Analytic functions

As was introduced in week 03.

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

  provided the limit exists.
- A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.
- A function f is **analytic in a domain** if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an **entire function** if it is analytic on the whole complex plane $\mathbb{C}$.

The Cauchy Riemann equations for $f(z) = u(x, y) + iv(x, y)$

When f is analytic at z_0 the following limit exists.

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

By considering the case when h is real and then purely imaginary we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

When u and v have continuous first partial derivatives on a domain D and the Cauchy Riemann equations are satisfied then the limit above exists and f is analytic on D.

The Cauchy Riemann equations in polars

Suppose

$$f(re^{i\theta}) = \tilde{u}(r,\theta) + i\tilde{v}(r,\theta).$$

$$\begin{aligned} f'(z) &= \frac{1}{e^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial r} + i\frac{\partial \tilde{v}}{\partial r}\right) \\ &= \frac{1}{ire^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial \theta} + i\frac{\partial \tilde{v}}{\partial \theta}\right) \end{aligned}$$

The Cauchy Riemann equations in polar coordinates are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

Functions which are analytic – $\exp(z)$

$$\exp(z) = \exp(x + iy) = \mathrm{e}^x \, \mathrm{e}^{iy} = \mathrm{e}^x(\cos(y) + i \sin(y)).$$

Here

$$u = \mathrm{e}^x \cos(y), \quad v = \mathrm{e}^x \sin(y).$$

The Cauchy Riemann equations are satisfied and

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z$$

as in the real case.

Observe that

$$|\mathrm{e}^z| = \mathrm{e}^x \quad \text{and} \quad \arg(\mathrm{e}^z) = y.$$

The definition of e^z gives the value in polar form. Also with $w = \mathrm{e}^z$, $x = \ln(|w|)$, $y = \arg(w)$.

Functions which are analytic – $\mathrm{Log}(z)$

$$\mathrm{Log}(z) = \ln\ r + i\mathrm{Arg}\, z = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x).$$

is analytic except on $\{z = x + iy:\ x \le 0,\ y = 0\}$.

$$\frac{\partial u}{\partial x} = \frac{x}{r^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{r^2}, \quad f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{x-iy}{r^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}.$$

Using the polar form of the Cauchy Riemann equations

$$\tilde{u} = \ln r, \quad \tilde{v} = \theta.$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} = \frac{1}{r}. \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r} = 0.$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}(z) = \frac{1}{\mathrm{e}^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial r} + i\frac{\partial \tilde{v}}{\partial r}\right) = \frac{1}{r\mathrm{e}^{i\theta}} = \frac{1}{z}.$$

The derivative is not analytic at $z = 0$ whereas Log(z) is also not analytic on the negative real axis.

Different representations of $f'(z)$ using u and v

$$
\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \text{(only involving derivatives with respect to } x\text{)},\\
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \quad \text{(only involving derivatives with respect to } y\text{)},\\
&= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad \text{(only involving } u\text{)},\\
&= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}, \quad \text{(only involving } v\text{)}.
\end{aligned}
$$

The different versions are because of the CR equations.

$f'(z)$ is thus completely determined by the gradient of u.

$f'(z)$ is thus completely determined by the gradient of v.

Harmonic functions and analytic function

- $\phi(x, y)$ is **harmonic** if

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

- If $f = u + iv$ is analytic then u and v are harmonic functions. v is said to be the **harmonic conjugate** of u.
- If u is known then we can attempt to get v as follows.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Partial integrate wrt x to get

$$v(x, y) = \text{some function} + g(y)$$

$$\frac{\partial v}{\partial y} = \text{deriv of some function} + g'(y) = \frac{\partial u}{\partial x}$$

This gives $g'(y)$ and then we get $g(y)$.

Harmonic functions and analytic function continued

- We can do things in a different order, i.e. with a harmonic function u given we can first use

  $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

  Partial integrate wrt to y to get

  $$v(x, y) = \text{some function} + h(x)$$

  $$\frac{\partial v}{\partial x} = \text{deriv of some function} + h'(x) = -\frac{\partial u}{\partial y}$$

  This gives $h'(x)$ and then we get $h(x)$.

The amount of work by each route will be about the same.

Example showing both order of operations

$u = x^2 - y^2 + 4xy$ is harmonic. Let v denote a harmonic conjugate.

$$\begin{aligned}
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = 2y - 4x, \\
v &= 2xy - 2x^2 + g(y), \\
\frac{\partial v}{\partial y} &= 2x + g'(y) = \frac{\partial u}{\partial x} = 2x + 4y,
\end{aligned}$$

$$g'(y) = 4y, \quad g(y) = 2y^2 + C.$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 2x + 4y, \\
v &= 2xy + 2y^2 + h(x), \\
\frac{\partial v}{\partial x} &= 2y + h'(x) = -\frac{\partial u}{\partial y} = 2y - 4x,
\end{aligned}$$

$$h'(x) = -4x, \quad h(x) = -2x^2 + C.$$

Expressing an analytic $f = u(x,y) + iv(x,y)$ in terms of z

In the case of only "polynomial terms" we can express in terms of z by using the finite Maclaurin series representation.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{etc.}$$

$$f(z) = f(0) + f'(0)z + \cdots + \frac{f^{(r)}(0)}{r!}z^r.$$

Examples of analytic functions and harmonic functions

$$\begin{aligned}
z &= x + iy, \\
z^2 &= (x^2 - y^2) + 2ixy, \\
z^3 &= (x^3 - 3xy^2) + i(3x^2y - y^3), \\
\frac{1}{z} &= \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}, \\
e^z &= e^x(\cos\, y + i \sin\, y), \\
\operatorname{Log} z &= \ln|z| + i \operatorname{Arg} z.
\end{aligned}$$

$\overline{z} = x - iy$ is an example of a function which is not analytic anywhere.

An analytic function $f(z)$ cannot depend on $\overline{z}$

Let $f = u + iv = u(x, y) + iv(x, y)$ and let

$$g(z, \overline{z}) = u\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) + iv\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right).$$

The Cauchy Riemann equations hold if and only if

$$\frac{\partial g}{\partial \overline{z}} = 0.$$

When f is not a polynomial an expression only involving z is given by the Taylor series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots .$$

In term 2 we show that a function analytic at z_0 always has a Taylor series which converges in a neighbourhood of z_0.

∇u and ∇v are orthogonal when $f'(z) \neq 0$

Suppose that $f = u + iv$ is analytic.

With vector calculus notation, the gradients of u and v are the vectors

$$\nabla u = \frac{\partial u}{\partial x}\underline{i} + \frac{\partial u}{\partial y}\underline{j} \quad \text{and} \quad \nabla v = \frac{\partial v}{\partial x}\underline{i} + \frac{\partial v}{\partial y}\underline{j}.$$

The dot product of these two vectors is

$$\begin{aligned} \nabla u \cdot \nabla v &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\ &= 0 \end{aligned}$$

using the Cauchy Riemann equations.

When $f'(z_0) \neq 0$ the gradient vectors ∇u and ∇v are non-zero.

Level curves of u and v are orthogonal when $f'(z) \neq 0$

The level curve for u passing through (x_0, y_0) is defined by

$$\Gamma^u = \{(x, y) :\ u(x, y) = u(x_0, y_0)\}$$

and the level curve for v passing through this point is defined by

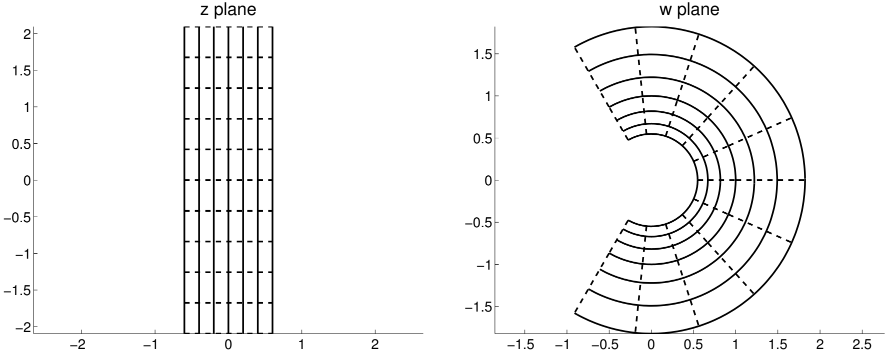
$$\Gamma^v = \{(x, y) :\ v(x, y) = v(x_0, y_0)\}.$$

∇u is normal to Γ^u and ∇v is normal to Γ^v.

The tangent to a curve is at right angle to a normal.

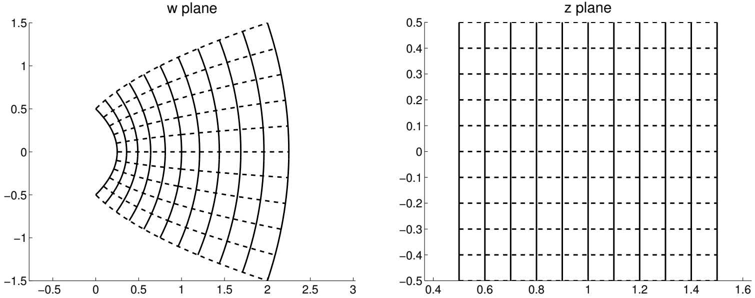
As the normals are orthogonal it follows that the tangent to a level curve of u is orthogonal to the tangent to a level curve of v at (x_0, y_0) when $f'(x_0 + iy_0) \neq 0$.

Mapping of $w = \exp(z)$, level curves of $z = \mathrm{Log}(w)$



The rectangular grid in the z-plane maps to the circular arcs and radial lines in the w-plane. The inverse function takes the curves in the w-plane to the grid in the z-plane. The circles and radial lines are thus curves where the real and imaginary parts of $\mathrm{Log}(w)$ are constant. These are orthogonal.

Mapping of $w = z^2$ near $z = 1$ and level curves of $z = \sqrt{w}$



Level curves in the w-plane are the real and imaginary parts of

$$z = g(w) = \sqrt{w}.$$

$f(z)$ is a conformal mapping when $f'(z) \neq 0$

Suppose we have 2 arcs described in parametric form as

$$z_1(t), \quad a_1 < t < b_1 \qquad \text{and} \qquad z_2(t), \quad a_2 < t < b_2.$$

Given an analytic function $f(z)$ we get 2 image curves

$$w_1(t) = f(z_1(t)) \qquad \text{and} \qquad w_2(t) = f(z_2(t)).$$

If the curves intersect at $z^* = z_1(t_1) = z_2(t_2)$ then the image curves intersect at $w^* = f(z^*)$. The direction of the tangents are the direction of $z_1'(t_1)$, $z_2'(t_2)$ $f'(z^*)z_1'(t_1)$ and $f'(z^*)z_2'(t_2)$. The angle between the curves in the z-plane is the angle of $z_1'(t_1)/z_2'(t_2)$ and similarly for the curves in the w-plane.

$$\frac{w_1'(t_1)}{w_2'(t_2)} = \frac{f'(z^*)z_1'(t_1)}{f'(z^*)z_2'(t_2)} = \frac{z_1'(t_1)}{z_2'(t_2)}.$$

When f is analytic and $f'(z^*) \neq 0$ angles are preserved.