involves letting

$$x = u + \frac{p}{u}, \quad p = -c/3.$$

$$\begin{aligned} x^3 = \left(u + \frac{p}{u}\right)^3 &= u^3 + 3pu + 3\frac{p^2}{u} + \frac{p^3}{u^3}, \\ cx = c\left(u + \frac{p}{u}\right) &= cu + \frac{cp}{u}, \\ x^3 + cx &= u^3 + \frac{p^3}{u^3}, \quad \text{when } 3p + c = 0. \end{aligned}$$

Thus

$$x^3 + cx + d = \frac{1}{u^3}\left(u^6 + du^3 + p^3\right), \quad \text{which involves a quadratic in } u^3.$$

Cardano's method continued

To find the roots use the quadratic formula to give

$$u^3 = \frac{-d \pm \sqrt{d^2 - 4p^3}}{2}.$$

From each u^3 we get 3 values of u and then for each u we have $x = u + p/u$ as a root. Only at most 3 distinct values are generated.

The method always works but often $d^2 - 4p^3 < 0$ when all the roots are real, i.e. to get all the real roots often requires computation involving i.

Example.

$$x^3 - 15x - 4 = (x-4)(x^2+4x+1) = (x-4)(x-(-2-\sqrt{3}))(x-(-2+\sqrt{3})).$$

Here $c = -15$, $p = 5$ and $d = -4$.

$$u^6 - 4u^3 + 5^3 = 0 \quad \text{and } d^2 - 4p^3 = 16 - 4 \times 5^3 = -4 \times 11^2.$$

$$u^3 = 2 \pm 11i.$$

Cardano's method continued

One solution to $u^3 = 2 + 11i$ is $u = 2 + i$.

To verify

$$\begin{aligned} (2+i)^2 &= (2+i)(2+i) = 3 + 4i, \\ (2+i)^3 &= (2+i)(2+i)^2 = (2+i)(3+4i) = 2 + 11i. \end{aligned}$$

Hence one solution is

$$x = u + \frac{p}{u} = (2+i) + \frac{5}{2+i} = (2+i) + (2-i) = 4.$$

Note the following triangles.