Laurent series representation

Let $f(z)$ be analytic in an annulus $r < |z - z_0| < R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}.$$

The series converge uniformly in any closed sub-annulus $r < \rho_1 \le |z - z_0| \le \rho_2 < R$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z,$$

where C is any positively orientated simple closed curve lying in the annulus which has z_0 as an interior point.

This indicates that the representation is unique.

Also note that in none of the examples did we obtain a_n by evaluating this integral as we had other ways to get them.

Isolated zeros of non-zero analytic functions

When $f(z)$ has a **zero of multiplicity** $m \geq 1$ **at** z_0 we have

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots = (z - z_0)^m g(z)$$

with $g(z)$ being analytic at z_0 and $g(z_0) = a_m \neq 0$. These properties of $g(z)$ imply that in a neighbourhood $\{z : |z - z_0| < \delta\}$, for some $\delta > 0$, $g(z)$ is non-zero and thus $f(z)$ is non-zero. The zeros of $f(z)$ are isolated.

As an example suppose that the Cauchy Riemann equations are used to show that the following is analytic.

$$f(x+iy) = (-2x^2-10xy+6x+2y^2+15y)+i(5x^2-4xy-15x-5y^2+6y).$$

$$f(x) = (-2x^2 + 6x) + i(5x^2 - 15x).$$

$$g(z) = (-2z^2 + 6z) + i(5z^2 - 15z).$$

$f(x + iy)$ and $g(z)$ are both analytic with $f(z) - g(z) = 0$ on the real line. Hence $f(z) = g(z)$ for all z.

Complex identity and the related real relation

The isolated zeros property of non-zero analytic functions is a way to quickly explain why many identities are also true in the complex plane. For example,

$$
\begin{aligned}
\cos^2(x) + \sin^2(x) - 1 &= 0, \\
\sin(2x) - 2\sin(x)\cos(x) &= 0,
\end{aligned}
$$

being true for all $x \in \mathbb{R}$ also hold for all $z \in \mathbb{C}$, i.e.

$$
\begin{aligned}
\cos^2(z) + \sin^2(z) - 1 &= 0, \\
\sin(2z) - 2\sin(z)\cos(z) &= 0.
\end{aligned}
$$

Laurent series: Classifying poles

If $f(z)$ has **a removable singularity** at z_0 then it has a Laurent series with no negative powers valid in $0 < |z - z_0| < R$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \lim_{z \to z_0} f(z) = a_0.$$

Example: $\sin(z)/z$ has a removable singularity at $z = 0$.

If $f(z)$ has a **pole of order** m then in $0 < |z - z_0| < R$ we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \frac{\phi(z)}{(z - z_0)^m}$$

with $\phi(z)$ being analytic at z_0 and $\phi(z_0) = a_{-m} \neq 0$.

An **essential singularity at** z_0 has infinitely many negative powers

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

Example: $\exp(1/z)$ with $z_0 = 0$.

Integrating a Laurent Series

Let $f(z)$ be analytic in an annulus with the following Laurent series representation.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

The coefficient a_{-1} is called the residue at z_0. We write $\mathrm{Res}(f, z_0)$.

Let Γ denote a loop traversed once in the anti-clockwise sense with z_0 inside Γ. Then term-by-term integration gives

$$\oint_\Gamma f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint_\Gamma (z - z_0)^n\,\mathrm{d}z = 2\pi i\, a_{-1}.$$

Start of Chap 8 on Residue theory

We begin with a review of earlier results which involve the following.

- The definition of analytic at a point. (chap 3)
- Loop integrals in the following situations.
  - When we have an anti-derivative. (chap 5)
  - Cauchy's theorem. When $f(z)$ is analytic inside a loop. (chap 5)
  - The generalised Cauchy integral formula. (chap 6)
  - The use of partial fractions to express $1/Q(z)$, $Q(z)$ being a polynomial, to deal with $f(z)/Q(z)$. (chap 4)
- Taylor's series in a disk. (chap 7)
- Laurent series in an annulus. (chap 7)

Definition of an analytic function

Complex derivative: Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided the limit exists. Note that h is complex.

A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.

Key results before chap 6 about analytic functions

The Cauchy Riemann equations for $f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Cauchy-Goursat theorem: If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

Results about analytic functions in term 2

Generalised Cauchy integral formula

With the same conditions as above and with z_0 inside Γ

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z, \quad n = 0, 1, 2, \ldots$$

Using the Cauchy integral formula we get series representations.

Taylor series: If $f(z)$ is analytic in the disk $|z-z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.$$

Laurent series: If $f(z)$ is analytic in $0 \le r < |z-z_0| < R$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}, \quad a_m = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} \, \mathrm{d}z,$$

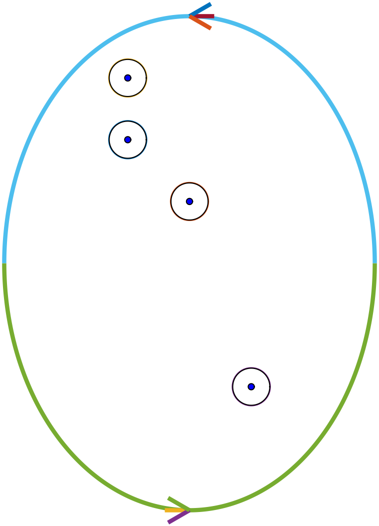
where C is simple closed loop in the annulus in the anti-clockwise sense. The series are unique once z_0 is specified.

A recap of some results about loop integrals

1. If f has an anti-derivative continuous on Γ then
$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$
f need not be analytic inside Γ, e.g. $f(z) = 1/z^2$.
2. If $f(z)$ is analytic on and inside Γ, i.e. we have no isolated singularities, then by Cauchy's theorem
$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$
3. If the integrand is of the form
$$f(z) = \frac{g(z)}{(z-z_0)^{m+1}},$$
where m is an integer and where $g(z)$ is analytic on and inside Γ, then by the generalised Cauchy integral formula
$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_\Gamma \frac{g(z)}{(z-z_0)^{m+1}}\,\mathrm{d}z = 2\pi i\frac{g^{(m)}(z_0)}{m!}$$

What if there are several isolated singularities?

Example: 4 isolated singularities of $f(z)$ inside Γ



From chap 4: The case of rational functions

Let

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}.$$

$$R(z) = \frac{p(z)}{q(z)} = \text{(some polynomial)} + \sum_{k=1}^{n} \frac{A_k}{z - z_k} + \text{(higher order poles)}.$$

Here A_k is the **residue** at z_k.

The polynomial part has an anti-derivative (another polynomial) and a $(z - z_k)^{-j-1}$ term has an anti-derivative $(z - z_k)^{-j}/(-j)$ when $j \geq 1$ and hence loop integrals of these part are 0.

$1/(z - z_k)$ has an anti-derivative throughout a loop when z_k is outside the loop and hence loop integrals of such terms are 0.

Residue theorem for rational functions

If $z_1, \ldots, z_m$ are points inside Γ at which $R(z)$ has poles then

$$\begin{aligned}
\oint_\Gamma R(z)\,\mathrm{d}z &= \sum_{k=1}^{m} A_k \oint_\Gamma \frac{\mathrm{d}z}{z - z_k} \\
&= 2\pi i \sum_{k=1}^{m} A_k \\
&= 2\pi i \sum_{k=1}^{m} \mathrm{Res}(R,\, z_k).
\end{aligned}$$

The answer just depends on the residues at the poles inside Γ.

A more general numerator

Suppose $Q(z)$ is a polynomial and $g(z)$ is analytic on and inside Γ.

$$f(z) = \frac{g(z)}{Q(z)}, \quad \text{with } Q(z) = (z - z_1)^{r_1} \cdots (z - z_n)^{r_n}.$$

By partial fractions we have the form

$$\frac{1}{Q(z)} = \sum_{k=1}^{n} \left(\frac{A_{1,k}}{z - z_k} + \cdots + \frac{A_{r_k,k}}{(z - z_k)^{r_k}} \right).$$

We can then separately determine

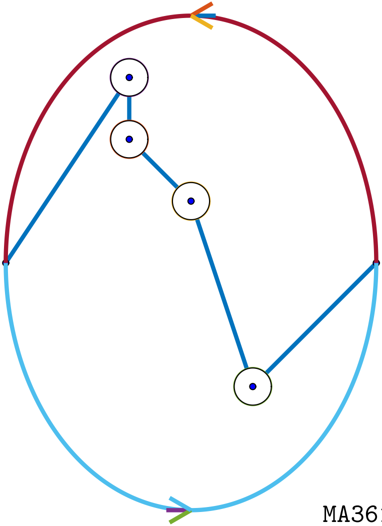
$$\oint_\Gamma \frac{g(z)}{(z - z_k)^{r_j}} \, \mathrm{d}z.$$

using the generalised Cauchy integral formula.

Dealing with a more general denominator

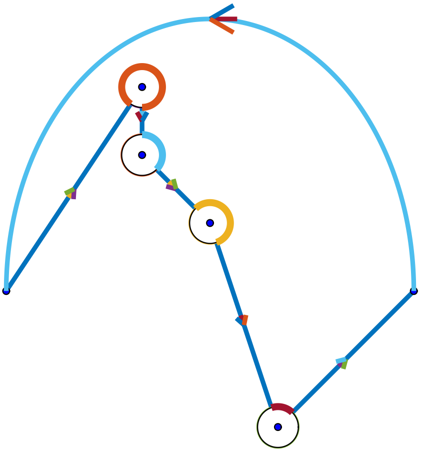
The following slides enable us to deal with any denominator which is analytic and has zeros.

Joining the points and dividing the domain



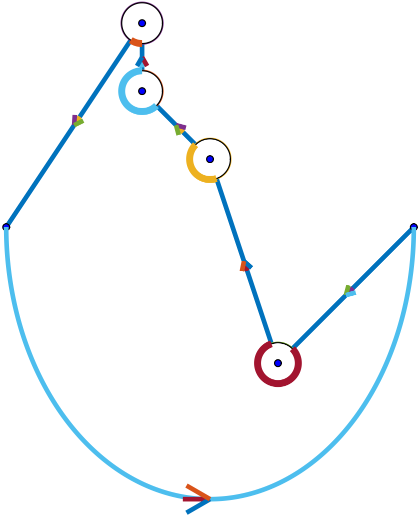
We have two simply connected domains, a top part and a bottom part. $f(z)$ is analytic in both simply connected domains.

The path Γ_t of the top part



$$\int_{\Gamma_t} f(z)\,\mathrm{d}z = 0.$$

The path Γ_b of the bottom part



$$\int_{\Gamma_b} f(z)\,\mathrm{d}z = 0.$$

The Residue theorem

If $z_1, z_2, \ldots, z_n$ are isolated singularities inside Γ and $C_1, C_2, \ldots, C_n$ are non-intersecting circles traversed once in the anti-clockwise direction then $\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)$ is the boundary of a region in which $f(z)$ is analytic and

$$\begin{aligned} \oint_\Gamma f(z)\,\mathrm{d}z &= \sum_{k=1}^{n} \oint_{C_k} f(z)\,\mathrm{d}z \\ &= 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f,\, z_k). \end{aligned}$$

Previously this was just shown to be true when we could split the integrand up using partial fractions for integrands which had a polynomial in the denominator. With the knowledge of Laurent series to describe the behaviour in the vicinity of each point we have now generalised to the above.

Techniques to calculate the residue

In the case of a **simple pole** of $f(z)$ at z_0 most examples for calculating the residue have involved calculating the limit

$$\operatorname{Res}(f,\, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

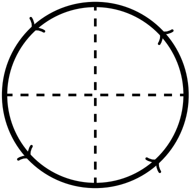
In many of the examples L'Hopital's rule has been used.

More generally when we have a **pole of order** $m \geq 1$ we can calculate the residue by using

$$\operatorname{Res}(f,\, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left((z - z_0)^m f(z) \right).$$

We need to know the order of the pole to use the above.

Trig integrals evaluated using residue theory



$$I = \int_0^{2\pi} R(\cos\,\theta, \sin\,\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z = 2\pi \sum_k \mathrm{Res}(F,\, z_k).$$

Here C is the unit circle and $F(z)$ is obtained by using

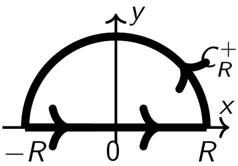
$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\,\theta = \frac{z + z^{-1}}{2}, \quad \sin\,\theta = \frac{z - z^{-1}}{2i}.$$

We determine I by the Residue theorem involving the residues of $F(z)$ at the poles z_k which are inside C. $F(z)$ is a rational function of z. Examples of these first appeared in chap 5.

Integrals on $(-\infty, \infty)$ evaluated using residue theory

With $P(z)$and $Q(z)$ being polynomials we consider

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{(weeks 23/24)} \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)} e^{miz}. \quad \text{(week 24)}$$



Suppose that $f(z)$ has poles at points $z_1, \ldots, z_n$ in the upper half plane. Suppose that $Q(z)$ has no zeros on the real axis.

With $\Gamma_R = [-R, R] \cup C_R^+$ denoting the closed contour

$$\oint_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f,\, z_k).$$

When the integral involving C_R^+ tends to 0 as $R \to \infty$ we get

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{or} \quad \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Want does an infinite integral mean?

Let $a \in \mathbb{R}$ we define

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$$

$$\int_{-\infty}^a f(x)\,\mathrm{d}x := \lim_{c\to-\infty}\int_c^a f(x)\,\mathrm{d}x.$$

When both limits exist

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x.$$

The principal value version only needs that the following exists.

$$\text{p.v.}\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^R f(x)\,\mathrm{d}x.$$

When our workings give the principal value a comment is made to justify when we do not put the p.v. notation as the integral exists in the "other sense" as well.

Example: When only the principal value exists

Consider

$$f(x) = \frac{i}{x+i}, \quad \int_{-R}^{0} f(x)\,\mathrm{d}x, \quad \int_{0}^{R} f(x)\,\mathrm{d}x, \quad \int_{-R}^{R} f(x)\,\mathrm{d}x.$$

$$f(x) = \frac{i}{x+i} = \frac{i(x-i)}{x^2+1} = \frac{1+ix}{x^2+1} \quad \text{when } x \in \mathbb{R}.$$

Let

$$F(z) = i\mathrm{Log}(z+i), \quad F'(z) = f(z) = \frac{i}{z+i}.$$

$$\int_{-R}^{0} f(x)\,\mathrm{d}x = F(0) - F(-R), \quad \int_{0}^{R} f(x)\,\mathrm{d}x = F(R) - F(0)$$

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = F(R) - F(-R) = \mathrm{Arg}(-R+i) - \mathrm{Arg}(R+i) \to \pi$$

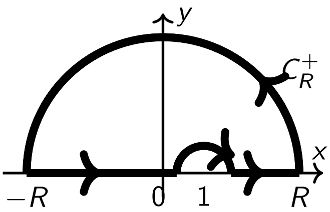
as $R \to \infty$. Both $|F(-R)|$ and $|F(R)|$ are unbounded as $R \to \infty$.

Singularities on $\mathbb{R}$ and Cauchy principal values

In the lectures and in the exercises of about weeks 24/25 we will also consider integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

when $f(x)$ has poles on the real axis. The integrals need to be considered in a principal valued sense. In the case of a singularity at 1 the indented contour is illustrated below.



The knowledge of the Laurent series enables us to determine the contribution from the smaller half circle as this shrinks to a point.

Examples this week

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + 2x + 2} = \pi.$$

Let

$$f(z) = \frac{1}{z^2 + 2z + 2}.$$

The magnitude on $|z| = R$ is of order $1/R^2$ when R is large. By the ML inequality the parts on C_R^+ tends to 0 as $R \to \infty$.

One simple pole at $z_1 = -1 + i$ in the upper half plane.

$$\mathrm{Res}(f, z_1) = \frac{1}{2i}.$$

Examples this week continued

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4 + 16}\, \mathrm{d}x = \frac{\pi\sqrt{2}}{16}.$$

Let

$$f(z) = \frac{1}{z^4 + 16}.$$

$f(z)$ has 4 simple poles in the complex plane and 2 of these are in the upper half plane at the points

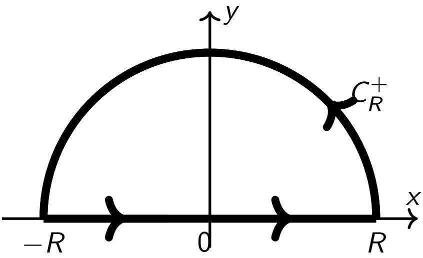
$$z_1 = 2\mathrm{e}^{i\pi/4} = \sqrt{2}(1+i), \quad z_2 = 2\mathrm{e}^{3i\pi/4} = \sqrt{2}(-1+i).$$

The magnitude on $|z| = R$ is of order $1/R^4$ when R is large. By the ML inequality the parts on C_R^+ tends to 0 as $R \to \infty$. In this case

$$I = 2\pi i \left(\mathrm{Res}(f,\, z_1) + \mathrm{Res}(f,\, z_2) \right).$$

Use of the *ML* inequality

Consider again the following contour.



The length of the half circle C_R^+ is πR. Suppose $M_R = \max\{|f(z)| : z \in C_R^+\}$. By the *ML* inequality.

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \pi R M_R.$$

This tends to 0 as $R \to \infty$ when $M_R \to 0$ sufficiently rapidly.

A sufficient condition for the C_R^+ part

Suppose that $f(z)$ is a rational function of the form

$$f(z) = \frac{P(z)}{Q(z)},$$

with $P(z) = a_p z^p + \cdots + a_1 z + a_0$ and $Q(z) = b_q z^q + \cdots + b_1 z + b_0$ where $a_p \neq 0$, $b_q \neq 0$. When $|z| = R$ is large

$$|f(z)| = \mathcal{O}\left(R^{p-q}\right) = \mathcal{O}\left(\frac{1}{R^{q-p}}\right).$$

$RM_R \to 0$ as $R \to \infty$ when $q - p \geq 2$, i.e. $q \geq p + 2$.

The integrals on C_R^+ when we have a e^{imz} term

After this week.

With $z = x + iy$, $imz = -my + imx$, $\mathrm{e}^{imz} = \mathrm{e}^{-my}\mathrm{e}^{imx}$. When $m > 0$, $|\mathrm{e}^{imz}| = \mathrm{e}^{-my} \le 1$ when $y \ge 0$.

When $\deg(Q) \ge \deg(P) + 2$ we have

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\,\mathrm{d}z \to 0 \quad \text{and} \quad \int_{C_R^+} \frac{P(z)}{Q(z)}\mathrm{e}^{imz}\,\mathrm{d}z \to 0$$

as $R \to \infty$ by using the ML inequality as in the case when $m = 0$.

When $\deg(Q) = \deg(P) + 1$ Jordan's lemma also gives

$$\int_{C_R^+} \frac{P(z)}{Q(z)}\mathrm{e}^{imz}\,\mathrm{d}z \to 0$$

as $R \to \infty$. Jordan's lemma should be covered next week.

If $m < 0$ then the lower half circle needs to be used for a similar result.