From chap 3: Definition of an analytic function

$f(z)$ is **complex differentiable at** z_0 if

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, i.e. the limit is independent of how $h \to 0$.

A function f is **analytic** at z_0 if f is differentiable at all points in some neighbourhood of z_0.

The Cauchy Riemann equations for $f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

With certain conditions on u and v, f is analytic if and only if these are satisfied.

Both of these were first met in chapter 3.

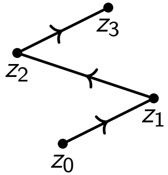
From chap 5: Smooth arcs and contours

A set $\gamma \subset \mathbb{C}$ is a smooth arc if the set can be described in the form

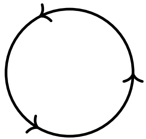
$$\{z(t): \ a \le t \le b\}, \quad z'(t) \ne 0 \text{ being continuous on } [a, b].$$

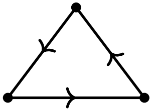
A contour is 1 point or a finite sequence of directed smooth arcs γ_k with the end of γ_k being the start of arc γ_{k+1}.

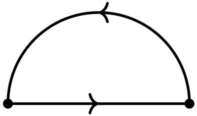
Examples of contours



Polygonal path with $n = 3$ arcs.


Circle, anti-clockwise.


Closed polygonal path with $n = 3$ arcs.


Closed path, $n = 2$ arcs.

Definitions of integrals along an arc

A very small change Δt in the parameter t gives a small change

$$\Delta z \approx \frac{\mathrm{d}z}{\mathrm{d}t}\Delta t.$$

The length of γ is

$$L = \int_a^b |z'(t)| \, \mathrm{d}t.$$

The contour integral of $f(z)$ is

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(z(t)) z'(t) \, \mathrm{d}t.$$

The ML inequality is

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| \le ML, \quad \text{where } M = \max_{z \in \gamma} |f(z)|.$$

Contour integrals in the complex plane

Definition of the contour integral;

$$\gamma = \{z(t) : \ a \le t \le b\}, \quad \int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

Independence of path when $f = F'$ (Note that F is analytic.)

If there exists an anti-derivative F along the path then

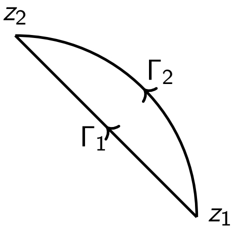
$$\frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t).$$

This is the integrand in the expression for the contour integral as given above and hence by the fundamental theorem of calculus

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(z(t))z'(t)\,\mathrm{d}t = F(z(b)) - F(z(a)).$$

This is also true for a contour which is a union of directed arcs.

Equivalent statements relating to path independence, loop integrals and anti-derivatives



$\Gamma_1 \cup (-\Gamma_2)$ is a closed loop.

The following are equivalent statements involving the integral of f.

(i) All loop integrals of f are 0.

(ii) The value of the integral of f only depends on the end points.

(iii) There exists an anti-derivative F, i.e. $F' = f$.

This was one of the last theory parts before the class test.

Cauchy's integral theorem

We now consider the following result and various ways it can be proved. You need to know the result but you will not be examined on any of the proofs discussed.

Recall that the following was mentioned in chapter 2.
Simply-connected: A domain which does not have holes.

Cauchy-Goursat theorem: If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

This is sometimes just known as Cauchy's integral theorem. Goursat's contribution was to show that the result could be proved assuming only that $f'(z)$ exists.

Contour integrals in $\mathbb{C}$ and line integrals in $\mathbb{R}^2$

Until 2019/0 year 2 students took a module in vector calculus and would have met line integrals and the following Green's formula which relates an area integral to an integral around the boundary.

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial S} P\mathrm{d}x + Q\mathrm{d}y. \qquad (*)$$

This is relevant to loop integrals in the complex plane as

$$\begin{aligned} \mathrm{d}z &= \mathrm{d}x + i\mathrm{d}y, \\ f(z)\mathrm{d}z &= (u+iv)(\mathrm{d}x + i\mathrm{d}y) = (u\mathrm{d}x - v\mathrm{d}y) + i(v\mathrm{d}x + u\mathrm{d}y). \end{aligned}$$

The real and imaginary parts here relate to the RHS of $(*)$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0.$$

The integrands in the area integrals are 0 by the Cauchy Riemann equations. This is the common proof given in text books of the Cauchy's integral theorem that I have seen.

Green's theorem to the multi-variable module

There are not too many steps from the area integrals in year 2 to be able to cover Green's theorem.

Firstly, for the jargon, the term line integral is used as with the vector dot product

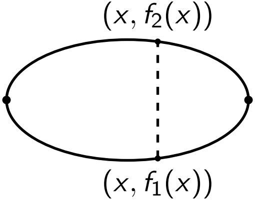
$$P\mathrm{d}x + Q\mathrm{d}y = \left(P\underline{i} + Q\underline{j}\right) \cdot \left(dx\underline{i} + dy\underline{j}\right).$$

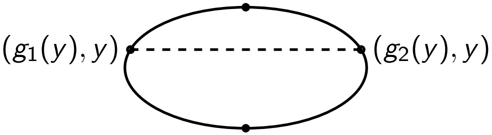
To show the result we can separately show

$$\iint_S \frac{\partial Q}{\partial x}\,\mathrm{d}x\mathrm{d}y = \oint_{\partial S} Q\,\mathrm{d}y, \quad \iint_S -\frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y = \oint_{\partial S} P\,\mathrm{d}x.$$

Describing the domain S and the boundary ∂S

We stick here to fairly simple geometries for which either x or y can be used as the parameter to describe the loop ∂S as the union of two parts. Consider the following two versions of ∂S.





$$\begin{aligned}
\text{Lower part of } \partial S \text{ anti-clockwise} &= \{(x, f_1(x)) :\ x_L \le x \le x_R\}, \\
\text{Upper part of } \partial S \text{ clockwise} &= \{(x, f_2(x)) :\ x_L \le x \le x_R\}, \\
\text{Left part of } \partial S \text{ clockwise} &= \{(g_1(y), y) :\ y_B \le y \le y_T\}, \\
\text{Right part of } \partial S \text{ anti-clockwise} &= \{(g_2(y), y) :\ y_B \le y \le y_T\}.
\end{aligned}$$

The line integrals

$$\begin{aligned}
\oint_{\partial S} P\,\mathrm{d}x &= \int_{x_L}^{x_R} P(x, f_1(x)) - P(x, f_2(x))\,\mathrm{d}x, \\
\oint_{\partial S} Q\,\mathrm{d}y &= \int_{y_B}^{y_T} Q(g_2(y), y) - Q(g_1(y), y)\,\mathrm{d}y.
\end{aligned}$$

The area integrals

$$\begin{aligned}
\iint_S -\frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y &= -\int_{x_L}^{x_R} \left(\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\,\mathrm{d}y \right) \mathrm{d}x, \\
&= \int_{x_L}^{x_R} P(x, f_1(x)) - P(x, f_2(x))\,\mathrm{d}x, \\
\iint_S \frac{\partial Q}{\partial x}\,\mathrm{d}x\mathrm{d}y &= \int_{y_B}^{Y_T} \left(\int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}\,\mathrm{d}x \right) \mathrm{d}y, \\
&= \int_{y_B}^{y_T} Q(g_2(y), y) - Q(g_1(y), y)\,\mathrm{d}y.
\end{aligned}$$

A proof involving deforming a contour to a point

Suppose for each s in $[0, 1]$ we have a loop and a value $I(s)$ as follows.

$$\Gamma_s = \{z(s,t) : \ 0 \le t \le 1\}, \quad \text{a loop for each fixed } s.$$

$$I(s) = \oint_{\Gamma_s} f(z)\,\mathrm{d}z = \int_0^1 f(z(s,t))\frac{\partial z}{\partial t}(s,t)\,\mathrm{d}t.$$

We assume that $z(s,t)$ is smooth in both s and t and note

$$I'(s) = \int_0^1 \left(f'(z(s,t))\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} + f(z(s,t))\frac{\partial^2 z}{\partial s \partial t} \right) \mathrm{d}t.$$

$$\frac{\partial}{\partial t}\left(f(z(s,t))\frac{\partial z}{\partial s} \right) = f'(z(s,t))\frac{\partial z}{\partial t}\frac{\partial z}{\partial s} + f(z(s,t))\frac{\partial^2 z}{\partial t \partial s}.$$

Detail is needed to justify all the steps. As we have a loop the values at $t = 0$ and $t = 1$ are the same, i.e. $z(s, 0) = z(s, 1)$ etc. and it follows that $I'(s) = 0$ and $I(s)$ does not vary with s. With a simply connected domain we can continuously deform the loop to a point giving $I(s) = 0$.

Comments on a proof given in Spiegel, p103–105

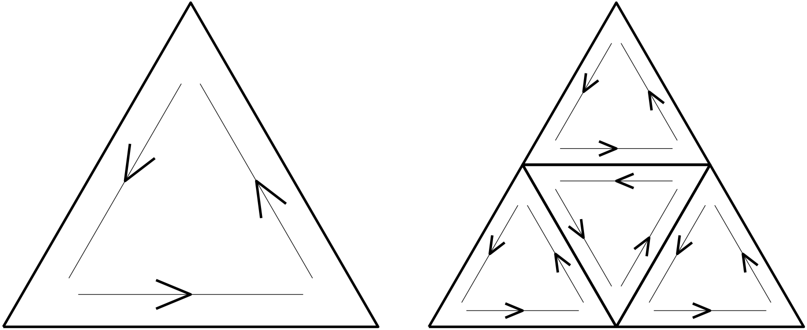
We wish to show that when $f(z)$ is analytic inside a loop Γ

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

An overview of the steps are as follows.

1. Show that it is true for a triangle of any size.
2. Deduce that it is true for any polygonal path. This follows as we can construct a mesh of triangles for the region inside a polygonal path.
3. As any loop can be approximated arbitrarily closely by a polygonal path it follows that it is true for any loop.

Dividing a triangle into 4 similar triangles



If Γ is the starting triangle and Γ_i^1, $i = 1, 2, 3, 4$ are the 4 smaller triangles then

$$I = \oint_{\Gamma} f(z)\,\mathrm{d}z = \sum_{i=1}^{4} I_i^1, \quad I_i^1 = \oint_{\Gamma_i^1} f(z)\,\mathrm{d}z.$$

This is because all internal edges appear exactly two times and the contributions cancel.

Use of the triangle inequality and repeatedly dividing

For at least one of the smaller triangles we have

$$|I| \le 4 \left| I^1_{k_1} \right|.$$

The triangle $\Gamma^1_{k_1}$ can be similarly divided and the process can be repeated to give

$$|I| \le 4^n \left| I^n_{k_n} \right|, \quad n = 1, 2, \ldots$$

Let z_0 be a common point of the nested triangles, let $\epsilon > 0$ and let n be sufficiently large that inside $\Gamma^n_{k_n}$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0), \quad \text{with } |\lambda(z)| < \epsilon.$$

As $f(z_0) + f'(z_0)(z - z_0)$ has an anti-derivative

$$I^n_{k_n} = \oint_{\Gamma^n_{k_n}} f(z)\,\mathrm{d}z = \oint_{\Gamma^n_{k_n}} \lambda(z)(z - z_0)\,\mathrm{d}z.$$

Completing the proof

From the previous steps

$$|I| \le 4^n \left| I^n_{k_n} \right|, \quad I^n_{k_n} = \oint_{\Gamma^n_{k_n}} \lambda(z)(z - z_0)\,\mathrm{d}z.$$

If L is the length of Γ then we have the following.

1. $L/2^n$ is the length of $\Gamma^n_{k_n}$.
2. For z inside the triangle $|z - z_0| < L/2^n$.
3. For z inside the triangle $|\lambda(z)| < \epsilon$.

Putting these parts together gives

$$|I| \le 4^n \left(\frac{L}{2^n}\right) \left(\frac{L}{2^n}\right) \epsilon = L^2 \epsilon.$$

$|I| < L^2 \epsilon$ for all $\epsilon > 0$ implies that $I = 0$.

Cauchy's integral theorem and a corollary

Recall that the following was mentioned in chapter 2.
Simply-connected: A domain which does not have holes.

Cauchy-Goursat theorem: If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 0.$$

Recall again the following equivalent statements.

(i) All loop integrals of f are 0.

(ii) The value of the integral of f only depends on the end points.

(iii) There exists an anti-derivative F, i.e. $F' = f$.

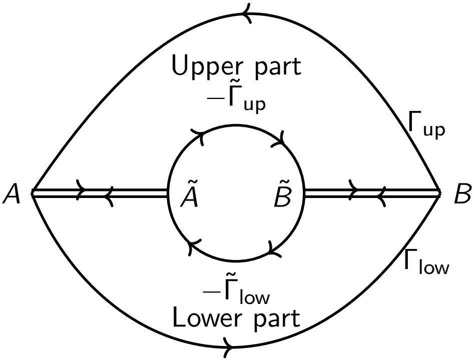
Corollary: If f is analytic in a simply connected region then there exists an **anti-derivative** F (which is analytic) such that $f(z) = F'(z)$.

A domain bounded by two loops

Suppose that $f(z)$ is just analytic between two loops.

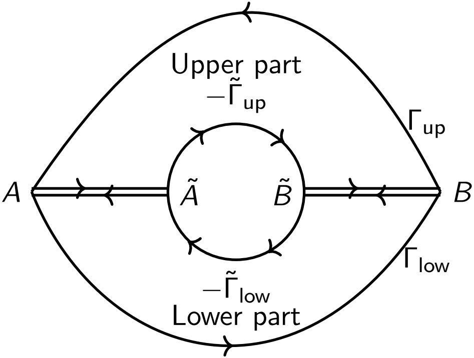
The domain between loops Γ and $\tilde{\Gamma}$ can be divided into two simply connected domains. In the diagram below we have an "upper part" and a "lower part". Note that

$$\Gamma = \Gamma_{up} \cup \Gamma_{low}, \quad \tilde{\Gamma} = \tilde{\Gamma}_{up} \cup \tilde{\Gamma}_{low}.$$



A domain bounded by two loops continued

$$\Gamma = \Gamma_{up} \cup \Gamma_{low}, \quad \tilde{\Gamma} = \tilde{\Gamma}_{up} \cup \tilde{\Gamma}_{low}.$$



The upper and lower domains have boundaries with 4 parts.

$$\Gamma_{\mathrm{up}} \cup [A, \tilde{A}] \cup (-\tilde{\Gamma}_{\mathrm{up}}) \cup [\tilde{B}, B],$$
$$\Gamma_{\mathrm{low}} \cup [B, \tilde{B}] \cup (-\tilde{\Gamma}_{\mathrm{low}}) \cup [\tilde{A}, A].$$

A domain bounded by two loops continued

$$\Gamma_{\rm up} \cup [A, \tilde{A}] \cup (-\tilde{\Gamma}_{\rm up}) \cup [\tilde{B}, B],$$
$$\Gamma_{\rm low} \cup [B, \tilde{B}] \cup (-\tilde{\Gamma}_{\rm low}) \cup [\tilde{A}, A].$$

As both of these bound simply connected domains where f is analytic the integral of $f(z)$ around each is 0. Also, the sum of both integrals is 0 and the integrals on

$$[A, \tilde{A}], \quad [\tilde{A}, A], \quad [B, \tilde{B}], \quad [\tilde{B}, B]$$

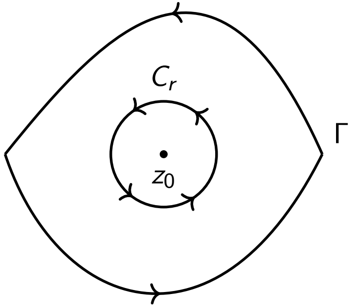
cancel. The remaining parts give

$$\int_{\Gamma_{\rm up}} f(z)\,{\rm d}z + \int_{-\tilde{\Gamma}_{\rm up}} f(z)\,{\rm d}z + \int_{\Gamma_{\rm low}} f(z)\,{\rm d}z + \int_{-\tilde{\Gamma}_{\rm low}} f(z)\,{\rm d}z = 0.$$

Rearranging and using $\Gamma = \Gamma_{\rm up} \cup \Gamma_{\rm low}$ and $\tilde{\Gamma} = \tilde{\Gamma}_{\rm up} \cup \tilde{\Gamma}_{\rm low}$ gives

$$\oint_{\Gamma} f(z)\,{\rm d}z = \oint_{\tilde{\Gamma}} f(z)\,{\rm d}z.$$

Results involving contour integration continued



When $f(z)$ is analytic between a contour Γ and a circle C_r we have

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = \oint_{C_r} f(z)\,\mathrm{d}z.$$

Note that this is true for all $r > 0$ such that $f(z)$ is analytic inside C_r. When we just have one isolated singularity we can hence consider the case $r \to 0$. Manipulation of this type is done a few times in the module.

Rational functions and the Residue theorem

The following was in the week 11 slides. Let

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}.$$

$$R(z) = \frac{p(z)}{q(z)} = (\text{some polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k} + (\text{higher order poles}).$$

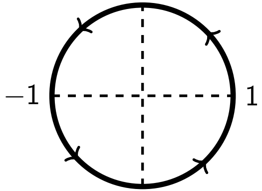
Here A_k is the **residue** at z_k. Note that with a rational function we can write it as a finite sum of terms which we can handle individually. If $z_1, \ldots, z_m$ are inside Γ then

$$\oint_\Gamma R(z)\,\mathrm{d}z = \sum_{k=1}^{m} A_k \oint_\Gamma \frac{\mathrm{d}z}{z - z_k} = 2\pi i \sum_{k=1}^{m} A_k.$$

We extend this in chapter 8 to any function analytic inside Γ except for a finite number of isolated singularities at $z_1, \ldots, z_m$.

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f,\, z_k).$$

Trig integrals evaluated using residue theory



$$I = \int_0^{2\pi} R(\cos\,\theta, \sin\,\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z.$$

Here C is the unit circle and $F(z)$ is obtained by using

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\,\theta = \frac{z + z^{-1}}{2}, \quad \sin\,\theta = \frac{z - z^{-1}}{2i}.$$

When R is a "rational function of its arguments" the function $F(z)$ is a rational function of z. We determine I by the Residue theorem involving the residues of $F(z)$ at the poles which are inside C.

Trig. integral examples

1.

$$I=\int_{-\pi}^{\pi}\frac{4\mathrm{d}\theta}{5+2\cos\,\theta}\quad\text{generates } F(z)=\frac{4}{z^2+5z+1}.$$

If z_1 denotes the zero of the quadratic inside the unit circle then

$$I=2\pi\mathrm{Res}(F,\,z_1)=2\pi\left(\frac{4}{2z_1+5}\right).$$

2.

$$I=\int_{0}^{2\pi}(\cos\,\theta)^{2n}\,\mathrm{d}\theta\quad\text{generates } F(z)=\frac{1}{2^{2n}}\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n}.$$

$$I=2\pi\mathrm{Res}(F,\,0).$$

Using the binomial expansion we can get the coefficient of $1/z$ which is the residue $\mathrm{Res}(F,\,0)$.

Trig. integral examples continued

3.

$$I = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{1+\sin^2\theta} \quad \text{generates } F(z) = \frac{4z}{-z^4+6z^2-1}.$$

Two simple poles inside the unit circle at $\pm z_1$ where $z_1^2 = 3-\sqrt{8}$.

$$I = 2\pi \left(\mathrm{Res}(F,\, z_1) + \mathrm{Res}(F,\, -z_1)\right).$$

Alternatively we can first let $t = 2\theta$ and write as

$$\frac{1}{1+\sin^2\theta} = \frac{2}{2+2\sin^2\theta} = \frac{2}{2+(1-\cos(2\theta))} = \frac{2}{3-\cos(t)}.$$

$$I = \frac{1}{2}\int_{-2\pi}^{2\pi} \frac{2\mathrm{d}t}{3-\cos(t)} = \int_{-\pi}^{\pi} \frac{2\mathrm{d}t}{3-\cos(t)}.$$

The expression we get for $F(z)$ in this case is simpler and we have just 1 simple pole in the unit circle.

4.

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos^2\theta}\mathrm{d}\theta = -\frac{\pi}{2} + \frac{9}{4}\int_0^{2\pi} \frac{\mathrm{d}t}{7+2\cos(t)}.$$

The second version follows from the first version by writing in terms of $\cos(2\theta)$ and then letting $t = 2\theta$. The integral term generates

$$F(z) = \frac{1}{z^2+7z+1}.$$

Chapter 6 comments
Cauchy's integral formula, consequences and bounds

Chapter 6 is a bit more theoretical than other chapters and after a certain point the material of the main notes is not examinable. The exercises, when available, should also clarify this a bit further.

The chapter has integral representations of functions and these are used to get the series representations which will be covered in chapter 7.

The Cauchy integral formula

If $f(z)$ is analytic in a domain D and Γ is a positively orientated loop in D and z is inside D then

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

It is from this representation that we can get a similar representation for f' and deduce that f' is also analytic. The reasoning can be continued to deduce that all derivatives also have representations and are all analytic. It can be shown that it is valid to differentiate through the integral to get the following as these representations.

The generalised Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta, \quad n = 0, 1, 2, \ldots$$

Steps to show the expression for $f'(z)$

By using the Cauchy integal form two times we have

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{and} \quad f(z+h) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - (z+h)}\, d\zeta.$$

We then get the representation

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - (z+h))}\, d\zeta.$$

The detail is to justify that

$$\oint_\Gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - (z+h))}\, d\zeta - \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$
$$= h \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2(\zeta - (z+h))}\, d\zeta \to 0 \quad \text{as } h \to 0.$$