Taylor's series

If $f(z)$ is analytic at z_0 then the Taylor series is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

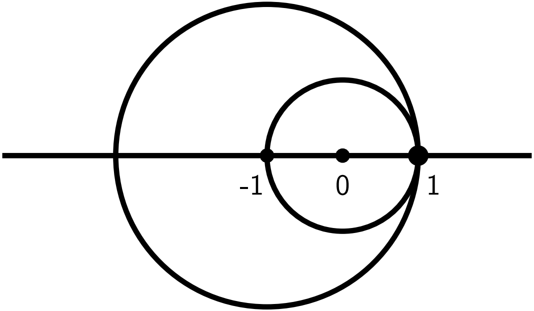
If $f(z)$ is analytic in $|z - z_0| < R$ then the series converges to $f(z)$ in this disk with uniform convergence in $|z - z_0| \le R' < R$ for all $R' < R$.

If $f(z)$ is not an entire function then the largest R is such that $f(z)$ has a non-analytic point on $|z - z_0| = R$.

Geometric series, examples of R

The following example was given at the start of lectures on chap 7 material.

$$f(z) = \frac{1}{1-z}.$$



The circles of convergence when we expand about $z_0 = -1$ has $R = 2$ and when we expand about $z_0 = 0$ has $R = 1$. The simple pole at $z = 1$ is on both circles.

Other examples of determining R

Consider the following function and expanding about $z_0 = 0$.

$$f(z) = \frac{1}{(1+e^{2z})(z^2-2)}$$

The non-analytic points (simple poles) are where

$$e^{2z} = -1 \quad \text{and when} \quad z^2 = 2.$$

$$e^{2z} = -1 \quad \text{when} \quad 2z = \mathrm{Log}(-1) = i\pi + 2k\pi i, \quad z = \frac{i\pi}{2} + k\pi i.$$

In the above $k \in \mathbb{Z}$.

The points at $\pm\sqrt{2}$ are nearer to $z_0 = 0$ than the points $\pm i\pi/2$ and thus $R = \sqrt{2}$.

A branch point case: $(1+z)^\alpha$, $z_0 = 0$, example of R

$$f(z) = (1+z)^\alpha$$

where the principal value is being used.

Apart from the cases where $\alpha \in \{0, 1, 2, \cdots\}$ there is a non-analytic point at $z = -1$. The non-analytic point is a pole if α is a negative integer but otherwise it is a branch point.

$$R = 1.$$

With the principal value meaning the branch cut is the set

$$\{z = x : \, x \le -1\}$$

and $f(z)$ is analytic when $|z| < 1$. The generalised binomial series representation is

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n + \cdots$$

Real coefficients, even functions, odd functions, etc

If $f(z) = u(x, y) + iv(x, y)$ is real when z is real then

$$v(x, 0) = 0 \quad \text{and} \quad f^{(n)}(0) = \left.\frac{\partial^n u(x, 0)}{\partial x^n}\right|_{x=0} \quad \text{is } \textbf{real}.$$

If R =radius of convergence and $0 < r < R$ then we have

$$\begin{aligned} \frac{f^{(n)}(0)}{n!} &= \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} f(re^{it})\, e^{-int}\, dt \\ &= \frac{1}{2\pi r^n} \int_{0}^{\pi} \left(f(re^{it}) + (-1)^n f(-re^{it})\right) e^{-int}\, dt. \end{aligned}$$

If $f(-z) = f(z)$ then the Maclaurin series only has **even** powers.

If $f(-z) = -f(z)$ then the Maclaurin series only has **odd** powers.

Series you are expected to know

Geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots, \quad \text{valid for } |z| < 1.$$

The following are **entire** functions:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \\ e^{-z} &= 1 - z + \frac{z^2}{2!} + \cdots + \frac{(-z)^n}{n!} + \cdots \end{aligned}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Remember that

$$e^{iz} = \cos(z) + i\sin(z), \qquad e^z = \cosh(z) + \sinh(z).$$

Some techniques with series

Inside the circle of convergence we can differentiate term-by-term and we integrate term-by-term, e.g. we can get $\sin(z)$ from $\cos(z)$ and conversely we can get $\cos(z)$ from $\sin(z)$ as $\cos(0) = 1$.

$$\begin{aligned} \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \end{aligned}$$

With knowledge of one series you can hence quickly get the other series. As examples obtained from the geometric series

$$\begin{aligned} \mathrm{Log}(1-z) &= -\int_0^z \frac{\mathrm{d}t}{1-t} = -\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n} + \cdots\right), \\ \frac{1}{(1-z)^2} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1-z}\right) = 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots . \end{aligned}$$

Any path in the disk from 0 to z is okay in the integral.

The Koebe function, de Branges' theorem and a conjecture

From the previous slide we immediately get the series for the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots .$$

This function has the property that $f(0) = 0$, $f'(0) = 1$. Also we could give an expression for the inverse to confirm that it is one-to-one in $|z| < 1$.

Suppose that you consider all functions $g(z)$ which are analytic in the unit disk, are one-to-one and satisfy $g(0) = 0$ and $g'(0) = 1$. Such functions have Maclaurin series of the form

$$g(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

In 1985 de Branges proved that $|a_n| \le n$.

In 1916 Bierberbach had proved that $|a_2| \le 2$ and he conjectured that $|a_n| \le n$ for all functions with the above properties. See a Wolfram web page for a history of the progress to prove this result which took nearly 70 years.

Multiplying series – the Cauchy product

If $f(z)$ and $g(z)$ are both analytic in $|z - z_0| < R$ then $h(z) = f(z)g(z)$ is also analytic in $|z - z_0| < R$.

To shorten the expressions let $z_0 = 0$.

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots, \\ g(z) &= b_0 + b_1 z + b_2 z^2 + \cdots, \\ h(z) &= c_0 + c_1 z + c_2 z^2 + \cdots. \end{aligned}$$

The following expression for c_n is known as the **Cauchy product**.

$$\begin{aligned} c_0 &= a_0 b_0, \\ c_1 &= a_0 b_1 + a_1 b_0, \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0, \\ \cdots & \quad \cdots \\ c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0. \end{aligned}$$

Leibnitz's formula for the nth derivative of a product

If we repeatedly use the product rule then we get

$$\begin{aligned} h &= fg, \\ h' &= f'g + fg', \\ h'' &= f''g + 2f'g' + fg'', \\ \cdots & \quad \cdots \\ h^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}. \end{aligned}$$

The last result is known as Leibnitz's rule for the nth derivative of a product.

The validity of the Cauchy product formula for the coefficients in the series for $h(z)$ about z_0 follows by noting the following.

$$h^{(n)}(z_0) = n!c_n, \quad f^{(k)}(z_0) = k!a_k, \quad g^{(n-k)}(z_0) = (n-k)!b_{n-k},$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Examples using the Cauchy product technique

$$\begin{aligned}\frac{e^z}{1-z} &= \left(1 + z + \cdots + \frac{z^n}{n!} + \cdots\right)(1 + z + \cdots + z^n + \cdots) \\ &= c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots .\end{aligned}$$

$$\begin{aligned}c_0 &= 1, \\ c_1 &= 1 + 1 = 2, \\ c_2 &= 1 + 1 + \frac{1}{2} = \frac{5}{2}, \\ c_n &= 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!}.\end{aligned}$$

We can get the series for $\tan(z) = \sin(z)/\cos(z)$ by first writing

$$\tan(z)\cos(z) = \sin(z).$$

We use the known series for $\cos(z)$ and $\sin(z)$ to deduce the terms for $\tan(z)$.

The generalised L'Hopital's rule

If we have

$$g(z_0) = g'(z_0) = \cdots = g^{(m-1)}(z_0) = 0 \quad \text{and } g^{(m)}(z_0) \ne 0$$

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

then for z near z_0 we have

$$\begin{aligned} f(z) &= a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots , \\ g(z) &= b_m(z - z_0)^m + b_{m+1}(z - z_0)^{m+1} + \cdots . \end{aligned}$$

$$\frac{f(z)}{g(z)} \to \frac{a_m}{b_m} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \quad \text{as } z \to z_0.$$

If the multiplicity of the zero of $g(z)$ at z_0 is greater than the multiplicity of the zero of $f(z)$ then there is no limit and $f(z)/g(z)$ has a singularity at z_0.

Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The terms a_0, a_1, ... are the coefficients of the power series.

The series always converges at $z = z_0$. When it converges at other points the region where it converges is a disk $\{z : |z - z_0| < R\}$ and it is analytic in the disk.

The largest R is the **radius of convergence**. When $R < \infty$ $\{z : |z - z_0| = R\}$ is the **circle of convergence**. In all cases

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

In our examples R is obtained using the ratio test or the root test.

$R = 0$ when we only have convergence at $z = z_0$.

$R = \infty$ when we have convergence for all z.

Obtaining R in the exercise sheet examples

$$\sum_{n=0}^{\infty} b_n, \qquad b_n = a_n (z - z_0)^n.$$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z - z_0|, \quad |b_n|^{1/n} = |a_n|^{1/n} |z - z_0|.$$

By the ratio test, when

$$\left|\frac{a_{n+1}}{a_n}\right| \to \alpha \quad \text{as } n \to \infty, \quad R = \frac{1}{\alpha}.$$

By the root test, when

$$|a_n|^{1/n} \to \alpha \quad \text{as } n \to \infty, \quad R = \frac{1}{\alpha}.$$

The $\limsup$ version deals with the case when the sequence $\left(|a_n|^{1/n}\right)$ does not converge but is bounded.

$$\alpha = \lim_{n \to \infty} c_n, \quad c_n = \sup\{|a_m|^{1/m} : \ m \geq n\}.$$

Why must the region where it converges be a disk?

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Suppose this converges at $z_1 \neq z_0$ and let $r = |z_1 - z_0| > 0$. The series may not converge at all points on $|z - z_0| = r$ but the following argument proves that the series converges uniformly in the region

$$\{z : \ |z - z_0| \le \tilde{r} < r\}.$$

The Proof

Convergence of the power series at z_1 means that

$$|a_n(z_1 - z_0)^n| = |a_n| r^n \to 0 \quad \text{as } n \to \infty.$$

This implies that the set $\{|a_n| r^n : \ n = 0, 1, 2, \ldots\}$ is bounded and we have

$$M = \sup\{|a_n| r^n : \ n = 0, 1, 2, \ldots\} < \infty.$$

If we take $\tilde{r} < r$ and take z such that $|z - z_0| \le \tilde{r}$ then

$$|a_n(z - z_0)^n| \le |a_n| \tilde{r}^n = |a_n| r^n \left(\frac{\tilde{r}}{r}\right)^n \le M \left(\frac{\tilde{r}}{r}\right)^n.$$

The right hand side is a term in a convergent geometric series and thus by the Weierstrass M-test the series converges uniformly in the disk $\{z : \ |z - z_0| \le \tilde{r}\}$.