Overview of chapter 7 about series

- Sections 7.1 and 7.2 are an introduction and revision about sequences and series of numbers in $\mathbb{C}$.
- Section 7.3 is about the uniform convergence of a series of analytic functions. We show that the limit is also analytic.
- Section 7.4 is about proving that a given function $f(z)$ which is analytic in $\{z : |z - z_0| < R\}$ is equal to its Taylor series representation about z_0. When $f(z)$ is not an entire function the largest R is such that $f(z)$ is not analytic at a point on $|z - z_0| = R$. This is the circle of convergence and R is the radius of convergence.

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots . \end{aligned}$$

Overview of chapter 7 continued

- Section 7.4 will also include some standard series and some manipulations such the Cauchy product technique to get the series for a product of analytic functions $f(z)g(z)$.
- Section 7.5 is concerned with the opposite of section 7.4 in that the starting point is a function defined by a series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

  We determine the circle of convergence and the radius of convergence from the coefficients $\{a_n\}$ by using the ratio test or root test when possible.

- Section 7.6 is concerned with showing that when f is analytic in $0 \le r < |z - z_0| < R$ we have a Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Overview of chapter 7 continued

- Section 7.7 is concerned with classifying an isolated singularity at z_0 by considering its Laurent series in $0 < |z - z_0| < R$.

$$\mathrm{Res}(f, z_0) = a_{-1}.$$

A key step in deriving the Taylor and Laurent series representations is in starting with the Cauchy integral representation and doing some manipulations with the following part of the integrand.

$$\frac{1}{\zeta - z}.$$

$$\begin{aligned}
\zeta - z &= (\zeta - z_0) - (z - z_0) \\
&= (\zeta - z_0)\left(1 - \left(\frac{z - z_0}{\zeta - z_0}\right)\right) \\
&= -(z - z_0)\left(1 - \left(\frac{\zeta - z_0}{z - z_0}\right)\right),
\end{aligned}$$

$$(1 - c)^{-1} = \frac{1}{1 - c} = 1 + c + c^2 + c^3 + \cdots \quad \text{when } |c| < 1.$$

Chapter 7: Definitions: sequences in $\mathbb{C}$

- A sequence $z_0, z_1, z_2, \ldots$ **converges** to z if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$|z_n - z| < \epsilon \quad \text{for all } n \ge N.$$

- A sequence $z_0, z_1, z_2, \ldots$ is a **Cauchy sequence** if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$|z_n - z_m| < \epsilon \quad \text{for all } n \ge N \text{ and } m \ge N.$$

Result about convergence

A sequence in $\mathbb{C}$ converges if and only if it is a Cauchy sequence.

In this module we do not directly use these definitions but we use results derived from them.

Definitions: series in $\mathbb{C}$

- Let c_0, c_1, c_2, ... denote a sequence. A **series** is an expression of the form

$$c_0 + c_1 + c_2 + \cdots \quad \text{and we write as} \quad \sum_{k=0}^{\infty} c_k.$$

The sequence of partial sums are given by

$$s_n = \sum_{k=0}^{n} c_k, \quad n = 0, 1, 2, \ldots$$

- The series **converges** if the sequence of partial sums converges and it **diverges** otherwise. When the series convergence **the sum of the series** is

$$s = \sum_{k=0}^{\infty} c_k$$

- If $\sum |c_k|$ converges then $\sum c_k$ is **absolutely convergent**.

Results about series in $\mathbb{C}$

- If a series $\sum c_k$ converges then $c_n \to 0$ as $n \to \infty$.
- If the series $\sum |c_k|$ converges then $\sum c_k$ converges.
- **Comparison test:** If there exists K such that $|c_k| \le M_k$ for all $k \ge K$ and $\sum M_k$ converges then $\sum c_k$ converges.
- From the identity

$$(1-c)(1+c+c^2+\cdots+c^n) = 1-c^{n+1}$$

  we have that the **geometric series**

$$\sum_{k=0}^{\infty} c^k = \frac{1}{1-c}, \quad \text{when } |c| < 1.$$

- **Ratio test:** If $|c_{k+1}/c_k| \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.
- **Root test:** If $|c_k|^{1/k} \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.

Series of functions

Suppose that $f_0(z)$, $f_1(z)$, $\ldots$ are all defined on D and let

$$F_n(z) = \sum_{k=0}^{n} f_k(z), \quad n = 0, 1, 2, \ldots$$

$\sum f_k(z)$ **converges pointwise on** D if $(F_n(z))$ converges $\forall z \in D$.

The sequence **converges uniformly to** $F(z)$ **on D** if

$$\sup_{z \in D} |F_n(z) - F(z)| \to 0 \quad \text{as } n \to \infty.$$

A sufficient condition for a series to converges uniformly is the **Weierstrass M-test**: If $|f_k(z)| \le M_k$ for all $z \in D$ and $\sum M_k$ converges then the series converges uniformly in D.

Uniform convergence preserves continuity: If $F_n(z)$, $n = 0$, 1, $2, \ldots$ are continuous in D and $F_n \to F$ uniformly on D then the limit function $F(z)$ is also continuous in D.

Uniform convergence and analytic functions

Theorem: Let $F_n(z)$ be a sequence of analytic function in a simply connected domain D and converging uniformly to $F(z)$ in D. Then $F(z)$ is analytic in D.

The main step in the proof is the following.

The uniform convergence of the sequence of functions enables us to deduce that for any loop Γ in D we have

$$\oint_\Gamma F(z)\,\mathrm{d}z = \lim_{n\to\infty} \oint_\Gamma F_n(z)\,\mathrm{d}z = 0.$$

As this is true for all loops Morera's theorem tells us that $F(z)$ is analytic.

Taylor series for analytic functions

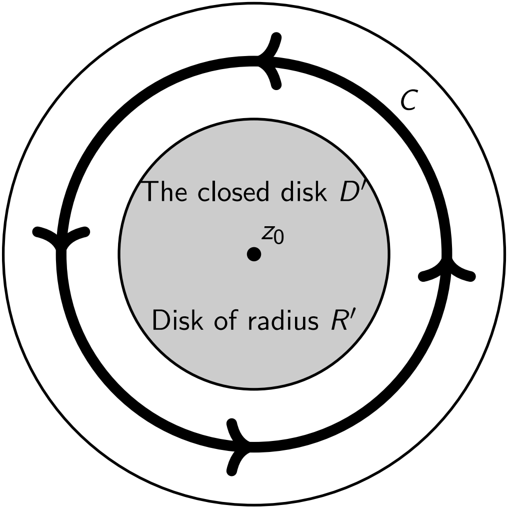
If $f(z)$ is analytic at z_0 then the series

$$f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

is called the **Taylor series** for $f(z)$ around z_0.

Theorem: If $f(z)$ is analytic in the disk $|z-z_0|<R$ then the Taylor series converges to $f(z)$ for all z in this disk and in any closed disk $|z-z_0|\le R'<R$ the convergence is uniform.

The circles used in the proof



Circle of radius R. With the largest R $f(z)$ is not analytic at 1 or more points on the circle.

z is in the shaded region. C is the circle in the loop integral, $\zeta \in C$. $f(z)$ is analytic inside the outer cirle.

Key formula in the proof of the Taylor series

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta.$$

$$\zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0) \left(1 - \left(\frac{z - z_0}{\zeta - z_0} \right) \right).$$

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} \left(1 + \alpha + \alpha^2 + \cdots + \alpha^n + \left(\frac{\alpha^{n+1}}{1 - \alpha} \right) \right), \quad \alpha = \frac{z - z_0}{\zeta - z_0}.$$

Note that with $|z - z_0| \le R'$ and $R' < |\zeta - z_0|$, $\zeta \in C$, we have

$$|\alpha| < 1.$$

Key formula in the proof continued

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0}\left(1 + \alpha + \alpha^2 + \cdots + \alpha^n + \left(\frac{\alpha^{n+1}}{1-\alpha}\right)\right), \quad \alpha = \frac{z - z_0}{\zeta - z_0}.$$

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\left(1 + \alpha + \alpha^2 + \cdots + \alpha^n\right)\, \mathrm{d}\zeta = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

Thus

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + T_n(z),$$

$$T_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)}\,\mathrm{d}\zeta.$$

It can be shown that $\max\{|T_n(z)| : |z - z_0| \le R'\} \to 0$ as $n \to \infty$.

Taylor's series, comments about R

If $f(z)$ is analytic at z_0 then the Taylor series is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

If $f(z)$ is analytic in $|z - z_0| < R$ then the series converges to $f(z)$ in this disk with uniform convergence in $|z - z_0| \le R' < R$ for all $R' < R$.

If $f(z)$ is not an entire function then the largest R is such that $f(z)$ has a non-analytic point on $|z - z_0| = R$.

Maclaurin series case

Maclaurin series is the case of Taylor series when $z_0 = 0$.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k.$$

If $f(z)$ is analytic in $|z| < R$ then the series converges to $f(z)$ in this disk with uniform convergence in $|z| \le R' < R$ for all $R' < R$.

As an example,

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

is analytic in $|z| < \pi/2$ but is not analytic at the points $\pm\pi/2$. In this case $R = \pi/2$.

Real coefficients, even functions, odd functions, etc

If $f(z) = u(x, y) + iv(x, y)$ is real when z is real then

$$v(x,0) = 0 \quad \text{and} \quad f^{(n)}(0) = \left. \frac{\partial^n u(x,0)}{\partial x^n} \right|_{x=0} \quad \text{is } \textbf{real}.$$

If $R =$radius of convergence and $0 < r < R$ then by considering the following integral on $[-\pi, 0]$ and $[0, \pi]$ involving the generalised CIF we have

$$\begin{aligned} \frac{f^{(n)}(0)}{n!} &= \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} f(re^{it})\, e^{-int}\, dt \\ &= \frac{1}{2\pi r^n} \int_{0}^{\pi} \left(f(re^{it}) + (-1)^n f(-re^{it}) \right) e^{-int}\, dt. \end{aligned}$$

If $f(-z) = f(z)$ then the Maclaurin series only has **even** powers.

If $f(-z) = -f(z)$ then the Maclaurin series only has **odd** powers.

Series you are expected to know

Geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots, \quad \text{valid for } |z| < 1.$$

The following are **entire** functions:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \\ e^{-z} &= 1 - z + \frac{z^2}{2!} + \cdots + \frac{(-z)^n}{n!} + \cdots \end{aligned}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Remember that

$$e^{iz} = \cos(z) + i\sin(z), \qquad e^z = \cosh(z) + \sinh(z).$$