Analytic functions

As was introduced last week (week 03).

- **Complex derivative:** Let f be a complex valued function defined in a neighbourhood of z_0. The **derivative of f at z_0** is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  provided the limit exists.
- A function f is **analytic at z_0** if f is differentiable at all points in some neighbourhood of z_0.
- A function f is **analytic in a domain** if f is analytic at all points in the domain.
- A function $f : \mathbb{C} \to \mathbb{C}$ is an **entire function** if it is analytic on the whole complex plane $\mathbb{C}$.

The Cauchy Riemann equations for $f(z) = u(x,y) + iv(x,y)$

When f is analytic at z_0 the following limit exists.

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

By considering the case when h is real and then purely imaginary we get

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Equating the real and imaginary parts gives the Cauchy Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Alternatively, when u and v have continuous first partial derivatives on a domain D and the Cauchy Riemann equations are satisfied then f is analytic on D.

A comment about directional derivatives

The following uses vector notation.

Let $\phi(x, y)$ be a scalar valued function and let

$$\underline{r} = x\underline{i} + y\underline{j}.$$

The gradient of ϕ is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\underline{i} + \frac{\partial\phi}{\partial y}\,\underline{j}.$$

The directional derivative of ϕ in the direction of a unit vector $\underline{n}$ is

$$\begin{aligned} \frac{\partial\phi}{\partial n}(\underline{r}) &= \frac{\mathrm{d}}{\mathrm{d}s}\phi(\underline{r} + s\underline{n})\bigg|_{s=0} \\ &= \left(n_1\frac{\partial\phi}{\partial x_1} + n_2\frac{\partial\phi}{\partial x_2}\right)(\underline{r}) = \underline{n}\cdot\nabla\phi(\underline{r}). \end{aligned}$$

When s is small

$$\phi(\underline{r} + s\underline{n}) - \phi(\underline{r}) \approx s\frac{\partial\phi}{\partial n}(\underline{r}) = (s\underline{n})\cdot\nabla\phi(\underline{r}).$$

The proof of the Cauchy Riemann equations

When the Cauchy Riemann equations hold

$$\begin{aligned} u(x_0+h_1, y_0+h_2) - u(x_0, y_0) &= \left(h_1 \frac{\partial u}{\partial x} + h_2 \frac{\partial u}{\partial y} \right)(x_0, y_0) + \mathcal{O}(|h|^2) \\ &= \left(h_1 \frac{\partial u}{\partial x} - h_2 \frac{\partial v}{\partial x} \right)(x_0, y_0) + \mathcal{O}(|h|^2), \\ v(x_0+h_1, y_0+h_2) - v(x_0, y_0) &= \left(h_1 \frac{\partial v}{\partial x} + h_2 \frac{\partial v}{\partial y} \right)(x_0, y_0) + \mathcal{O}(|h|^2) \\ &= \left(h_1 \frac{\partial v}{\partial x} + h_2 \frac{\partial u}{\partial x} \right)(x_0, y_0) + \mathcal{O}(|h|^2). \end{aligned}$$

With $z_0 = x_0 + iy_0$ and $h = h_1 + ih_2$

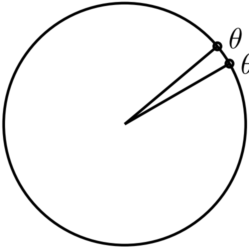
$$\begin{aligned} f(z_0+h) - f(z_0) &\approx \left(\left(h_1 \frac{\partial u}{\partial x} - h_2 \frac{\partial v}{\partial x} \right) + i \left(h_1 \frac{\partial v}{\partial x} + h_2 \frac{\partial u}{\partial x} \right) \right)(z_0) \\ f(z_0+h) - f(z_0) &= (h_1 + ih_2) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)(z_0) + \mathcal{O}(|h|^2). \end{aligned}$$

Dividing by $h = h_1 + ih_2$ and letting $h \to 0$ shows that the limit exists.

Remarks about polars

$$z = re^{i\theta}, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2, \quad \tan\theta = \frac{y}{x}.$$

$$\frac{\partial z}{\partial r} = e^{i\theta}, \quad \frac{\partial z}{\partial \theta} = ire^{i\theta}.$$



$\Delta\theta$=change in θ

If r is fixed and $g(\theta) = re^{i\theta}$ then

$$\begin{aligned} g(\theta+\Delta\theta) - g(\theta) &= g'(\theta)\Delta\theta + \frac{g''(\theta)}{2}\Delta\theta^2 + \cdots \\ &= re^{i\theta}\left(i\Delta\theta - \Delta\theta^2/2 + \cdots\right). \end{aligned}$$

Partial derivatives of θ and r wrt x and y

$$r^2 = x^2 + y^2, \quad 2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y.$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

If $\theta = \arg(z)$ then

$$\tan(\theta) = \frac{y}{x}, \quad \cot(\theta) = \frac{x}{y}.$$

We can partially differentiate either wrt x or y to get, after about two intermediate lines,

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}.$$

The expressions are valid on the axis when $x^2 + y^2 > 0$.

The Cauchy Riemann equations in polars

Suppose

$$f(re^{i\theta}) = \tilde{u}(r,\theta) + i\tilde{v}(r,\theta).$$

$$\begin{aligned} f'(z) &= \frac{1}{e^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial r} + i\frac{\partial \tilde{v}}{\partial r}\right) \\ &= \frac{1}{ire^{i\theta}}\left(\frac{\partial \tilde{u}}{\partial \theta} + i\frac{\partial \tilde{v}}{\partial \theta}\right) \end{aligned}$$

The Cauchy Riemann equations in polar coordinates are

$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

Functions which are analytic

$$\exp(z) = \exp(x + iy) = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Here

$$u = e^x \cos(y), \quad v = e^x \sin(y).$$

The Cauchy Riemann equations are satisfied and

$$\frac{d}{dz} e^z = e^z$$

as in the real case.

Observe that the value of e^z is in polar form and thus

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y.$$

$$\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z$$

is analytic except on $\{z = x + iy : \; x \le 0, \; y = 0\}$ and

$$\frac{d}{dz} \operatorname{Log}(z) = \frac{1}{z}.$$