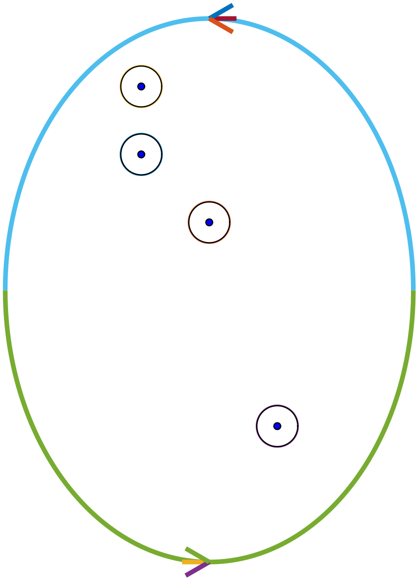
Several isolated singularities of $f(z)$ inside Γ



The Residue Theorem

If $z_1, z_2, \ldots, z_n$ are isolated singularities inside Γ and $C_1, C_2, \ldots, C_n$ are non-intersecting circles traversed once in the anti-clockwise direction then $\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)$ is the boundary of a region in which $f(z)$ is analytic and

$$\begin{aligned} \oint_\Gamma f(z)\,\mathrm{d}z &= \sum_{k=1}^{n} \oint_{C_k} f(z)\,\mathrm{d}z \\ &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k). \end{aligned}$$

With the knowledge of Laurent series to describe the behaviour of $f(z)$ in the vicinity of each point z_k we get the above result.

Earlier results with 0 or 1 isolated singularities

Week 13: **Cauchy-Goursat theorem**: If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

No singularities inside Γ.

Week 18: **The generalised Cauchy integral formula**:
If f is analytic in a simply connected domain D and Γ is any loop and z_0 is inside Γ then

$$\frac{f^{(m)}(z_0)}{m!} = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{m+1}}\,\mathrm{d}z, \quad m = 0, 1, 2, \ldots$$

1 singularity inside Γ.

The earlier results as a special case of the Residue Theorem

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k).$$

- When $f(z)$ is analytic inside Γ we have no isolated singularities inside Γ, i.e. $n = 0$.
- When $n = 1$ and we have a pole at z_1 of order m

$$\mathrm{Res}(g,\, z_1) = \frac{f^{(m)}(z_1)}{m!}, \quad \text{when } g(z) = \frac{f(z)}{(z - z_1)^{(m+1)}}.$$

The earlier results were of course needed to establish the residue theorem result.

Techniques to calculate the residue

In the case of a **simple pole** of $f(z)$ at z_0 most examples for calculating the residue have involved calculating the limit

$$\mathrm{Res}(f,\, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

In many of the examples L'Hopital's rule has been used.

More generally when we have a **pole of order** $m \ge 1$ we can calculate the residue by using

$$\mathrm{Res}(f,\, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left((z - z_0)^m f(z) \right).$$

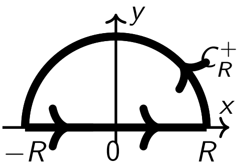
We need to know the order of the pole to use the above.

It is sometimes possible to simplify the expression for $(z - z_0)^m f(z)$ before differentiation is done.

Integrals on $(-\infty, \infty)$ evaluated using residue theory

With $P(z)$ and $Q(z)$ being polynomials we consider

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{(week 23)} \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)} e^{imz}. \quad \text{(week 24)}$$



Suppose that $f(z)$ has poles at points $z_1, \ldots, z_n$ in the upper half plane. Suppose that $Q(z)$ has no zeros on the real axis.

With $\Gamma_R = [-R, R] \cup C_R^+$ denoting the closed contour

$$\oint_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k).$$

When the integral involving C_R^+ tends to 0 as $R \to \infty$ we get

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{or} \quad \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Examples in the lectures

In week 23.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi.$$

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4 + 16}\, dx = \frac{\pi\sqrt{2}}{16}.$$

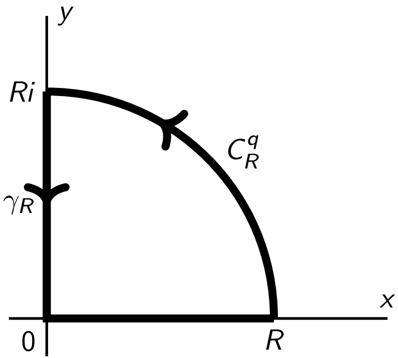
In week 24 (this week). The first integral is on the exercise sheet. Let $a > 0$.

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1 + x^2}\, dx = \pi e^{-a}.$$

$$\int_{-\infty}^{\infty} \frac{x \sin(x)}{1 + x^2}\, dx = \pi e^{-1}.$$

The last example will need Jordan's lemma to justify that the contribution from C_R^+ tends to 0 as $R \to \infty$.

Other loops in the exercises



$$f(z) = \frac{1}{z^4 + 16}$$

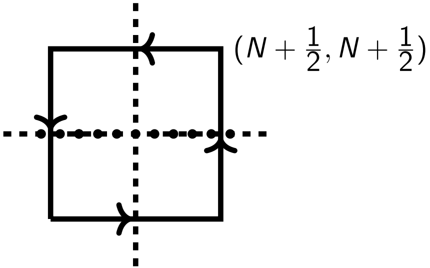
has one simple pole at $z_0 = 2e^{\pi i/4}$ inside this loop when $R > 2$. With an upper half circle instead as the loop we have 2 simple poles inside the loop at z_0 and $2e^{3\pi i/4}$ as in the slide 7.

A square as a loop in the exercises

In the context of the sum of a series

$$\sum_{n=1}^{N} f(n), \quad f(z) \text{ being even,}$$

the following loop Γ_N, which is a square, is used.



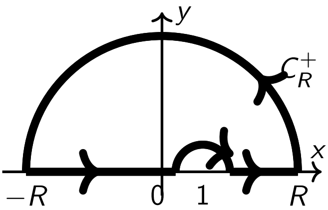
This has length $L_N = 4(2N+1)$. $M_N = \max\{|f(z)| : z \in \Gamma_N\}$.
We need $M_N L_N \to 0$ as $N \to \infty$.

Singularities on $\mathbb{R}$ and Cauchy principal values

In the lectures and in the exercises of this week and next week we will also consider integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

when $f(x)$ has poles on the real axis. The integrals need to be considered in a principal valued sense. In the case of a singularity at 1 the indented contour is illustrated below.



The knowledge of the Laurent series enables us to determine the contribution from the smaller half circle.

A sufficient condition for the C_R^+ part to tend to 0

In week 23 we proved the following.

Suppose that $f(z)$ is a rational function of the form

$$f(z) = \frac{P(z)}{Q(z)},$$

with

$$\begin{aligned} P(z) &= a_p z^p + \cdots + a_1 z + a_0, \\ Q(z) &= b_q z^q + \cdots + b_1 z + b_0 \end{aligned}$$

where $a_p \neq 0$, $b_q \neq 0$. When $|z| = R$ is large

$$|f(z)| = \mathcal{O}\left(R^{p-q}\right) = \mathcal{O}\left(\frac{1}{R^{q-p}}\right).$$

$RM_R \to 0$ as $R \to \infty$ when $q - p \geq 2$, i.e. $q \geq p + 2$.

The integrals on C_R^+ when we have a e^{imz} term

With $z = x + iy$, $imz = -my + imx$, $e^{imz} = e^{-my}e^{imx}$. When $m > 0$, $|e^{imz}| = e^{-my} \le 1$ when $y \ge 0$.

When $\deg(Q) \ge \deg(P) + 2$ we have

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\,dz \to 0 \quad \text{and} \quad \int_{C_R^+} \frac{P(z)}{Q(z)} e^{imz}\,dz \to 0$$

as $R \to \infty$ by using the ML inequality.

When $\deg(Q) = \deg(P) + 1$ Jordan's lemma also gives

$$\int_{C_R^+} \frac{P(z)}{Q(z)} e^{imz}\,dz \to 0$$

as $R \to \infty$.

Jordan lemma comments

When $\deg(Q) = \deg(P) + 1$ there is a constant $A \geq 0$ such that for part of the integrand

$$\left| \frac{P(Re^{i\theta}) i R e^{i\theta}}{Q(Re^{i\theta})} \right| \leq A, \quad \text{for sufficiently large } R.$$

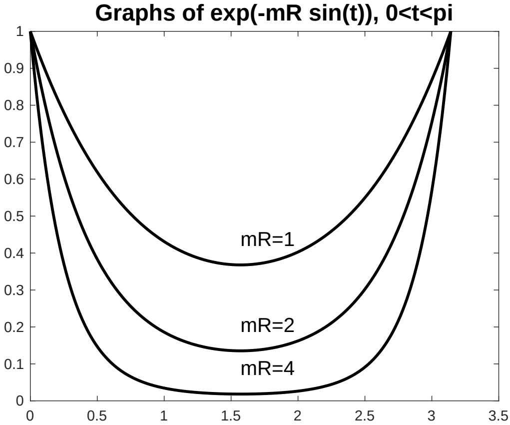
Much of the detail is showing that for the other part to be considered

$$\int_0^{\pi} \exp(-mR\sin\theta)\,\mathrm{d}\theta \to 0 \quad \text{as } R \to \infty.$$

Firstly, $\sin(\theta) = \sin(\pi - \theta)$ and

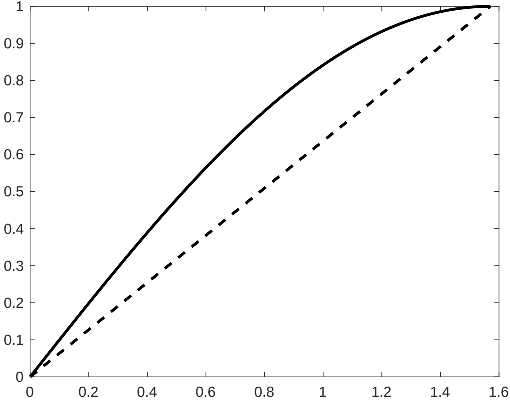
$$\int_0^{\pi} \exp(-mR\sin\theta)\,\mathrm{d}\theta = 2\int_0^{\pi/2} \exp(-mR\sin\theta)\,\mathrm{d}\theta.$$

Graphs of $\exp(-mR\sin(\theta))$, $mR = 1$, 2 **and** 4



The value is 1 at $\theta = 0$ and $\theta = \pi$ but small in the middle part.

A lower bound for $\sin(\theta)$ on $[0, \pi/2]$



$\sin(\theta)$ is above the linear interpolant using $x = 0$, $x = \pi/2$.

$$\sin(\theta) \geq \frac{2}{\pi}\theta.$$

Jordan's lemma, completing the detail

$$\sin(\theta) \geq \frac{2}{\pi}\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$$exp(-R\sin(\theta)) \leq exp(-k\theta), \quad \text{with } k = \frac{2R}{\pi}.$$

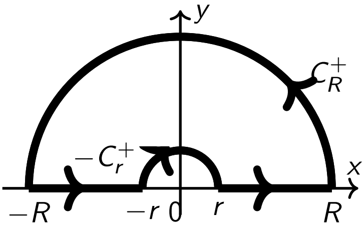
$$\begin{aligned}\int_0^{\pi/2} \exp(-R\sin\theta)\,\mathrm{d}\theta &\leq \int_0^{\pi/2} \exp(-k\theta)\,\mathrm{d}\theta \\ &\leq \int_0^{\infty} \exp(-k\theta)\,\mathrm{d}\theta = \frac{1}{k} \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

Singularities on $\mathbb{R}$ and Cauchy principal values

Suppose $f(z)$ has a simple pole on $\mathbb{R}$ and we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

The integrals need to be considered in a principal valued sense. In the case of a pole at $z = 0$ we need an indented contour as illustrated below.



The knowledge of the Laurent series enables us to determine the contribution from the smaller half circle.

The principal value for a singularity on $\mathbb{R}$

When we have a singularity of $f(z)$ at $x_0 \in \mathbb{R}$ the principal value means

$$\text{p.v.} \int_{-R}^{R} f(x) \mathrm{d}x = \lim_{r \to 0} \left(\int_{-R}^{x_0 - r} f(x) \mathrm{d}x + \int_{x_0 + r}^{R} f(x) \mathrm{d}x \right)$$

In the above the part of the real line can be described as $[-R, R] \setminus (x_0 - r, x_0 + r)$. The part of $[-R, R]$ that we are excluding has x_0 exactly in the middle.

The C_r^+ contribution as $r \to 0$

When $f(z)$ has a simple pole at 0 it has a Laurent series of the following form for z sufficiently close to 0.

$$f(z) = \frac{a_{-1}}{z} + g(z) \quad \text{where } g(z) = \text{analytic function.}$$

$$\int_{C_r^+} f(z)\,\mathrm{d}z = a_{-1} \int_{C_r^+} \frac{\mathrm{d}z}{z} + \int_{C_r^+} g(z)\,\mathrm{d}z.$$

$z(\theta) = re^{i\theta}$, $0 \le \theta \le \pi$ describes C_r^+ and the length of C_r^+ is πr.

$$\int_{C_r^+} \frac{\mathrm{d}z}{z} = \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}}\,\mathrm{d}\theta = i \int_0^\pi \mathrm{d}\theta = \pi i.$$

As a function $g(z)$ analytic on and near C_r^+ it is bounded there exists K such that $|g(z)| \le K$ in the region. ($K = 2|g(0)|$ will do if $g(0) \ne 0$ when r is sufficiently small.) Using the ML inequality we have

$$\left| \int_{C_r^+} g(z)\,\mathrm{d}z \right| \le K\pi r \to 0 \quad \text{as } r \to 0. \quad \lim_{r \to 0} \int_{C_r^+} f(z)\,\mathrm{d}z = \pi i \mathrm{Res}(f, 0).$$

Examples which use indented contours

We show the following.

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x = \pi, \quad I_2 = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x = \pi.$$

We do these by using an indented contour and the following expressions.

$$I_1 = \mathrm{Im}\left\{ \mathrm{p.v} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \right\}.$$

$$I_2 = \mathrm{Re}\left\{ \mathrm{p.v} \int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{2ix}}{2x^2}\,\mathrm{d}x \right\}.$$

I_1 and I_2 exist in the usual sense, it is just intermediate quantities which need the principal value meaning.

Term 1 exercises involving p_n'/p_n, q'/q

Let $z_1, z_2, \ldots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Prove by induction on n that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

Let

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2}\cdots(z - z_n)^{r_n}$$

where $z_1, \ldots, z_n$ are distinct points. What can you say about the multiplicity of the zeros of $q'(z)$ at the points $z_1, \ldots, z_n$? Using a derivation based on partial fractions show that

$$\frac{q'(z)}{q(z)} = \frac{r_1}{z - z_1} + \frac{r_2}{z - z_2} + \cdots + \frac{r_n}{z - z_n}.$$

Note that the rational functions p_n'/p_n and q'/q have simple poles and the residues are positive integers. We generalise this next.

Counting zeros and poles

Suppose that $f(z)$ is analytic in a domain except for a finite number of poles. Let

$$G(z) = \frac{f'(z)}{f(z)}.$$

Let z_0 be a zero of $f(z)$ of multiplicity m and let z_p be a pole of $f(z)$ of order n. It can quickly be shown that

$$\text{Res}(G, z_0) = m, \quad \text{and} \quad \text{Res}(G, z_p) = -n.$$

Let $f(z)$ be analytic inside a simple loop Γ and let $N_0(f)$ be the number of zeros of $f(z)$ inside Γ. By the residue theorem

$$N_0(f) = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

If $g(z)$ is also analytic inside C and $|g(z)| < |f(z)|$ on Γ then

$$N_0(f+g) = N_0(f).$$

This is Rouche's theorem. A smaller enough change to $f(z)$ on Γ does not change the integer.

The fundamental theorem of algebra

Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0$$

denote a polynomial of degree n. Let

$$f(z) = a_n z^n, \quad g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

For R sufficiently large $|f(z)| > |g(z)|$ on the circle $|z| = R$. As $f(z)$ has a zero at $z = 0$ of multiplicity n the use of Rouche's theorem implies that $p(z) = f(z) + g(z)$ also has n zeros inside $|z| = R$. This is the fundamental theorem of algebra and the proof here is independent of the proof given in chapter 6.

Another example using Rouche's theorem

Let

$$\begin{aligned} h(z) &= z^5 + 3z^3 - 1 = z^5 \left(1 + \frac{3}{z^2} - \frac{1}{z^5}\right) \\ &= z^5 \tilde{h}(w), \quad \tilde{h}(w) = 1 + 3w^2 - w^5, \quad w = \frac{1}{z}. \end{aligned}$$

$$h(z) = f(z) + g(z), \quad \text{with} \quad f(z) = z^5, \quad g(z) = 3z^3 - 1.$$

On the circle $|z| = 2$ we have
$|g(z)| \le 3(8) + 1 = 25 < 32 = |f(z)|$. $f(z)$ has one zero of multiplicity 5 at 0. Thus by Rouche's theorem $h(z)$ has 5 zeros inside the circle $|z| = 2$.

Similary by considering $\tilde{h}(w)$ with $\tilde{f}(w) = -w^5$, $\tilde{g}(w) = 1 + w^2$ and the circle $|w| = 2$ we get all the roots of $\tilde{h}(w)$ satisfy $|w| < 2$.

Conclusion: All the roots of $f(z)$ satisfy $1/2 < |z| < 2$.