MA3614
Complex variable methods and applications

Lecture Notes by M.K. Warby in 2023/2024
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Assessment dates and assessment information

Class test: Likely to be in the winter exam weeks. (the format and timing etc to be confirmed). (20%).

Final exam: May exam period, 3 hours (80%).

Recommended reading and my sources

There is no essential text to obtain for this module although there are many texts which cover at least most of the material and among the sources for the notes that I will generate are the following books by Saff and Snider, Osborne, Wunsch and Spiegel.

1. E. B. Saff and A. D. Snider. *Fundamentals of Complex Analysis with applications to engineering and sciences (Third edition)*. Prentice Hall, 2003. QA300.S18.

2. A. D. Osborne. *Complex variables and their applications*. Addison-Wesley, 1999. QA331.7.O83.

3. A. D. Wunsch. *Complex Variables with Applications (3rd edition)*. Addison-Wesley, 2005. QA331.W86.

4. Murray R. Spiegel. *Schaum's Outline of Theory and Problems of Complex Variables*. McGraw-Hill, 1974. QA331.S68.

When I took the module with the same title in 2012/3 the module code was MA3914 and it started as MA3614 in 2013/4. The text that I have used the most when creating the notes is the book by Saff and Snider.

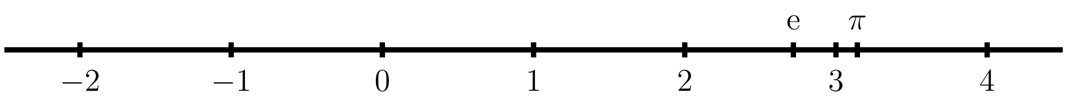
Chapter 1

An introduction to the module and revision of previous study of complex numbers

The material in this chapter should mostly be a reminder of how complex numbers are defined and represented in the complex plane as well as an introduction to some of the topics in the module. In section 1.8 the notes also contain some brief details of work by Cardano in the 16th century about finding the roots of cubics which is generally regarded as a starting point for the interest in complex numbers.

1.1 The definition of complex numbers and the complex plane

Let $\mathbb{R}$ denote the real numbers. It is usual to regard a specific real number x as a point on the real line as indicated below.



Complex number are defined after first introducing a symbol i which has the property

$$i^2 = -1$$

and we write $i = \sqrt{-1}$. The space of complex numbers is defined by

$$\mathbb{C} = \{x + iy : \quad x, y \in \mathbb{R}\}.$$

We get a one-one correspondence between points $(x, y) \in \mathbb{R}^2$ and complex numbers $z = x + iy$. For some terminology, $x = \mathrm{Re}(z)$ is known as the **real part** and $y = \mathrm{Im}(z)$ is known as the **imaginary part** and the representation of z in this form is often referred to as the **cartesian** representation of z. As has just been illustrated, a real number is represented by a point on the real line and similarly a complex number can be represented by a point in the complex plane. The representation of complex numbers in this way is known as an **argand diagram**. Now a point in $\mathbb{R}^2$ can also be represented in **polar coordinates** and thus for complex numbers we also have

$$z = x + iy = r(\cos\theta + i\sin\theta), \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arg z. \tag{1.1.1}$$

$r \geq 0$ is known as the **magnitude** or **absolute value** of z and $\theta = \arg z$ is any angle for which (1.1.1) is true. The phrase any angle is used here as if we add any integer multiple of 2π to θ then we get the same point and this is something which will be discussed at various times in the module. When there is a need to uniquely determine θ the usual convention is to take the **principal argument**, which is denoted by Arg$z$, which satisfies

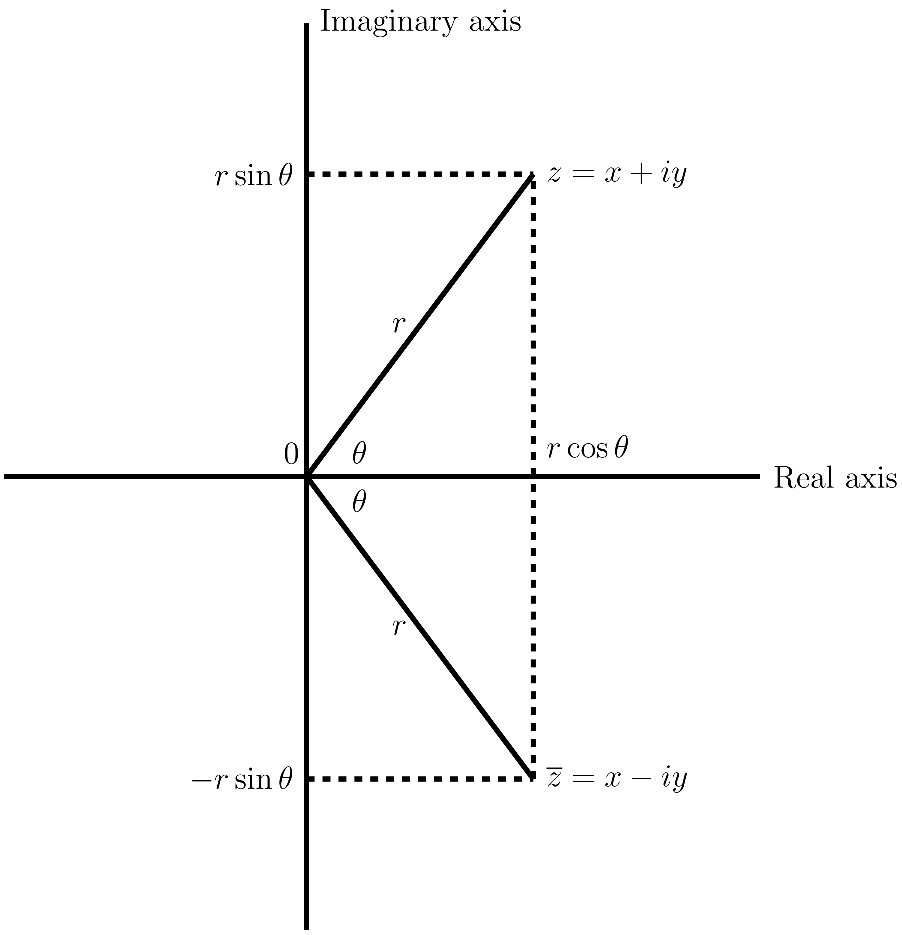
$$\text{Arg}z \in (-\pi, \pi].$$

Note that Arg$z$ defines a function which is discontinuous as you cross the negative real axis with a jump discontinuity of magnitude 2π. Also note that $\arg z$ and Arg$z$ are not defined when $z = 0$.

With z defined the **complex conjugate** is denoted by $\overline{z}$ and is defined by

$$\overline{z} = x - iy = r(\cos(-\theta) + i\sin(-\theta)).$$

The number $\overline{z}$ is the reflection of z in the real line and we can represent z and $\overline{z}$ in the following diagram.



1.2 Addition, multiplication, division and complex conjugate of complex numbers

We can add (and subtract), multiply and divide complex numbers and you can represent the result in cartesian and polar form.

Let

$$\begin{aligned} z_1 &= x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1), \\ z_2 &= x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2), \end{aligned}$$

be complex numbers with x_k, y_k, $r_k \ge 0$ and θ_k, $k = 1, 2$ being real.

Addition

Using the cartesian representation we have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Multiplication

Using the cartesian representation we have

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + i^2 y_1 y_2 + i(x_1 y_2 + y_1 x_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

Note that we expand in the usual way and whenever i^2 appears we can replace it by -1.

It is also beneficial to do the multiplication using the polar forms as we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

where in the last step we have used the expansion formulas for the cosine and the sine functions. In the polar form we just multiply the magnitudes and we add the angles.

Complex conjugate $\overline{z} = x - iy$ of $z = z + iy$

The complex conjugate has already been mentioned but we make a few more comments here. Firstly, note that

$$|\overline{z}| = |z| = \sqrt{x^2 + y^2}, \quad \mathrm{Arg}\overline{z} = -\mathrm{Arg} z,$$

and also

$$\begin{aligned} z\overline{z} &= |z|^2 = x^2 + y^2, \\ z + \overline{z} &= 2x \in \mathbb{R}, \\ z - \overline{z} &= 2iy \text{ (purely imaginary when } y \ne 0). \end{aligned}$$

From the first part of the equations we have when $z \neq 0$

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$$

and this is used in a moment to deal with division. Note that in polars this is

$$\frac{1}{z} = \frac{\cos(\theta) - i\sin(\theta)}{r}.$$

Other points to note here are that a complex number $z = x + iy$ is real if and only if $y = 0$ which is if and only if $z = \overline{z}$ and we can interchange the operation of taking the complex conjugate with the operations of addition, multiplication, division and powers in the sense that

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 z_2} &= (\overline{z_1})\,(\overline{z_2}), \\ \overline{(z_1/z_2)} &= \overline{z_1}/\overline{z_2}, \\ \overline{z^n} &= (\overline{z})^n \end{aligned}$$

The verification of these is left as exercises.

1.2.1 Division

From the previous discussion about multiplication and the complex conjugate we can cope with division as follows.

$$\frac{z_1}{z_2} = z_1\left(\frac{\overline{z_2}}{z_2\overline{z_2}}\right) = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{1}{|z_2|^2} z_1\overline{z_2}$$

and to complete the operation we have to do the product of z_1 and $\overline{z_2}$. If we do the operation using the polar form then we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2) \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)). \end{aligned}$$

With division we get the angle of the result by subtracting the two angles, i.e. $\theta_1 - \theta_2$ is one of the values of the argument of z_1/z_2.

1.3 Powers: z^n for $n = 2, 3, \ldots$

We have already considered multiplication using the polar form which involves adding the angles and thus for the powers we have

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta), \\ z^2 &= r^2(\cos 2\theta + i\sin 2\theta), \\ z^3 &= r^3(\cos 3\theta + i\sin 3\theta), \\ \cdots & \quad \cdots \\ z^n &= r^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

With $r = 1$ this gives us **DeMoivre's theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Later in the module (probably in this term) we will generalise the taking of a power of a complex number to z^α for any complex number α.

1.4 The $r\mathrm{e}^{i\theta}$ representation

For one or more years you will have been using the exponential function with a real argument and you would have seen that it has the Maclaurin expansion

$$f:\mathbb{R}\to\mathbb{R},\quad f(x) = \mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots.$$

Later in the module we will see that we can just replace x by z for any z in the complex plane. Before we get to that stage we will assume that it is valid to replace x by $i\theta$ to be able to define $\mathrm{e}^{i\theta}$ and manipulation of the absolutely convergent series leads to the identity

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta.$$

With this more compact notation the results considered earlier can now be summarised as follows.

$$\begin{aligned}
\mathrm{e}^{i\theta_1}\mathrm{e}^{i\theta_2} &= \mathrm{e}^{i(\theta_1+\theta_2)},\\
\frac{1}{\mathrm{e}^{i\theta}} &= \mathrm{e}^{-i\theta},\\
\frac{\mathrm{e}^{i\theta_1}}{\mathrm{e}^{i\theta_2}} &= \mathrm{e}^{i(\theta_1-\theta_2)},\\
\overline{\mathrm{e}^{i\theta}} &= \mathrm{e}^{-i\theta},\\
\left(\mathrm{e}^{i\theta}\right)^n &= \mathrm{e}^{ni\theta},\quad n = 0, \pm 1, \pm 2, \ldots.
\end{aligned}$$

1.5 The triangle inequality $|z_1 + z_2| \le |z_1| + |z_2|$

With real numbers x_1 and x_2 we have the triangle inequality

$$|x_1 + x_2| \le |x_1| + |x_2|.$$

Similarly with complex numbers $z_1 = x_1 + iy_1 = r_1\mathrm{e}^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2\mathrm{e}^{i\theta_2}$ we have

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = r_1^2 + (z_1\overline{z_2} + z_2\overline{z_1}) + r_2^2.$$

For the middle term

$$(z_1\overline{z_2} + z_2\overline{z_1}) = 2\mathrm{Re}(z_1\overline{z_2}) = 2r_1r_2\cos(\theta_1 - \theta_2) \le 2r_1r_2.$$

Putting everything together gives

$$|z_1 + z_2|^2 \le r_1^2 + 2r_1r_2 + r_2^2 = (r_1 + r_2)^2 = (|z_1| + |z_2|)^2$$

and we have shown that the triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2|$$

is also true for complex numbers.

There is another version of this result which follows by first replacing z_2 by $z_2 - z_1$ giving

$$|z_2| \le |z_1| + |z_2 - z_1|$$

or equivalently

$$|z_2 - z_1| \ge |z_2| - |z_1|$$

and if we swap z_1 and z_2 we also have

$$|z_2 - z_1| = |z_1 - z_2| \ge |z_1| - |z_2|.$$

The two different lower bounds can be combined into one expression as follows. Note that $|z_1 - z_2| \ge 0$ with $|z_1 - z_2| = 0$ only if $z_1 = z_2$. When $z_1 \ne z_2$ one of the right hand side bounds is positive and one is negative and the sharpest result is obtained if we write

$$|z_2 - z_1| \ge ||z_2| - |z_1||.$$

There is an exercise question related to this result and in particular to interpreting when we have equality in this case and also when do we have $|z_1 + z_2| = |z_1| + |z_2|$?

1.6 Convergence of a sequence of complex numbers

At a number of stages in this module we will consider series and to understand this you need to know something about convergence and in particular the convergence of a sequence of partial sums. The convergence of a sequence of complex numbers $z_1, z_2, \ldots$ is defined in a similar way to the convergence of a sequence of real numbers.

Definition 1.6.1 Convergence of a sequence of numbers. $z_1, z_2, \ldots$ *converges to* z^* *if for every* $\epsilon > 0$ *there exists* $N = N(\epsilon)$ *such that*

$$|z_n - z^*| < \epsilon \quad \textit{for all } n \ge N.$$

All the results about combining convergent sequences hold and we are unlikely to meet a case when we need to return to this ϵ–N definition to prove convergence.

As an example, the sequence $z, z^2, z^3, z^4, \ldots, z^n, \ldots$ converges to 0 as $n \to \infty$ if and only if $|z| < 1$.

1.7 Comments about functions of a complex variable

In previous study you consider functions defined on $\mathbb{R}$ (or on part of $\mathbb{R}$), e.g.

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := \mathrm{e}^{2x} - 3\mathrm{e}^{-x} + x^5 + x^4 - 2x \tag{1.7.1}$$

and you will have considered differentiation and integration of such functions when this is possible. In this particular example we can differentiate and integrate infinitely many times. Much of this module is concerned with extending these ideas and considering functions of a complex variable. As we will see, it is possible to generalise (1.7.1) and consider

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) := \mathrm{e}^{2z} - 3\mathrm{e}^{-z} + z^5 + z^4 - 2z \tag{1.7.2}$$

with the meaning of the exponential with a complex argument to be discussed later. As we will see later we will take

$$\mathrm{e}^{x+iy} = \exp(x+iy) = \exp(x)\exp(iy) = \exp(x)(\cos y + i \sin y).$$

An obvious question is the following.

Why would you want to generalise (1.7.1) to (1.7.2)?

A partial answer, which will become apparent later on, is that you often learn more about the function which helps to understand the real case better. In a sense the complex plane $\mathbb{C}$ is the more natural domain for the function than just restricting to $\mathbb{R}$. One property in particular that we will meet is that of a function being **analytic** which is concerned with being able to differentiate the function in a complex sense at all points in a region. As we will see, the complex derivative is the same as the real derivative when both exist. In the case of (1.7.2) the function is analytic in the entire complex plane and

$$f'(z) = 2\mathrm{e}^{2z} + 3\mathrm{e}^{-z} + 5z^4 + 4z^3 - 2.$$

What better understanding is obtained?

We can attempt to answer this with examples which involve power series.

(i) One of the simplest series is the geometric series which we can derive by noting the identity

$$(1 + x + x^2 + x^3 + \cdots + x^n)(1 - x) = 1 - x^{n+1}.$$

It is valid to replace x by z where z can be any complex number. Then provided $z \neq 1$ we have

$$1 + z + z^2 + z^3 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

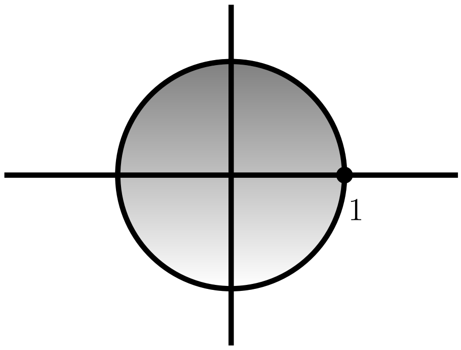
If $|z| < 1$ then the series on the left hand side converges and we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

The series defines a function in the disk $\{z \in \mathbb{C} : |z| < 1\}$ with $R = 1$ being the radius of convergence. The radius of convergence that you meet in earlier modules on analysis does indeed refer to the radius of a circle in the complex plane. As we will see,

$$g(z) := \frac{1}{1-z}$$

is analytic in $\mathbb{C}$ except at the point $z = 1$ which determines the radius of convergence.



This example hence explains why we use the term radius of convergence.

As we will see later, the function $g(z)$ is analytic in the entire complex plane except at the point $z = 1$ and can be represented by a Laurent series in $|z| > 1$. The Laurent series for $|z| > 1$ in this example can be obtained with very little effort. If we write

$$1 - z = -z\left(1 - \frac{1}{z}\right) \quad \text{so that} \quad \frac{1}{1-z} = \left(\frac{-1}{z}\right)\left(1 - \frac{1}{z}\right)^{-1}.$$

As $|z| > 1$, $1/|z| < 1$ and we have the geometric series representation

$$\frac{1}{1-z} = \left(\frac{-1}{z}\right)\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) = -\left(\frac{1}{z} + \frac{1}{z^2} + \cdots\right).$$

The series representation here involves negative powers of z. More general Laurent series can have both positive and negative powers and it will be covered in term 2.

(ii) We now replace z in the previous example with $-z^2$ and we similarly have

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots + (-z)^{2n} + \cdots, \quad |z| < 1.$$

Let now

$$g(z) := \frac{1}{1+z^2}.$$

If we just consider the real case, i.e.

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \frac{1}{1+x^2}$$

then we have a function which is infinitely differentiable on $\mathbb{R}$, it is bounded on $\mathbb{R}$, but yet the power series about $x = 0$ only converges for $|x| < 1$. When the function is considered as a function of a complex variable it becomes clearer why this is the case as $g(z) \to \infty$ as $z \to i$ or as $z \to -i$. As we will see the function has the property of being analytic at all points except $\pm i$.

This example hence explains that to understand the value for the radius of convergence it is often necessary to consider the function with a complex variable.

(iii) In earlier modules you consider Taylor's series for functions which are continuously differentiable a sufficient number of times. Suppose a function f is $n+1$ times continuously differentiable in an interval which contains a and x. When we have these conditions

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t, \end{aligned}$$

This is Taylor's series with an integral form of the remainder. We also have

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1}, \end{aligned}$$

where $\eta = \eta(x)$ is some value between a and x. This is Taylor's series with a Lagrange form of the remainder. Do not worry if you have not previously done this in year 2 as these will not be used in this module. At almost all stages in this module and we will consider functions which can be differentiated infinitely many times and we will be concerned when it is valid to write

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &\quad + \frac{f^{(n)}}{n!}(x-a)^n + \cdots \end{aligned}$$

for x sufficiently close to a. What you would not have done before is to consider the properties that f needs to have for the power series to be equal to the function. It is not just sufficient that the function is infinitely differentiable at $x=a$ in the real sense as we now illustrate with an example.

Let

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} \exp(-1/x^2), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The value at $x=0$ is the same as the limit as $x \to 0$ and thus the function is continuous at $x=0$. For $x \neq 0$ we have

$$f'(x) = \frac{2}{x^3}\exp(-1/x^2)$$

and it can be shown that $f'(x) \to 0$ as $x \to 0$. In fact

$$f^{(n)}(x) \to 0, \quad \text{as } x \to 0 \text{ for } n = 1, 2, 3, \ldots$$

as a consequence of how rapidly the exponential term tends to 0. Thus f is infinitely differentiable (in the real sense) at $x=0$ with all the derivatives having the value 0. Thus if we take the Taylor series about $x=0$ using the derivatives considered in the real sense then we get the zero function, the radius of convergence is ∞ but the series is only the same as $f(x)$ at $x=0$.

The problem with this function f is that it is not analytic at $z = 0$ when we consider it as a function of complex variable. If we let

$$f(z) := \exp(-1/z^2)$$

and consider what happens when we take $z = iy$, $y \in \mathbb{R}$ then

$$f(iy) = \exp(1/y^2) \to \infty \quad \text{as } y \to 0.$$

As this case shows the limiting value depends on which direction we tend to 0. When a function is analytic the value in a limit must be independent of the direction in which we tend to the limit. Thus this function does not have a Taylor series about $z = 0$ in the sense considered in this module but it does have a Laurent series representation about $z = 0$ and Laurent series will be considered in term 2.

What we will show later in this module is that

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$$

in a neighbourhood of $z = a$ provided f is analytic at $z = a$. Conversely we will also show that a convergent power series defines an analytic function.

Thus to summarize, this example shows that it is not sufficient for a function to be infinitely differentiable (in the real sense) in order to have a convergent power series representation but we need the stronger property that it is analytic.

1.8 Some other results in the module: roots of polynomials

Polynomials are among the simpler functions that you consider and earlier in your study of mathematics you meet (and derive) the formula for solving a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{R}, \text{ and } a \neq 0.$$

The roots are

$$\alpha_1 = \frac{-b-\Delta}{2a}, \quad \alpha_2 = \frac{-b+\Delta}{2a}, \quad \text{where } \Delta = \sqrt{b^2 - 4ac}.$$

When $b^2 - 4ac \geq 0$ the term $\Delta \geq 0$ is real and we have real roots. When $b^2 - 4ac < 0$ we have a complex conjugate pair of roots

$$\alpha_1 = \frac{-b-i\delta}{2a}, \quad \alpha_2 = \frac{-b+i\delta}{2a}, \quad \text{where } \delta = \sqrt{4ac - b^2}.$$

Introducing the symbol $i = \sqrt{-1}$ enables us to solve all quadratics with real coefficients and we can factorise the quadratic as

$$ax^2 + bx + c = a(x - \alpha_1)(x - \alpha_2).$$

The **fundamental theorem of algebra** (which was proved by Gauss in 1799) generalises the result in the sense that a polynomial of any degree can be factorised in this way. Specifically, a polynomial of degree n can always be factorised in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

where now $a_0, a_1, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. The points $\alpha_1, \alpha_2, \ldots, \alpha_n$, known as the roots or the zeros, need not be distinct. The proof will be done in this module and it uses properties of functions which are analytic in the entire complex domain. The result provides no information as to where the zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ are located but just that they must exist. Thus in your previous study of linear algebra, when you have a real or complex $n \times n$ matrix A it follows that there are n eigenvalues, when you count them as above, since there must exist values λ_1, $\lambda_2, \ldots, \lambda_n$ such that the characteristic polynomial can be written as

$$\det(tI - A) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n).$$

A historical note about solving cubics

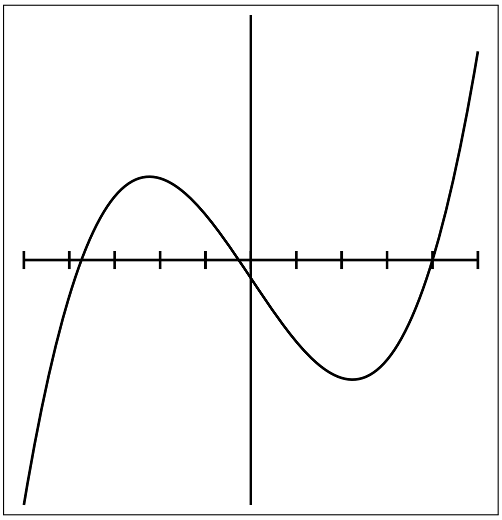


Figure 1.1: A plot of $y = (x^3 - 15x - 4)/10$ on $-5 \le x \le 5$.

It might be thought that complex number were first introduced to be able to solve quadratics. However, this does not seem to be the case and this may be because just introducing $\sqrt{-1}$ in order to get non-real solutions was not too interesting. One of the things which it is believed to have started interest in complex numbers was work by Cardano (1501–1576) who had constructed a method to find the roots of cubics. When we have a cubic with real coefficients there must always be at least one real root as we have a continuous function which takes all values in $(-\infty, \infty)$. The graph of a typical cubic is shown in figure 1.1 on this page. The problem that Cardano found with his method is that there were examples in which you could only make sense of the manipulations to get the real roots if there was such a thing as the square root of negative numbers. Briefly the method of Cardano involves the following.

Suppose we have

$$x^3 + cx + d = 0.$$

(A general cubic equation can always be transformed to an equivalent problem with no x^2 term by using a substitution.) The method then involves the substitution

$$x = u + \frac{p}{u}$$

with at the moment p being arbitrary. With this substitution we get

$$\begin{aligned} x^3 + cx + d &= \left(u + \frac{p}{u}\right)^3 + c\left(u + \frac{p}{u}\right) + d \\ &= \left(u^3 + 3pu + 3\frac{p^2}{u} + \frac{p^3}{u^3}\right) + c\left(u + \frac{p}{u}\right) + d \\ &= u^3 + 3p\left(u + \frac{p}{u}\right) + \frac{p^3}{u^3} + c\left(u + \frac{p}{u}\right) + d \\ &= u^3 + (3p + c)\left(u + \frac{p}{u}\right) + \frac{p^3}{u^3} + d. \end{aligned}$$

Now if we choose p so that $3p + c = 0$, i.e. $p = -c/3$ then the expression simplifies in that we have

$$x^3 + cx + d = u^3 + \frac{p^3}{u^3} + d = \frac{1}{u^3}\left(u^6 + du^3 + p^3\right).$$

The part $u^6 + du^3 + p^3$ in the last expression is a quadratic in u^3 and by the quadratic formula we can make it equal to 0 by taking

$$u^3 = \frac{-d \pm \sqrt{d^2 - 4p^3}}{2}.$$

Thus to summarise the method, we obtain 2 values of u^3 from this formula, for each value of u^3 we obtain 3 possible values of u and for each value of u we form

$$x = u + \frac{p}{u} = u - \frac{c}{3u}$$

as a root of the cubic. Although this gives 6 different values for u we only actually get 3 possibly different values for x. The problem that Cardano encountered was that there are examples for which

$$d^2 - 4p^3 = d^2 + 4c^3/27 < 0$$

so that u^3 is complex and indeed finding u from u^3 also requires complex quantities. At the time the method was created complex numbers had not yet been invented and the report is that Cardano described the method as needing to pass through “alien territory” (i.e. involving the square root of negative numbers) to generate a meaningful answer. Cardano is believed to have described the square root of negative numbers as “useless” yet his formula demonstrated that they are useful.

For a specific example consider the case shown in figure 1.1.

$$x^3 - 15x - 4 = 0, \quad (c = -15,\, d = -4).$$

By inspection $x = 4$ is a root and in fact

$$x^3 - 15x - 4 = (x-4)(x^2 + 4x + 1)$$

and the quadratic factor also has real roots (which of course is consistent with the graph on page 1-11 which we can see crosses the real axis at 3 distinct points.) In this case $p = -c/3 = 5$ and Cardano's method gives

$$u^6 - 4u^3 + 5^3 = 0 \quad \text{and } d^2 - 4p^3 = 16 - 4 \times 5^3 = -4 \times 11^2.$$

Hence

$$u^3 = \frac{4 \pm i\sqrt{4 \times 11^2}}{2} = 2 \pm 11i.$$

If we just consider one of the values and use the polar form then we have

$$u^3 = 2 + 11i = r\mathrm{e}^{i\theta}, \quad r = \sqrt{5^3}, \quad 0 < \theta < \frac{\pi}{2}$$

and one possible value for u is

$$u = r^{1/3}\mathrm{e}^{i\theta/3}$$

and the corresponding value x is given by

$$\begin{aligned} x = u + \frac{p}{u} &= \sqrt{5}\mathrm{e}^{i\theta/3} + \frac{5}{\sqrt{5}}\mathrm{e}^{-i\theta/3} \\ &= 2\sqrt{5}\cos(\theta/3). \end{aligned}$$

Without a little investigation it is not immediately obvious that in this example this is the root $x = 4$ and next we verify this by considering powers of $2 + i$ as follows.

$$\begin{aligned} (2+i)^2 &= 3 + 4i, \\ (2+i)^3 &= (2+i)(2+i)^2 = (2+i)(3+4i) = 2 + 11i. \end{aligned}$$

Thus if $u^3 = 2 + 11i$ then one of the solutions is $u = 2 + i$ and

$$2 + i = \sqrt{5}(\cos\phi + i\sin\phi) \quad \text{with } \cos\phi = \frac{2}{\sqrt{5}}.$$

Hence with $\phi = \theta/3$ we have

$$x = 2\sqrt{5}\cos\phi = 4.$$

In this particular case we could have also more directly written

$$x = u + \frac{5}{u} = (2+i) + \frac{5}{2+i} = (2+i) + \frac{5(2-i)}{5} = 4.$$

As this example shows, Cardano's method does indeed lead to a root of the cubic but it requires an understanding of complex numbers to work in some cases when all the roots are real.

1.9 Roots: Solutions of $z^n = \zeta$

In sections 1.3 and 1.4 we showed that if we had the polar form $z = r\mathrm{e}^{i\theta}$ then $z^n = r^n \mathrm{e}^{ni\theta}$. We now consider the reverse operation in the sense that if ζ is known then what possible values of z gives that value, i.e. we wish to solve for z the equation

$$z^n - \zeta = 0.$$

As an observation, as we are finding the roots of a polynomial of degree n there can be at most n distinct values are these can be obtained as follows.

Let ζ have the polar form

$$\zeta = \rho\mathrm{e}^{i\alpha}, \quad \rho \ge 0, \quad \alpha = \mathrm{Arg}\,\zeta.$$

Hence

$$z^n = \zeta \quad \text{implies that } r^n \mathrm{e}^{ni\theta} = \rho\mathrm{e}^{i\alpha}.$$

By taking the absolute value of the expression we get

$$r^n = \rho, \quad r = \sqrt[n]{\rho} \ge 0$$

and thus all the solutions have the same magnitude. It remains then to find all solutions of

$$\mathrm{e}^{ni\theta} = \mathrm{e}^{i\alpha}.$$

Since $1 = \mathrm{e}^0 = \mathrm{e}^{2\pi i} = \mathrm{e}^{4\pi i} = \cdots$ we have

$$\mathrm{e}^{ni\theta} = \mathrm{e}^{i\alpha + 2k\pi i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

and hence all possible values of θ are

$$\frac{\alpha + 2k\pi}{n}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

There are infinitely many possible values of θ but this only generates n different values of z which we obtain by taking n consecutive values of k and our n roots are

$$\sqrt[n]{\rho}\exp(i\alpha/n)\exp(i2k\pi/n), \quad k = 0, 1, \ldots, n-1.$$

It is handy here to let

$$\omega = \mathrm{e}^{i2\pi/n}$$

which is a root of unity and let

$$z_0 = \sqrt[n]{\rho}\exp(i\alpha/n)$$

denote one of the roots. When this is done all the roots can now be neatly written as

$$z_0\omega^k, \quad k = 0, 1, \ldots, n-1$$

which gives n equally spaced points on a circle with centre at 0 and radius $|z_0| = \sqrt[n]{\rho}$ in the complex plane. This shows that to find all the roots of any number just involves finding one of the roots and then combining with the n roots of unity. In the case that

$\zeta = 1$ the n roots of unity, which are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, are shown in figure 1.2 in the cases of $n = 2$, 3, 4, 5, 6 and 7. The factorizations in the cases $n = 2, \ldots, 5$ correspond to

$$\begin{aligned}
z^2 - 1 &= (z-1)(z+1), \\
z^3 - 1 &= (z-1)(z-\beta_3)(z-\overline{\beta_3}), \quad \beta_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \\
z^4 - 1 &= (z-1)(z-i)(z+1)(z+i), \\
z^5 - 1 &= (z-1)(z-\beta_{5,1})(z-\overline{\beta_{5,1}})(z-\beta_{5,2})(z-\overline{\beta_{5,2}}), \\
\beta_{5,1} &= \cos(2\pi/5) + i\sin(2\pi/5), \ \beta_{5,2} = \cos(4\pi/5) + i\sin(4\pi/5).
\end{aligned}$$

In terms of expressions involving square roots the expression for the real and imaginary parts of ω become increasingly more complicated as n increases. At the time of writing these notes the wikipedia page on the roots of unity gives the expressions for $n = 2, 3, \ldots, 7$. As is shown above, we can quite easily express the roots when $n = 3$ in both polar and cartesian form. To get a feel for the increase in complexity as n increases it is worth briefly mentioning the case $n = 5$. This is manageable but the answer is not immediate. If $z = \mathrm{e}^{i2\pi/5}$ then the real part is $c = \cos(2\pi/5)$ and as

$$z^4 + z^3 + z^2 + z + 1 = 0$$

taking the real part gives

$$\cos(8\pi/5) + \cos(6\pi/5) + \cos(4\pi/5) + \cos(2\pi/5) + 1 = 0.$$

Using symmetry (or that cosine is even) and a double angle trig. formula we have

$$\cos(8\pi/5) = \cos(-2\pi/5) = \cos(2\pi/5) = c$$

and

$$\cos(6\pi/5) = \cos(-4\pi/5) = \cos(4\pi/5) = 2c^2 - 1.$$

Using this result in the previous one give us that c satisfies

$$2(c + 2c^2 - 1) + 1 = 4c^2 + 2c - 1 = 0, \quad c = \frac{-1+\sqrt{5}}{4}.$$

The other root of the quadratic gives $\cos(4\pi/5)$ and for the imaginary part

$$\sin(2\pi/5) = \sqrt{1-c^2} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

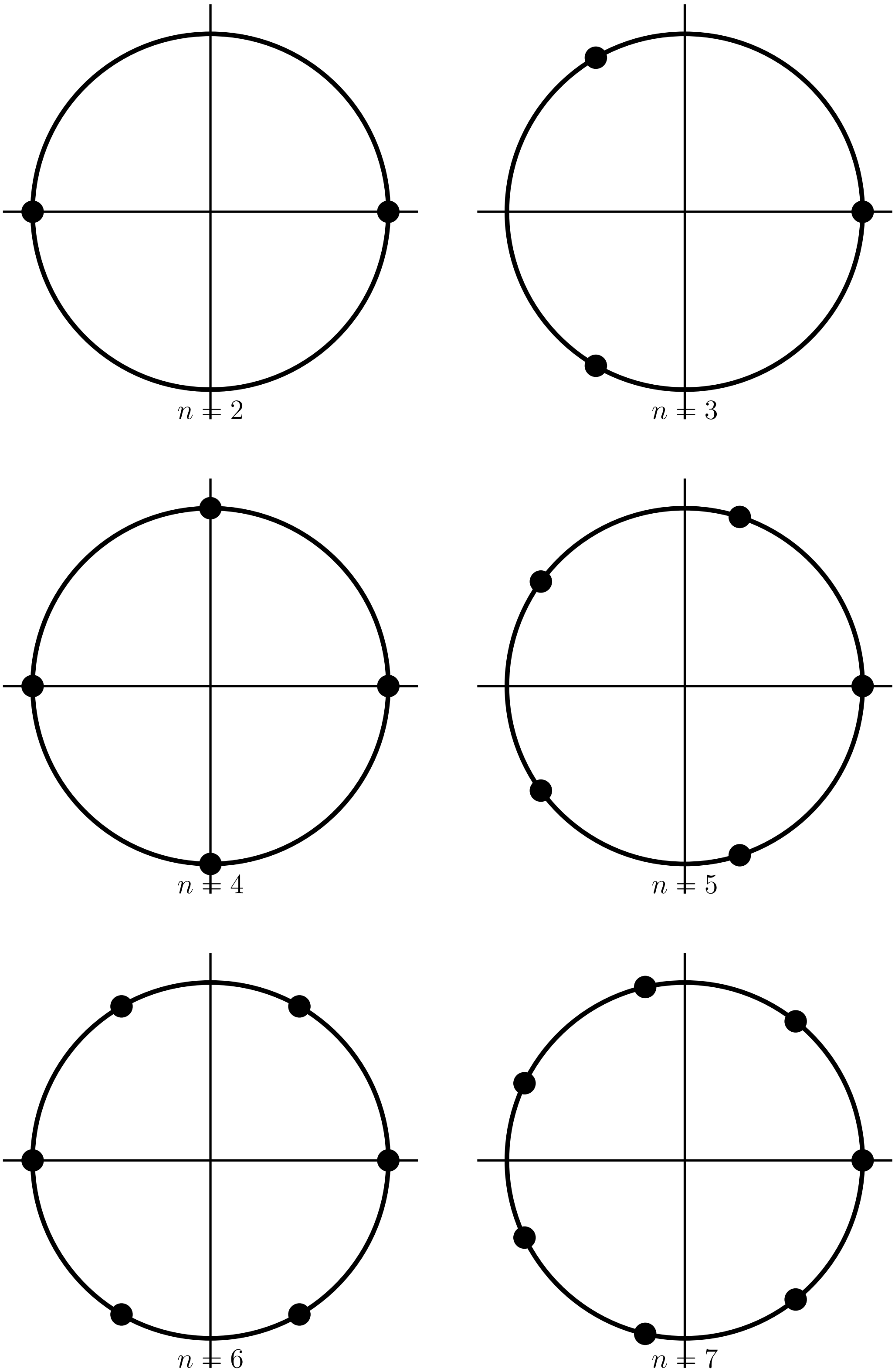


Figure 1.2: Roots of unity for $n = 2, 3, 4, 5, 6, 7$. All the circles have centre at 0 and radius= 1.

Chapter 2

Functions of a complex variable – the domain and continuity

This chapter will be covered fairly quickly in the lectures and it does not contain much material which is directly examinable. The purpose of the chapter is mainly to introduce a number of terms connected with functions of a complex variable and in particular to define what we mean by continuity which in turn needs the definition of a limit. As we will see, the requirement for a limit to exist in the complex sense is more restrictive than it is in the real sense and this will have many implications later on. Limits are needed for continuity and in the next chapter we need limits to define the term complex differentiable from which we will then define the term analytic. From the next chapter onwards most of the functions that we consider in this module are analytic at most points.

2.1 The domain of a function – terminology

In mathematics a function $f : A \to B$ is a rule which assigns to each element $a \in A$ an element $b = f(a) \in B$. The set A is known as the **domain of definition of** f and the set

$$\{f(a) \in B : \ a \in A\}$$

is the **image** of f on A which is a subset of B. The image is sometimes written as $f(A)$.

In your previous modules you would have considered sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ and typically the sets would have been intervals such as the following.

Unbounded open intervals: $\mathbb{R} = (-\infty, \infty)$, $(-\infty, a)$, (a, ∞) where $a \in \mathbb{R}$.

Bounded open interval: $(a, b) = \{x \in \mathbb{R}, \ a < x < b\}$.

Closed bounded interval: $[a, b] = \{x \in \mathbb{R}, \ a \le x \le b\}$.

The last two intervals given only differ in whether or not the end points are included and this is important in a number of results. For example, if $f : [a, b] \to \mathbb{R}$ is a continuous function then f is bounded, f has a minimum and maximum in $[a, b]$ and it takes every value between the extreme function values. The extreme values may occur at the end points and with intervals there are only two end points to consider.

When functions of a complex variable are considered the set A is now a subset of $\mathbb{C}$ and now there a few more possibilities for the sets that can be considered which is discussed below. The extra complication in considering subsets of $\mathbb{C}$ compared with intervals in $\mathbb{R}$ is that the "boundary" of the set is now a curve in $\mathbb{C}$ whereas we only had the two end points when we considered intervals such as (a, b). The following is a list of terms to describe the types of sets in $\mathbb{C}$ that we will be considering leading to what will be meant by a "domain" and what will be meant as a "region".

An **open disk** is a set of the form $\{z \in \mathbb{C} : \ |z - z_0| < \rho\}$. Here z_0 is the centre of the disk and $\rho > 0$ is the radius.

The **unit disk** is $\{z \in \mathbb{C} : \ |z| < 1\}$.

A **neighbourhood** of a point z_0 means a disk of the form $\{z \in \mathbb{C} : \ |z - z_0| < \rho\}$ for some $\rho > 0$.

A point $z_0 \in A$ is said to be an **interior point** of A if there is a neighbourhood of z_0 which is contained in A, i.e. for sufficiently small $\rho > 0$ we have $\{z \in \mathbb{C} : \ |z - z_0| < \rho\} \subset A$.

A set in $\mathbb{C}$ is **open** if every point is an interior point.

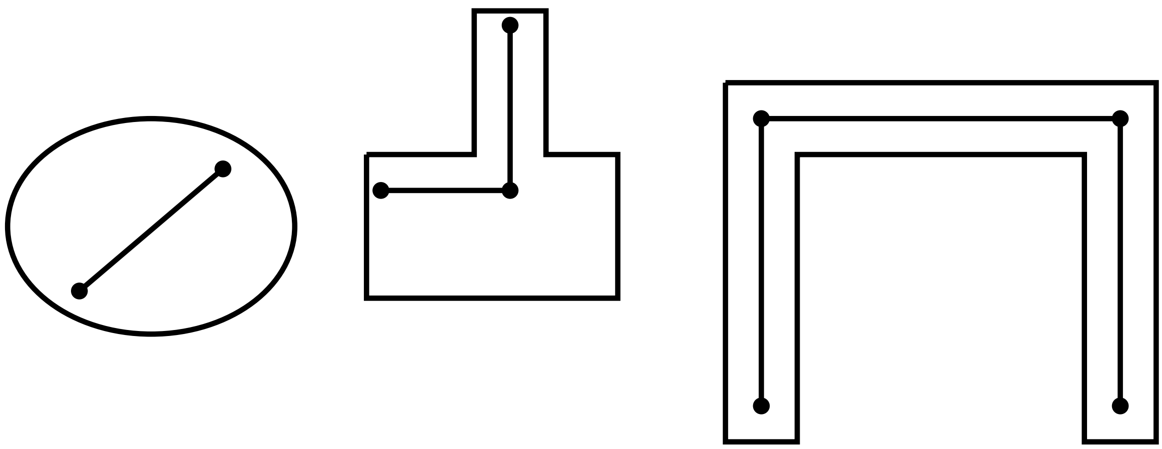
Given a set A, $z_0 \in \mathbb{C}$ is a **boundary point** if every neighbourhood of z_0 contains points which are in A and also contains points which are not in A.

The **boundary** of A is the set of all boundary points.

In a moment we define what we mean by a connected region which in turn needs the definition of a polygonal path.

Let $w_1, w_2, \ldots, w_{n+1}$ be points in $\mathbb{C}$ and let I_k be the straight line segment joining w_k to w_{k+1}. The successive line segments $I_1, I_2, \ldots, I_n$ is a **polygonal path** joining w_1 to w_{n+1}.

A set A is **connected** if every pair of points z_1 and z_2 in A can be joined by a polygonal path which is contained in A. For many sets that we consider only one segment is needed to join z_1 and z_2 but it is easy to create examples of sets where more than one line segment is needed as is shown below.



Throughout this module it should be immediately evident whether or not a set is connected but in mathematics such terms need to be precisely defined.

In this module a **domain** refers to an open connected set when we are considering a function of a complex variable.

A **region** is a bit more general and refers to a domain or to a domain together with some or all of the boundary points.

Throughout this module we consider functions defined on regions and there is still a bit more jargon associated with the regions concerned with whether or not they are bounded and also to their degree of connectivity.

A set A is **bounded** if there exists $R > 0$ such that the set is contained in the disk $\{z : \ |z| < R\}$.

An **unbounded** set is a set which is not bounded.

This is not a precise mathematical definition but for the purpose of this section a domain is **simply connected** if it does not contain any holes and it is **multi-connected** otherwise. In this module we will consider an annulus which is an example of a doubly connected domain and this is in the following list of examples.

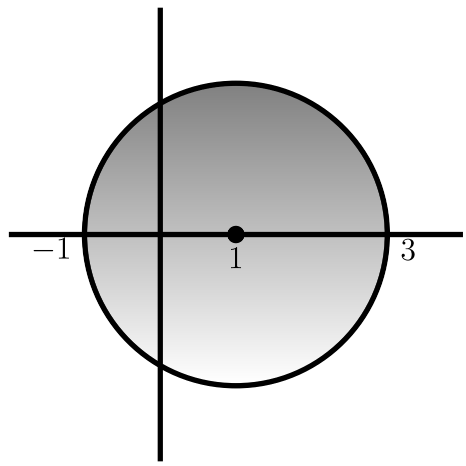
Some examples of subsets of $\mathbb{C}$

We consider now some examples and give diagrams of domains which are bounded.

1. $A = \mathbb{C}$, i.e. the entire complex plane. This is an unbounded domain. We will often consider functions defined on $\mathbb{C}$ and in the previous chapter there were several examples, e.g. polynomials and the exponential function.

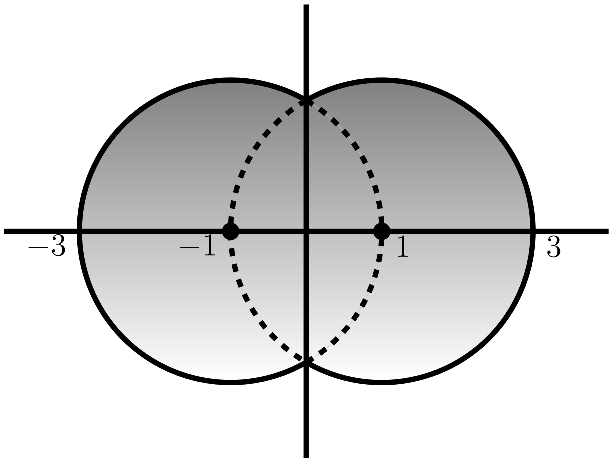
2. The real line $\mathbb{R}$. This is not a domain in $\mathbb{C}$ in the sense defined above as any neighbourhood of $x \in \mathbb{R}$ contains points with non-zero imaginary part which are not in $\mathbb{R}$.

3. A disk, e.g. $\{z \in \mathbb{C} : \ |z - 1| < 2\}$, is one of the simplest bounded simply connected domains. The boundary of a disk is a circle. The disk is hence the domain which is interior to the circle



4. If we take the union of two or more disks then we get a domain provided they all intersect. The following is hence a simply connected domain.

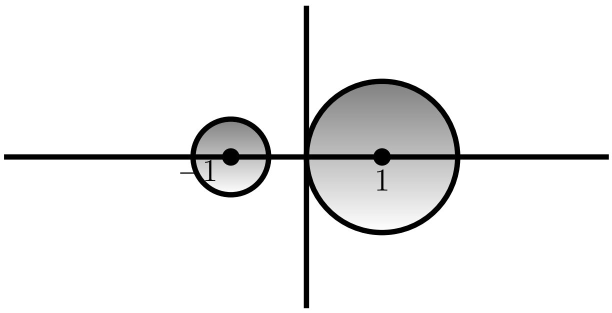
$$A = \{z : \ |z - 1| < 2\} \cup \{z : \ |z + 1| < 2\}$$



However the following set is not connected

$$A = \{z : \ |z-1| < 1\} \cup \{z : \ |z+1| < 0.5\}.$$

The region is not connected as we cannot join points in the left hand disk with points in the right hand disk by a polygonal path which does not leave the set A.

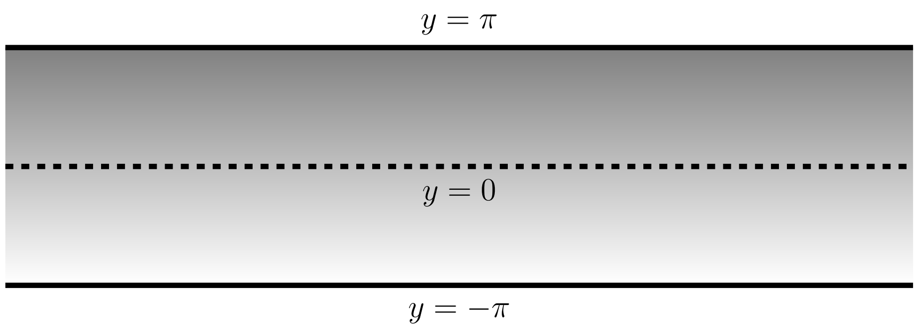


Throughout this module we will just consider connected sets.

5. The infinite strip $A = \{z = x + iy : \ -\infty < x < \infty, \ -\pi < y \le \pi\}$ is an unbounded region. It does not qualify as a domain as points with $y = \pi$ are not interior points. This is the natural region to consider for the exponential function

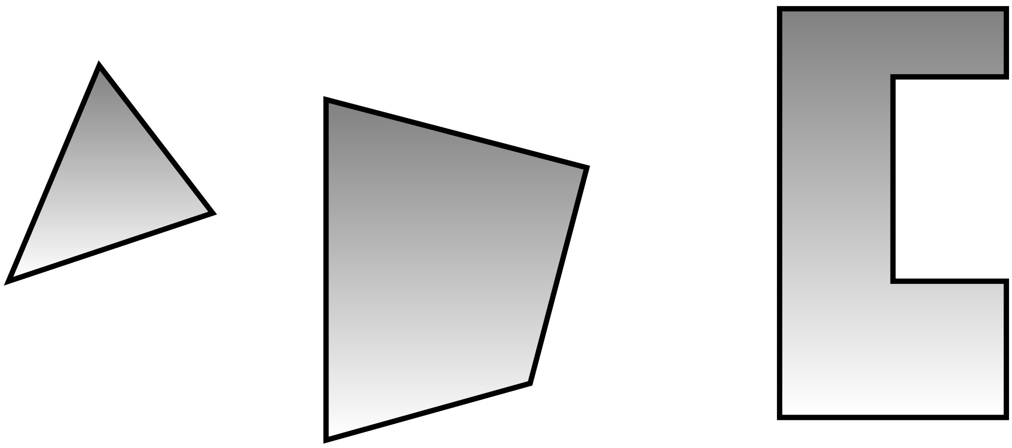
$$\mathrm{e}^{x+iy} = \exp(x+iy) = \exp(x)(\cos y + i \sin y)$$

which is periodic with period $2\pi i$, i.e. $\exp(z + 2\pi i) = \exp(z)$ for all $z \in \mathbb{C}$.



The infinite strip is actually an example of an unbounded polygonal region in that the boundary is union of straight lines.

6. The interior of **polygons** gives us domains and examples with 3, 4 and 8 sides are shown below where in all the cases shown the domains are bounded and simply connected. Note the terminology here, the polygon is the boundary and the region interior to it is the bounded domain.



There is a formula, known as the Schwarz-Christoffel formula, for mapping the upper half plane onto the interior of a polygon (bounded or unbounded).

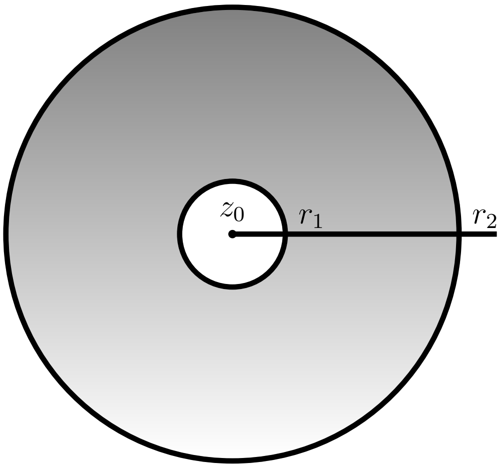
7. An **annulus** is a domain of the form

$$A = \{z \in \mathbb{C} : \; r_1 < |z - z_0| < r_2\},$$

i.e. it is region between two circles with the same centre. The region is bounded if r_2 is finite. The domain is not simply connected as it has a hole. This type of domain will be considered in this module when isolated singularities are considered, for example we will consider functions such as $g(z) = 1/(z - z_0)$ in an annulus and more generally one of the topics of the module is **Laurent series** which involves series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Note that a power series is a special case of a Laurent series corresponding to $a_n = 0$ for $n < 0$. As we will see later in the module when the inner radius $r_1 = 0$ the coefficient a_{-1} is known as the **residue of** $f(z)$ **at** $z = z_0$ and it is what we are most interested in when we consider integration along closed paths in the complex plane. When $r_1 > 0$ we never get arbitrary close to the centre z_0 and in this case we just get a way of representing the function in such a region.



2.2 The domain implied by the formula for f

In many cases the domain of a function is implied once the formula for f is given as the convention is to take the domain as the largest it can be for which the formula makes sense. For example, if

$$f(z) = \frac{1}{z}$$

then this makes sense for all $z \in \mathbb{C}$ except for $z = 0$, i.e. we have the annulus

$$\{z : \ 0 < |z| < \infty\}.$$

Similarly, if

$$f(z) = \frac{z^2 + z + 1}{(z-1)(z-2)^2(z-3)^3}$$

then this makes sense for all $z \in \mathbb{C}$ except for the points $z = 1$, $z = 2$ and $z = 3$ at which the denominator is 0. As other examples,

$$\begin{aligned} \mathrm{e}^{x+iy} = \exp(x+iy) &= \exp(x)(\cos y + i \sin y), \\ \cos(z) &= \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}, \\ \sin(z) &= \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i} \end{aligned}$$

are all defined for all $z \in \mathbb{C}$. The function

$$\cot z = \frac{\cos z}{\sin z}$$

is defined for all $z \in \mathbb{C}$ except at the points where $\sin z = 0$ and these are the points $\pm k\pi$, $k = 0, 1, 2, \ldots$.

2.3 Plotting a function of a complex variable

When you consider a real valued function of one variable you can graphically represent the function in two dimensions with the x-direction for the dependent variable and the y-direction for the function value and typically we write $y = f(x)$. For example the cubic $f(x) = (x^3 - 15x - 4)/10$ considered in the discussion of Cardano's method for finding the roots of cubics was represented in this way and it shown again in figure 2.1.

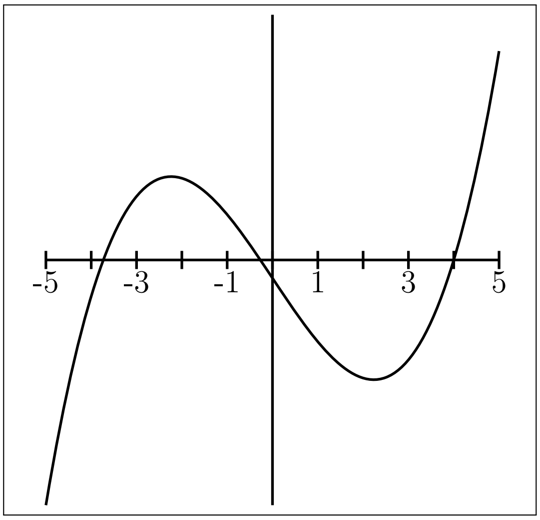


Figure 2.1: A plot of $y = (x^3 - 15x - 4)/10$ on $-5 \le x \le 5$.

It is more complicated to attempt to represent a complex valued function $w = f(z)$ of a complex variable z as we now need a plane to represent z and another plane to represent w. If we consider the real and imaginary part of z and w, i.e.

$$\begin{aligned} z &= x + iy, \\ w = f(z) &= u + iv, \end{aligned}$$

then u and v are two real valued functions of x and y, i.e.

$$\begin{aligned} u &= u(x, y), \\ v &= v(x, y). \end{aligned}$$

One possibility is to attempt to show a surface for u and to show another surface for v. An alternative to this is to give two copies of the complex plane and to give a curve or curves in the z-plane and to show the image of the curve or curves in the w-plane. We consider this approach next in the case of two functions.

Plotting $f(z) = z^2$

In this case it is better to take a region which does not include both a point $z \neq 0$ and $-z$ as these both map to the same $w = f(z)$. With this in mind the plots given in figure 2.2

show a radial mesh of part of the unit disk in the z-plane in the left hand side plot with image in the w-plane in the right hand side plot.

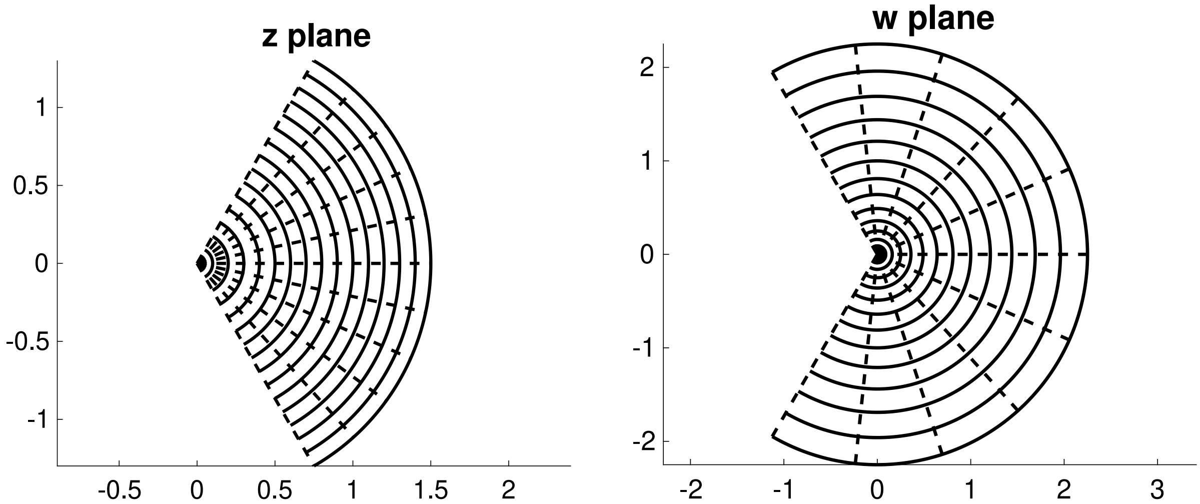


Figure 2.2: $w = f(z) = z^2$, $z = r\mathrm{e}^{i\theta}$, $|\theta| \le \pi/3$, $r \le 1.5$.

The radial lines in the z-plane are the straight lines with the origin as an end point and these map to radial lines in w-plane and the angle between any two radial lines is doubled. In this case the circles shown in the z-plane map to circles in the w-plane with a different radius although note that this is because all the circles in the z-plane have the origin as the centre. In all cases the radial lines and the circles are orthogonal where they intersect in both planes and thus the angle between curves is preserved everywhere except for the radial lines which intersect at 0. A mapping which preserves angles is known as a **conformal mapping** and thus $f(z) = z^2$ is a conformal mapping at all points with the exception of $z = 0$.

In figure 2.3 we similarly show a radial mesh of a disk with centre at 1 and radius 1/2 in the z-plane together with the image mesh in the w-plane.

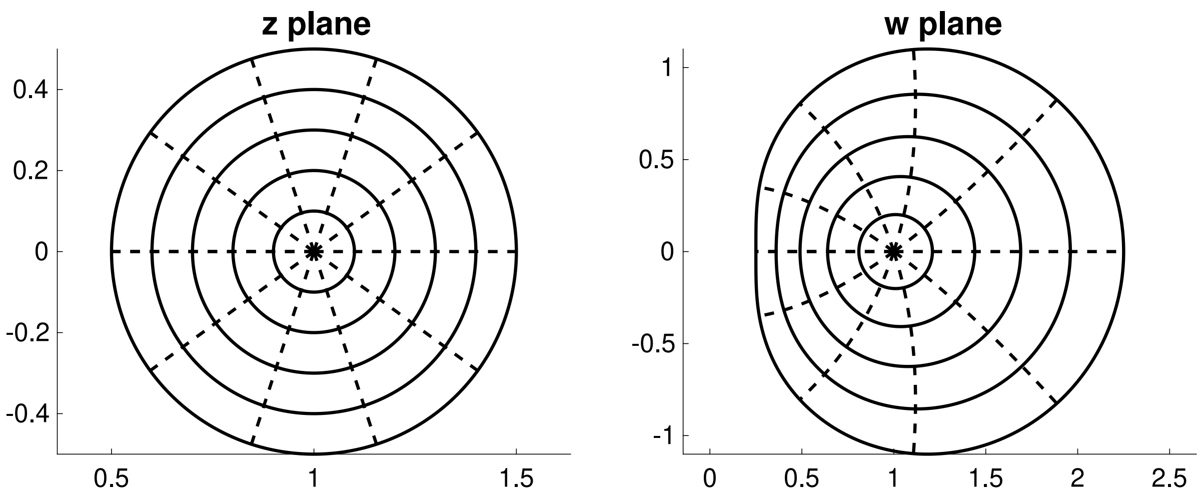


Figure 2.3: $w = f(z) = z^2$, $|z - 1| \le 0.5$.

In this case the image of the circles in the z-plane are not circles in the w-plane although the circle with the smallest radius surrounding 1 is “close” to a circle and this can be quite easily explained as follows. Firstly, with any polynomial $f(z)$ of degree 2 we have

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2}(z-1)^2$$

and thus in this case

$$z^2 = 1 + 2(z-1) + (z-1)^2 \approx 1 + 2(z-1)$$

when z is close to 1. The function $f_1(z) = 1 + 2(z-1)$ maps a circle centered at 1 in the z-plane to a circle centred at 1 in the w-plane and thus z^2 approximately does this when z is close to 1.

Again the plots suggest that angles are preserved at all intersection points and this can be proved to be the case.

Plotting $f(z) = (z - z_0)/(1 - \overline{z_0} z)$

Given some point $z_0 \in \mathbb{C}$ the function

$$w = f(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

is a particular **bilinear function** (the ratio of two linear polynomials). The term **Möbius transformation** is also used for functions of this type. If $|z_0| < 1$ and we restrict z to the unit disk then the function is bounded on the unit disk as we have kept away from the point $z = 1/\overline{z_0}$ which has magnitude greater than 1. In figure 2.4 we show a radial mesh of the unit disk in the z-plane together with the image mesh in the w-plane. (The plot was generated using a computer program.)

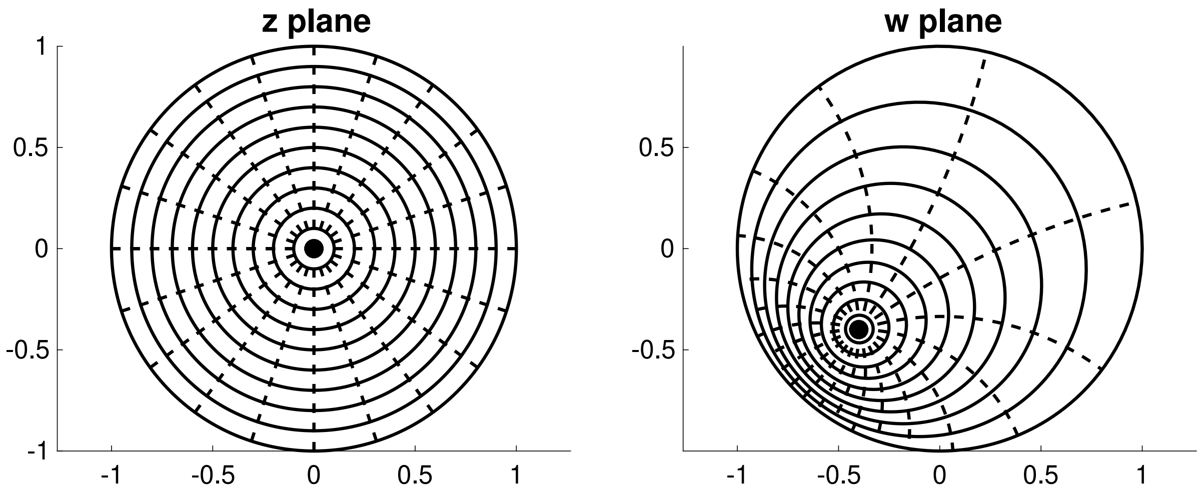


Figure 2.4: $w = f(z) = \dfrac{z - z_0}{1 - \overline{z_0} z}$, with $z_0 = 0.4(1+i)$ and $|z| \le 1$.

The plot suggests that the image of a circle is a circle and this can be proved to be the case and indeed in the w-plane all the curves shown are parts of circles or are parts of a straight line. On the first exercise sheet one of the questions asks you to verify that if $|z| = 1$ then $|w| = 1$ which explains why the unit circle maps to the unit circle.

2.4 The limit of a function and continuity

When a function of a real variable is considered the terms limit and continuity at a point are defined with the limit of $f(x)$ at $x = a$, when it exists, being the value that $f(x)$ tends to as x tends to a and continuity is concerned with a function having a limiting value which is the same as $f(a)$. Continuity is concerned with $f(x)$ being close to $f(a)$ whenever x is close to a and the 'closeness' part is given in terms of $\epsilon > 0$ and $\delta > 0$. We can similarly define these terms for complex valued functions of a complex variable with the distance between values being the absolute value.

Definition 2.4.1 The limit of $f(z)$ as $z \to z_0$. *Let f be defined in a neighbourhood of z_0 and let $f_0 \in \mathbb{C}$. If for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that*

$$|f(z) - f_0| < \epsilon \quad \textit{for all } z \textit{ satisfying } 0 < |z - z_0| < \delta$$

then we say that

$$\lim_{z \to z_0} f(z) = f_0.$$

Definition 2.4.2 The limit of $f(z)$ as $z \to \infty$. *Let f be defined in a region of the form $\{z : \ |z| > \rho\}$. If for every $\epsilon > 0$ there exists a real number $r > 0$ such that*

$$|f(z) - f_0| < \epsilon \quad \textit{for all } z \textit{ satisfying } |z| > r$$

then we say that

$$\lim_{z \to \infty} f(z) = f_0.$$

As an example of a function having a limit at ∞ consider the ratio of two polynomials when the denominator has at least the degree of the numerator, e.g.

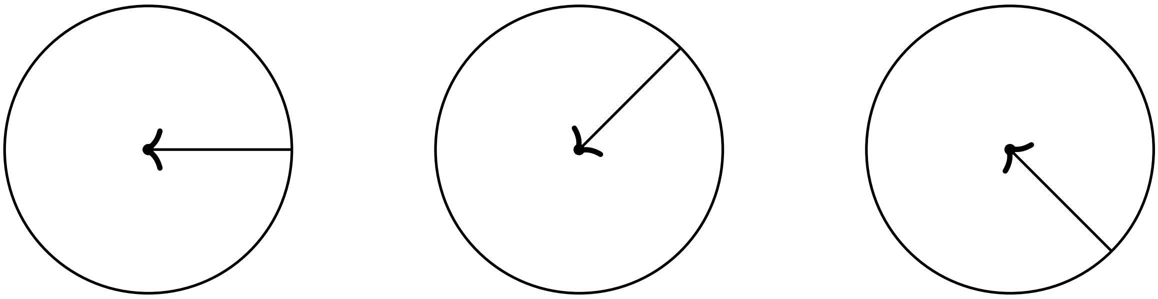
$$\frac{1}{z} \to 0 \quad \text{as } z \to \infty \qquad \text{and} \quad \frac{z+1}{2z+1} = \frac{1 + (1/z)}{2 + (1/z)} \to \frac{1}{2} \quad \text{as } z \to \infty.$$

Note that we did not need to use ϵ and δ to get these limiting values.

Definition 2.4.3 Continuity. *A function $w = f(z)$ is continuous at $z = z_0$ provided $f(z_0)$ is defined and*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The definitions in the complex case hence involves statements which are the same as are used in the real case apart from now having z_0, f_0, z and $f(z)$ as complex numbers and with absolute value now meaning the absolute value of a complex number. This last observation about the quantities now being complex numbers makes the conditions for a function to have a limit and to be continuous a bit stricter a requirement on f than in the real case as there are now more possibilities as to how $z \to z_0$ and the limit value must be independent of this. To attempt to visualise some of the possibilities for the trajectory of z as it approaches z_0 consider the following. The trajectory could be along any radial line, i.e. $z(t) = z_0 + te^{i\alpha}$, as $t \to 0$ for any $-\pi < \alpha \le \pi$, or the trajectory might be a spiral of the form $z(t) = z_0 + te^{i\alpha t}$ as $t \to 0$ for any $\alpha \in \mathbb{R}$. Trajectories of this type are shown in figure 2.5.



Examples of approaching a point along a radial line.

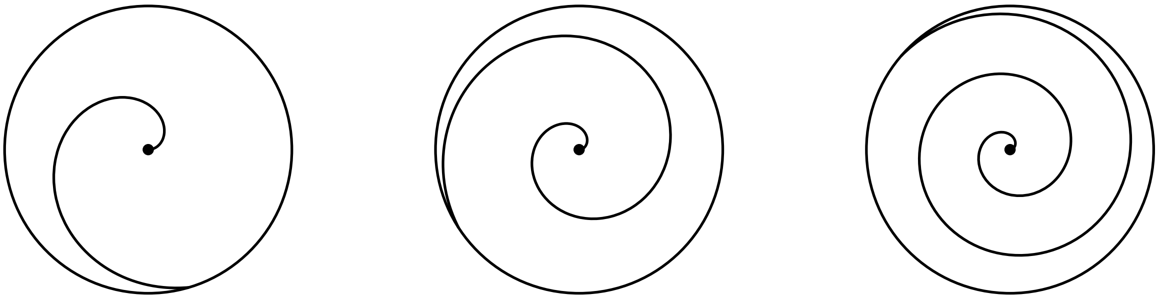


Figure 2.5: Examples of approaching a point along various spirals.

In the introduction chapter it was shown that how z tends a point can make a difference as was the case with

$$f(z) = \exp(-1/z^2).$$

If we approach $z = 0$ along the real axis then we have $f(x) \to 0$ as $x \to 0$ ($x \in \mathbb{R}$) but if we approach $z = 0$ along the imaginary axis then we have $f(iy) \to \infty$ as $y \to 0$ ($y \in \mathbb{R}$). As a function of a complex variable this function does not have a limit as $z \to 0$ and it is not bounded either.

For another example of a function which has already been mentioned which does not have a limit as $z \to 0$ we have $\operatorname{Arg} z$.

Most of the functions considered in this module are continuous at most points and the proofs are very similar to the proof in the case of functions of a real variable and these are not repeated here. For example,

$$\begin{aligned} f_1(z) &= z, \\ f_2(z) &= \exp(z), \end{aligned}$$

are both continuous on $\mathbb{C}$. As in the real case, once we have a few standard functions which are continuous then these can be combined in various ways to prove that many more functions have the continuity property. We have the following involving adding, multiplying and dividing.

Theorem 2.4.1 *Suppose that $f(z)$ and $g(z)$ are continuous at z_0. Then*

(i) $f(z) \pm g(z)$ and $f(z)g(z)$ are continuous at z_0.

(ii) $f(z)/g(z)$ *is continuous at* z_0 *provided* $g(z_0) \neq 0$.

We can also consider functions of a function.

Theorem 2.4.2 *Suppose that* $f(z)$ *is continuous at* z_0 *and* $g(z)$ *is continuous at* $f(z_0)$ *then* $g(f(z))$ *is continuous at* z_0.

Continuity of the function can also be deduced from the continuity of the real and imaginary parts and vice versa and we state this as a theorem.

Theorem 2.4.3 *Let* $f(z) = u(x,y) + iv(x,y)$. *If* f *is continuous at* $z_0 = x_0 + iy_0$ *then* u *and* v *are both continuous as functions on* $\mathbb{R}^2$ *at* (x_0, y_0). *Conversely, if* u *and* v *are both continuous at* (x_0, y_0) *then* f *is continuous at* $z_0 = x_0 + iy_0$.

As a consequence of the above results we have that any polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

is continuous on $\mathbb{C}$ and any rational function of the form

$$\frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}, \quad b_m \neq 0,\ m \geq 1,\ a_n \neq 0$$

is continuous except at points where the denominator is 0. The fundamental theorem of algebra tells us that there must be points where the denominator vanishes and unless such points are also zeros of the numerator this type of function will generally not have a limit at every $z \in \mathbb{C}$. For a function such as this the domain is all of $\mathbb{C}$ except for a finite number of points.

Examples

1. Let $z_0 \neq 0$ and let

$$f(z) = \begin{cases} \dfrac{z^4 - z_0^4}{z - z_0}, & z \neq z_0, \\ 4z_0^3, & z = z_0. \end{cases}$$

From the previous theorems this function is continuous at all values of $z \neq z_0$ as it is a combination of continuous functions. To determine whether or not it is also continuous at z_0 we need to consider the limit as $z \to z_0$. In the next chapter we will see that L'Hopital's rule that you may have used for real-valued differentiable functions can also be used in this complex case when we only want the limit but before then we show that we can get the limit by just considering properties of polynomials. To do this note that the numerator $z^4 - z_0^4$ does vanish at $z = z_0$ and hence $z - z_0$ is a factor and we need to determine the other factor. If you cannot spot what the other factor is then you might note that

$$z^4 - z_0^4 = z_0^4 \left(\left(\frac{z}{z_0} \right)^4 - 1 \right) \quad \text{and} \quad z - z_0 = z_0 \left(\left(\frac{z}{z_0} \right) - 1 \right)$$

Thus with $w = z/z_0$ we have the geometric series

$$w^4 - 1 = (w - 1)(w^3 + w^2 + w + 1)$$

and

$$\frac{z^4 - z_0^4}{z - z_0} = z_0^3(w^3 + w^2 + w + 1) \to 4z_0^3 \quad \text{as } z \to z_0$$

as $w \to 1$ as $z \to z_0$. The function value is the same as the limit and hence the function is continuous on $\mathbb{C}$. The function is just the cubic polynomial

$$f(z) = z^3 + z_0 z^2 + z_0^2 z + z_0^3.$$

2. Let

$$f(z) = \frac{\overline{z}}{z}.$$

As in the previous case this function is continuous for all $z \neq 0$ by the combination of continuous functions result and thus we just need to consider if a limit exists as $z \to 0$. If we let $z = t\mathrm{e}^{i\alpha}$ with $t \in \mathbb{R}$ and $t \neq 0$ then

$$\frac{\overline{z}}{z} = \frac{t\mathrm{e}^{-i\alpha}}{t\mathrm{e}^{i\alpha}} = \mathrm{e}^{-2i\alpha}.$$

The right hand side does not depend on t and in particular this tells us that we have a limit as $t\mathrm{e}^{i\alpha} \to 0$, i.e. when we approach 0 on a radial line. However the result depends on which radial line is used and every value on the unit circle is attained for some z which is arbitrarily close to $z = 0$. For a limit to exist there must be just one value and thus this function does not have a limit as $z \to 0$. This situation will be met again when we use limits to define complex differentiability and we are trying to determine which functions are analytic and which functions are not analytic.

Chapter 3

The complex derivative and analytic functions

3.1 Definition of an analytic function

In the previous chapter a neighbourhood of a point z_0 was defined and for a function f defined in such a neighbourhood the limit $\lim_{z\to z_0} f(z)$ and the continuity of f at z_0 were also defined. Continuity is about $f(z)$ being close to $f(z_0)$ whenever z is close to z_0 where in the complex case this means all z in a disk centred at z_0. In the previous chapter it was also noted that the continuity requirement is a stricter requirement on a function than is the case of continuity for a real valued function of a real variable. This chapter is concerned with differentiability in the complex sense which, as we will see, is a much stricter requirement on a function than is the corresponding case with real valued functions. We start with some definitions.

Definition 3.1.1 Complex derivative. *Let f be a complex valued function defined in a neighbourhood of z_0. The* **derivative of f at** z_0 *is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

provided the limit exists. When the limit exists f is said to be **differentiable at** z_0.

Definition 3.1.2 Analytic at a point. *A function f is* **analytic at** z_0 *if f is differentiable at all points in some neighbourhood of z_0.*

Note: The term **holomorphic** is also commonly used for this property. It might not seem much at this stage but to emphasise what has just been stated we do need the differentiable property to hold in a neighbourhood of a point for the function to be analytic at the point.

Definition 3.1.3 Analytic in a domain. *A function f is* **analytic in a domain** *if f is analytic at all points in the domain.*

Definition 3.1.4 Entire function. *A function $f : \mathbb{C} \to \mathbb{C}$ is an* **entire function** *if it is analytic on the whole complex plane $\mathbb{C}$.*

The expression used to define the derivative is the same as in the real case but remember that there are now more possibilities for how $h \to 0$ and the implication of this will be discussed shortly when the Cauchy Riemann equations are considered.

One immediate consequence of the above definitions is that if $f(z)$ is analytic at z_0 then it is continuous in a neighbourhood of z_0. This is because

$$f(z) - f(z_0) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0)$$

and we have a product with both terms having a limit as $z \to z_0$ and we can hence immediately deduce that $\lim_{z \to z_0} f(z) = f(z_0)$. It further follows that we can define the function

$$\lambda(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0, \\ 0, & z = z_0 \end{cases}$$

and this is continuous in the neighbourhood. Thus in particular this shows that

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0) \\ &\approx f(z_0) + f'(z_0)(z - z_0) \quad \text{when } z \approx z_0. \end{aligned}$$

3.2 Examples of functions which are analytic

We consider next some functions which can be shown to be analytic by directly using the definition and in all cases the details are virtually identical to the corresponding real case.

1. Let $f(z) := z$. We trivially have

$$\frac{f(z+h) - f(z)}{h} = \frac{z + h - z}{h} = 1.$$

   Thus $f'(z) = 1$ as in the real case.

2. Let $f(z) := z^2$. We have

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h \to 2z$$

   as $h \to 0$. Thus $f'(z) = 2z$ as in the real case.

3. Let $f(z) := z^n$ for any $n = 1, 2, 3, \ldots$.

$$f(z+h) - f(z) = (z+h)^n - z^n = nhz^{n-1} + \cdots + h^n$$

   by the binomial theorem. Again

$$\frac{f(z+h) - f(z)}{h} = nz^{n-1} + \mathcal{O}(h) \to nz^{n-1} \quad \text{as } h \to 0.$$

   Thus $f'(z) = nz^{n-1}$ as in the real case.

4. Let $f(z) = 1/z$. If $z \neq 0$ and h is sufficiently small such that $z + h \neq 0$ then

$$f(z+h) - f(z) = \frac{1}{z+h} - \frac{1}{z} = \frac{z-(z+h)}{(z+h)z} = \frac{-h}{(z+h)z}.$$

It then follows that

$$\frac{f(z+h)-f(z)}{h} = \frac{-1}{(z+h)z} \to -\frac{1}{z^2} \quad \text{as } h \to 0.$$

Again we have the same expression for the derivative as in the real case.

In all the above cases we obtain the same expression for the derivative as in the real case with virtually identical workings and hence it is perhaps not too surprising that the rules that you learned for differentiating finite sums, products, quotients and functions of a function also hold in the complex case and these are just stated next without any proofs.

3.3 Combining analytic functions

Theorem 3.3.1 Combining differentiable functions. *Let f and g be differentiable at z_0. We have the following.*

(i)

$$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0).$$

(ii)

$$(cf)'(z_0) = cf'(z_0)$$

for all constants $c \in \mathbb{C}$.

(iii)

$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

This is the product rule.

(iv)

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \quad \textit{if } g(z_0) \neq 0.$$

This is the quotient rule.

(v) Let now f be a function which is differentiable at $g(z_0)$. Then

$$\left.\frac{d}{dz} f(g(z))\right|_{z=z_0} = f'(g(z_0))g'(z_0).$$

This is known as the function of a function rule or the chain rule.

From the examples considered earlier and the above theorem we deduce that polynomials

$$a_n z^n + \cdots + a_1 z + a_0$$

are entire functions and rational functions of the form

$$\frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}, \quad a_n \neq 0, \quad b_m \neq 0$$

are analytic in $\mathbb{C}$ except at points at which the denominator is 0.

Another function that has been mentioned which is an entire function is the exponential function

$$\exp(x + iy) = \mathrm{e}^x(\cos y + i \sin y).$$

This will be shown later after we have considered what the analytic property means for the real and imaginary parts of f.

3.4 L'Hôpital's rule

When derivatives are available for two functions f and g we can often make progress in determining whether or not a limit exists of a quotient

$$\frac{f(z)}{g(z)}$$

in the case that $f(z_0) = g(z_0) = 0$. For $z \neq z_0$ we have

$$\frac{f(z)}{g(z)} = \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \frac{(f(z) - f(z_0))/(z - z_0)}{(g(z) - g(z_0))/(z - z_0)}.$$

Now if $g'(z_0) \neq 0$ then letting $z \to z_0$ gives

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

This result is known as L'Hôpital's rule. In term 2 we will extend the result to deal with cases when $f'(z_0) = g'(z_0) = 0$, and also possibly of higher derivatives, after we have shown that the analytic property of a function actually implies that derivatives of all order also exist and are analytic. This property of analytic functions requires results about integration along paths in the complex plane which will occupy a large part of the module involving the latter part of term 1 and much of term 2.

Example

Suppose we want to compute

$$\lim_{z \to i} \frac{i + z^{11}}{i + z^{15}}.$$

In this case

$$f(z) = i + z^{11} \quad \text{and} \quad g(z) = i + z^{15}$$

and we have $f(i) = g(i) = 0$. For the derivatives we have

$$f'(z) = 11z^{10} \quad \text{and} \quad g'(z) = 15z^{14}$$

and observe that $g'(i) \neq 0$. Thus

$$\lim_{z \to i} \frac{i + z^{11}}{i + z^{15}} = \frac{f'(i)}{g'(i)} = \frac{11i^{10}}{15i^{14}} = \frac{11}{15}, \quad \text{as } i^4 = 1.$$

3.5 Examples of functions which are not analytic anywhere

You do not need to search very far to find functions which are not analytic at any points with one of the simplest examples being

$$f(z) = \overline{z}.$$

This function is continuous but if we consider

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{h}}{h}$$

then we have an expression with no limit as $h \to 0$. This was the expression encountered in the example on page 2-13. We get a limit as $h \to 0$ along radial lines but the limit obtained depends on which radial line is used. For example if $h = h_1$ is real then we get 1 whilst if $h = ih_2$ is pure imaginary (which requires h_2 to be real) then we get -1. This is sufficient to explain why no limit exists as $h \to 0$.

Other examples that immediately follow of functions which are not analytic are

$$f(x + iy) = x \quad \text{and} \quad f(x + iy) = y.$$

In both cases these follow from the relations

$$2x = z + \overline{z}, \quad 2iy = z - \overline{z}.$$

If x or y were analytic then this immediately implies that $\overline{z}$ is analytic (as it is a combination involving z) but this contradicts what was done above in showing that $\overline{z}$ is not analytic anywhere.

The case $f(z) = \overline{z}$ suggests other examples of this type. If $f(z)$ is analytic at $\overline{z_0}$ and $f'(\overline{z_0}) \neq 0$ and we let $g(z) = f(\overline{z})$ then

$$\begin{aligned} \frac{g(z) - g(z_0)}{z - z_0} &= \frac{f(\overline{z}) - f(\overline{z_0})}{z - z_0} \\ &= \left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right) \left(\frac{\overline{z} - \overline{z_0}}{z - z_0} \right). \end{aligned}$$

The first term tends to $f'(\overline{z_0}) \neq 0$ as $z \to z_0$ but the second term does not have a limit as $z \to z_0$ and thus a limit does not exist. There will be a result at the end of this chapter which further explains this case.

3.6 The Cauchy Riemann equations

When continuity was discussed in the previous chapter we considered the real and imaginary parts of $f(z)$, i.e.

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy, \quad x, y, u, v \in \mathbb{R},$$

and we had that f is continuous at $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{R}$, if and only if u and v are continuous at (x_0, y_0). We now consider what existence of $f'(z_0)$ means for the first partial derivatives of u and v at (x_0, y_0).

Recall the definition

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

If we consider the limit taking $h = h_1$ being real then

$$\begin{aligned} f(z_0 + h_1) - f(z_0) &= u(x_0 + h_1, y_0) + iv(x_0 + h_1, y_0) \\ &\quad -(u(x_0, y_0) + iv(x_0, y_0)) \\ &= (u(x_0 + h_1, y_0) - u(x_0, y_0)) \\ &\quad +i(v(x_0 + h_1, y_0) - v(x_0, y_0)) \end{aligned}$$

and

$$\begin{aligned} f'(z_0) &= \lim_{h_1\to 0} \left(\frac{u(x_0 + h_1, y_0) - u(x_0, y_0)}{h_1} \right. \\ &\quad \left. +i\frac{v(x_0 + h_1, y_0) - v(x_0, y_0)}{h_1} \right) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right)(x_0, y_0). \end{aligned}$$

If we consider the limit taking $h = ih_2$ being purely imaginary then

$$\begin{aligned} f'(z_0) &= \lim_{h_2\to 0} \left(\frac{u(x_0, y_0 + h_2) - u(x_0, y_0)}{ih_2} \right. \\ &\quad \left. +i\frac{v(x_0, y_0 + h_2) - v(x_0, y_0)}{ih_2} \right) \\ &= \left(\frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right)(x_0, y_0) = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right)(x_0, y_0). \end{aligned}$$

Equating the two different representations for $f'(z_0)$ gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and these are known as the Cauchy-Riemann equations. We summarize what has just been shown in a theorem.

Theorem 3.6.1 *If $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0$ then at (x_0, y_0)*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence if we have expressions for u and v and these equations do not hold then we can immediately deduce that f is not complex differentiable. The Cauchy-Riemann equations are thus a **necessary condition** for f to be analytic.

If all the first partial derivatives of u and v are continuous at (x_0, y_0) then we next show that the converse is true in that if the Cauchy-Riemann equations are satisfied then $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0$. That is, subject to the continuity of the derivatives of u and v, the Cauchy Riemann equations are also a **sufficient condition**. This is a bit harder to do as we have to show that the limit existing using two different directions is sufficient to show that the same limit works with all possible directions.

Proof of the sufficiency of the Cauchy Riemann equations:

The proof of the sufficiency of the Cauchy Riemann equations is not examinable and in the following we give a short version, without all the details, to partially justify the result and then this is followed by a full text book type justification.

A partial justification using directional derivatives

The difficulty in proving the result is that when we consider the difference $f(z_0+h) - f(z_0)$ we need to consider this for all possible ways that $h \to 0$. Let $h = h_1 + ih_2$ denote the cartesian form of h with $h_1, h_2 \in \mathbb{R}$. We have

$$\begin{aligned} f(z_0 + h) - f(z_0) &= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) \\ &\quad + i(v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)). \end{aligned}$$

To express the difference of u at (x_0, y_0) with u at the near by point $(x_0 + h_1, y_0 + h_2)$ can be done in terms of the directional derivative of u in the direction of (h_1, h_2) and this in turn involves the dot product of ∇u with a unit vector in the direction involved. (If you have not met the term gradient and directional derivative before then we present in any case what this means in terms of what is covered in MA2612. For information the gradient vector of $u = u(x, y)$ is the vector

$$\nabla u = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix}.$$

Gradient vectors were part of a vector calculus module MA2741 which last ran in 2019/0.) The outcome of this is that when u is sufficiently smooth we can write

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = \left(h_1 \frac{\partial u}{\partial x} + h_2 \frac{\partial u}{\partial y} \right)(x_0, y_0) + \mathcal{O}(|h|^2).$$

We can similarly do this for the function v and when we consider $f = u + iv$ we have

$$\begin{aligned} f(z_0 + h) - f(z_0) &= \left(\left(h_1 \frac{\partial u}{\partial x} + h_2 \frac{\partial u}{\partial y} \right) \right. \\ &\quad \left. + i \left(h_1 \frac{\partial v}{\partial x} + h_2 \frac{\partial v}{\partial y} \right) \right)(x_0, y_0) + \mathcal{O}(|h|^2). \end{aligned}$$

Now we use the Cauchy Riemann equations to express all the partial derivatives in terms of partial derivatives with respect to x and note that the expression has a factor of

$h = h_1 + ih_2$, that is we have

$$\begin{aligned} & f(z_0+h) - f(z_0) \\ & \quad = \left(\left(h_1 \frac{\partial u}{\partial x} - h_2 \frac{\partial v}{\partial x} \right) + i \left(h_1 \frac{\partial v}{\partial x} + h_2 \frac{\partial u}{\partial x} \right) \right) (x_0, y_0) + \mathcal{O}(|h|^2) \\ & \quad = (h_1 + ih_2) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (x_0, y_0) + \mathcal{O}(|h|^2). \end{aligned}$$

Dividing by $h = h_1 + ih_2$ and letting this tend to 0 gives

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (x_0, y_0).$$

We have shown that the limit exists and we have an expression for the limit.

A full text book type version of the proof

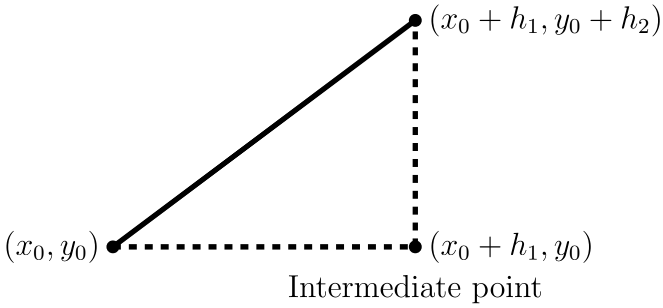
Taking one of the representations for $f'(z_0)$ we need to consider the following for arbitrary $h = h_1 + ih_2$

$$\frac{f(z_0+h) - f(z_0)}{h} - \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (x_0, y_0).$$

That is we have to show that our candidate for the limit, obtained by considering one direction, works as the limit when we consider all possible directions. Part of the expression just given involves

$$\begin{aligned} f(z_0+h) - f(z_0) &= u(x_0+h_1, y_0+h_2) - u(x_0, y_0) \\ &\quad + i(v(x_0+h_1, y_0+h_2) - v(x_0, y_0)) \end{aligned}$$

and we start by just considering the part of this which just involves u. As we have a change in both the x and y directions it helps to introduce an intermediate point for the comparison.



Now we use the mean value theorem twice to give

$$\begin{aligned} & (u(x_0+h_1, y_0+h_2) - u(x_0+h_1, y_0)) + (u(x_0+h_1, y_0) - u(x_0, y_0)) \\ = \; & h_2 \frac{\partial u}{\partial y}(x_0+h_1, y_0+\alpha_1 h_2) + h_1 \frac{\partial u}{\partial x}(x_0+\alpha_2 h_1, y_0) \end{aligned}$$

for some $\alpha_1, \alpha_2 \in (0,1)$. By the continuity of the partial derivatives we can relate them to the derivatives at (x_0, y_0) and write

$$\begin{aligned}\frac{\partial u}{\partial x}(x_0+\alpha_2 h_1, y_0) &= \frac{\partial u}{\partial x}(x_0,y_0)+\epsilon_2\\ \frac{\partial u}{\partial y}(x_0+h_1, y_0+\alpha_1 h_2) &= \frac{\partial u}{\partial y}(x_0,y_0)+\epsilon_1\end{aligned}$$

with $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $h \to 0$. Putting the terms involving u together gives

$$u(x_0+h_1,y_0+h_2)-u(x_0,y_0) = h_1\frac{\partial u}{\partial x}(x_0,y_0)+h_2\frac{\partial u}{\partial y}(x_0,y_0)+h_1\epsilon_2+h_2\epsilon_1.$$

Similar reasoning for v leads to the existence of ϵ_3 and ϵ_4 which both tend to 0 as $h \to 0$ with

$$v(x_0+h_1,y_0+h_2)-v(x_0,y_0) = h_1\frac{\partial v}{\partial x}(x_0,y_0)+h_2\frac{\partial v}{\partial y}(x_0,y_0)+h_1\epsilon_4+h_2\epsilon_3.$$

As the candidate for the limit only involves partial derivatives with respect to x we use the Cauchy Riemann equations to write all the partial derivatives in terms of derivatives with respect to x and the last two equations become

$$\begin{aligned}u(x_0+h_1,y_0+h_2)-u(x_0,y_0) &= \left(h_1\frac{\partial u}{\partial x}-h_2\frac{\partial v}{\partial x}\right)(x_0,y_0)+h_1\epsilon_2+h_2\epsilon_1,\\ v(x_0+h_1,y_0+h_2)-v(x_0,y_0) &= \left(h_1\frac{\partial v}{\partial x}+h_2\frac{\partial u}{\partial x}\right)(x_0,y_0)+h_1\epsilon_4+h_2\epsilon_3.\end{aligned}$$

Now we need to combine these parts appropriately to have the difference of $u+iv$ at the two points and this gives

$$\begin{aligned}&\left(h_1\frac{\partial u}{\partial x}-h_2\frac{\partial v}{\partial x}\right)(x_0,y_0)+i\left(h_1\frac{\partial v}{\partial x}+h_2\frac{\partial u}{\partial x}\right)(x_0,y_0)\\ &\quad=(h_1+ih_2)\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(x_0,y_0),\end{aligned}$$

i.e. the expression is the product of two terms. This is one of the key stages of the proof as $h = h_1 + ih_2$ and for f we thus have

$$\begin{aligned}f(z_0+h)-f(z_0) &= (h_1+ih_2)\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(x_0,y_0)\\ &\quad+(h_1\epsilon_2+h_2\epsilon_1)+i(h_1\epsilon_4+h_2\epsilon_3).\end{aligned}$$

Thus

$$\frac{f(z_0+h)-f(z_0)}{h}-\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(x_0,y_0)=\frac{(h_1\epsilon_2+h_2\epsilon_1)+i(h_1\epsilon_4+h_2\epsilon_3)}{h}$$

and it just remains to show that the right hand side tends to 0 as $h \to 0$. Now since $h = h_1 + ih_2$ and $|h|^2 = h_1^2 + h_2^2$ we have

$$\left|\frac{h_1}{h}\right| \le 1 \quad \text{and} \quad \left|\frac{h_2}{h}\right| \le 1$$

and the result follows by the triangle inequality and that $\epsilon_k \to 0$ for $k = 1, 2, 3, 4$, that is we have

$$\begin{aligned}\left|\frac{(h_1\epsilon_2 + h_2\epsilon_1) + i(h_1\epsilon_4 + h_2\epsilon_3)}{h}\right| &\le \left|\frac{h_1}{h}\right| |\epsilon_2| + \left|\frac{h_2}{h}\right| |\epsilon_1| + \left|\frac{h_1}{h}\right| |\epsilon_4| + \left|\frac{h_2}{h}\right| |\epsilon_3| \\ &\le |\epsilon_2| + |\epsilon_1| + |\epsilon_4| + |\epsilon_3| \to 0 \quad \text{as } h \to 0.\end{aligned}$$

Thus after more than 1.5 pages we have shown that our candidate for the limit is correct. We summarize what has been done in the following theorem.

Theorem 3.6.2 *When u and v have continuous partial derivatives in a domain D the function $f = u + iv$ is analytic on D if and only if the Cauchy Riemann equations are satisfied throughout D.*

Examples

1. If
$$f(z) = \overline{z} = x - iy \quad \text{then } u(x, y) = x \text{ and } v(x, y) = -y.$$
We immediately have for the partial derivatives
$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1.$$
As
$$\frac{\partial u}{\partial x} \ne \frac{\partial v}{\partial y}$$
one of the Cauchy Riemann equations is not satisfied which confirms what has already been done that this function is not analytic.

2. The complex exponential is defined by
$$\exp(x + iy) = \mathrm{e}^x(\cos y + i \sin y)$$
and thus
$$u(x, y) = \mathrm{e}^x \cos y, \quad \text{and} \quad v(x, y) = \mathrm{e}^x \sin y.$$
All the first partial derivatives are as follows.
$$\begin{aligned}\frac{\partial u}{\partial x} &= \mathrm{e}^x \cos y = u, & \frac{\partial v}{\partial y} &= \mathrm{e}^x \cos y = u, \\ \frac{\partial u}{\partial y} &= -\mathrm{e}^x \sin y = -v, & \frac{\partial u}{\partial y} &= \mathrm{e}^x \sin y = v.\end{aligned}$$
The Cauchy Riemann equations are satisfied everywhere and further if $f = u + iv$ then
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv = f(z)$$
which is the same as the corresponding result in the real case, i.e.
$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z.$$

Some remarks about the representations of $f'(z)$

By using the Cauchy Riemann equations we can represent the derivative f' of an analytic function $f = u + iv$ in several different ways as follows.

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \text{(only involving partial derivatives with respect to } x\text{)}, && (3.6.1)\\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \quad \text{(only involving partial derivatives with respect to } y\text{)}, && (3.6.2)\\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad \text{(only involving } u\text{)}, && (3.6.3)\\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}, \quad \text{(only involving } v\text{)}. && (3.6.4) \end{aligned}$$

$f'(z)$ is thus completely determined by the gradient of u (which is denoted by ∇u) and it is also completely determined by the gradient of v (which is denoted by ∇v).

The Cauchy Riemann equations in polar form

In some situations it is simpler to use polar coordinates r and θ instead of cartesian coordinates x and y and consider a function in the form

$$f(z) = u(x, y) + iv(x, y) = \tilde{u}(r, \theta) + i\tilde{v}(r, \theta).$$

To obtain the expression for $f'(z)$ in terms of the partial derivatives of $\tilde{u}$ and $\tilde{v}$ and the form of the Cauchy Riemann equations in this case we can directly use the definition of the derivative with suitable choices for h. At a value $z = r\mathrm{e}^{i\theta}$ a change in the radial direction (i.e. r direction) corresponds to

$$h = (r + h_r)\mathrm{e}^{i\theta} - r\mathrm{e}^{i\theta} = h_r\mathrm{e}^{i\theta}$$

and a change in the θ direction corresponds to

$$h = r\mathrm{e}^{i(\theta+\Delta\theta)} - r\mathrm{e}^{i\theta} = r\mathrm{e}^{i(\theta)}\left(\mathrm{e}^{i(\Delta\theta)} - 1\right) = r\mathrm{e}^{i(\theta)}\left(i\Delta\theta + \mathcal{O}\left((\Delta\theta)^2\right)\right).$$

The change in the r direction gives

$$f'(z) = \frac{1}{\mathrm{e}^{i\theta}}\left(\frac{\partial\tilde{u}}{\partial r} + i\frac{\partial\tilde{v}}{\partial r}\right) = \mathrm{e}^{-i\theta}\left(\frac{\partial\tilde{u}}{\partial r} + i\frac{\partial\tilde{v}}{\partial r}\right).$$

For the change in the θ direction we take

$$h = r\mathrm{e}^{i(\theta+\Delta\theta)} - r\mathrm{e}^{i\theta} = r\mathrm{e}^{i\theta}\left(i\Delta\theta + \mathcal{O}\left((\Delta\theta)^2\right)\right) = r\mathrm{e}^{i(\theta)}i\Delta\theta\left(1 + \mathcal{O}\left(\Delta\theta\right)\right)$$

and consider

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{ir\mathrm{e}^{i\theta}}\left(\frac{u(r, \theta+\Delta\theta) - u(r, \theta)}{\Delta\theta} + i\frac{v(r, \theta+\Delta\theta) - v(r, \theta)}{\Delta\theta}\right)\phi(\Delta\theta),$$

where

$$\phi(\Delta\theta) = \frac{1}{(1 + \mathcal{O}\left(\Delta\theta\right))} \to 1 \quad \text{as } \Delta\theta \to 0.$$

Thus
$$\frac{f(z+h)-f(z)}{h} \to \mathrm{e}^{-i\theta}\left(-\frac{i}{r}\frac{\partial \tilde{u}}{\partial \theta}+\frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}\right) \quad \text{as } \Delta\theta \to 0.$$

Both the expressions have the factor $\mathrm{e}^{-i\theta}$ and as they must be the same we equate the real and imaginary parts of the term in brackets to get the Cauchy Riemann equations in polar form
$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}, \quad \frac{1}{r}\frac{\partial \tilde{u}}{\partial \theta} = -\frac{\partial \tilde{v}}{\partial r}.$$

Example

With $z = r\mathrm{e}^{i\theta}$, $r > 0$ and taking $\theta = \operatorname{Arg} z$ let
$$f(z) = \ln r + i\theta$$
and consider this on the domain
$$G = \left\{ z = r\mathrm{e}^{i\theta} : \ r \neq 0 \text{ and } -\pi < \theta < \pi \right\}.$$

The domain is thus the complex plane with the non-positive real axis excluded (i.e. we are excluding 0 and the negative real axis). In the notation above we have
$$\tilde{u}(r,\theta) = \ln r \quad \text{and} \quad \tilde{v}(r,\theta) = \theta.$$

The Cauchy Riemann equations are satisfied as
$$\frac{\partial \tilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} = \frac{1}{r} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \theta} = \frac{\partial \tilde{v}}{\partial r} = 0$$
and hence $f(z)$ is analytic on G. Further note that
$$f'(z) = \mathrm{e}^{-i\theta}\left(\frac{\partial \tilde{u}}{\partial r} + i\frac{\partial \tilde{v}}{\partial r}\right) = \frac{1}{r}\mathrm{e}^{-i\theta} = \frac{1}{r\mathrm{e}^{i\theta}} = \frac{1}{z}.$$

$f(z)$ is the principal branch of the complex logarithm. Observe that $f'(z)$ is analytic for all $z \neq 0$ but for $f(z)$ to be analytic we needed to exclude the non-positive real axis to ensure the analytic property because $\mathrm{Im} f(z) = \mathrm{Arg} z$ has a jump discontinuity across this half line. To introduce the notation that we use later, the principal valued complex logarithm is defined by
$$\operatorname{Log} z := \ln |z| + i\operatorname{Arg} z.$$

3.7 Harmonic functions

Definition 3.7.1 *A function $\phi(x,y)$ is said to be* **harmonic** *in a domain D if it satisfies Laplace's equation*
$$\Delta\phi \equiv \nabla^2\phi \equiv \nabla \cdot (\nabla\phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

As we now demonstrate, there is a strong connection between harmonic functions and analytic functions which follows quite quickly by using the Cauchy Riemann equations. Suppose that we have an analytic function

$$f(z) = u(x, y) + iv(x, y)$$

and suppose that both u and v are twice continuously differentiable which in particular implies that mixed partial derivatives can be done in any order, i.e.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right). \tag{3.7.1}$$

This twice continuously differentiable property has been the case in all the examples encountered and as will be shown later in the module it is actually unnecessary to add this requirement as the analytic property of f actually implies that derivatives of all order are continuous. Now if we consider the property (3.7.1) for u and use the Cauchy Riemann equations then we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial y^2}.$$

Equating the last two expressions gives us that v satisfies Laplace's equation, i.e.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Similar reasoning gives that u satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Both u and v are harmonic functions with the function v said to be the **harmonic conjugate** of u.

Thus the above has shown that given an analytic function we get a pair of related harmonic functions. With some restrictions we can do the reverse of generating an analytic function with a given harmonic function u as its real part. The restriction which makes this possible is that the domain being considered should be simply connected (i.e. it has no holes) and the ease or difficulty of obtaining f is concerned with the ease or difficulty of obtaining the harmonic conjugate function v. When the domain is simply connected and the functions involved can be integrated relatively easily we can obtain v from u as follows. If we are given u then we can differentiate it and by using the Cauchy Riemann equations we have expressions for the partial derivatives of v, i.e.

$$\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x}. \end{aligned}$$

If this is going to determine v then it can only determine v up to an additive constant. Our procedure is to partially integrate the first relation with respect to x to get a form for v with the constant of integration being some function of y which we denote by $g(y)$. We can then partially differentiate with respect to y and use the other Cauchy Riemann equation to attempt to determine $g(y)$. We consider this approach with some examples.

Examples

1. Suppose

$$u(x,y) = (x^2 - y^2) + 4xy.$$

For the first partial derivatives we have

$$\frac{\partial u}{\partial x} = 2x + 4y, \quad \frac{\partial u}{\partial y} = -2y + 4x.$$

As a check, u does indeed satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0.$$

To determine v consider the Cauchy Riemann equation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-2y + 4x) = 2y - 4x.$$

Remember that we have partial derivatives here and thus if we integrate with respect to x then the 'constant term' is a function which does not involve x, i.e. any function of y. With this in mind, this gives

$$v = 2yx - 2x^2 + g(y)$$

for some function $g(y)$. We next differentiate this with respect to y and use the other Cauchy Riemann equation to give

$$\frac{\partial v}{\partial y} = 2x + g'(y) = \frac{\partial u}{\partial x} = 2x + 4y$$

which requires that

$$g'(y) = 4y, \quad \text{giving } g(y) = 2y^2 + A$$

where A is a constant. The general form of the harmonic conjugate v is

$$v(x,y) = 2yx - 2x^2 + g(y) = 2yx - 2x^2 + 2y^2 + A$$

and f is

$$\begin{aligned} f = u + iv &= (x^2 - y^2) + 4xy + i(2yx - 2x^2 + 2y^2 + A) \\ &= (x^2 - y^2)(1 - 2i) + 2xy(2 + i) + iA. \end{aligned}$$

This can be written in terms of z alone in several ways and the one that we present here involves considering the derivatives and a finite Maclaurin expansion. The details of this are as follows. Firstly note that we have already shown that $f(z)$ is analytic and hence differentiating in any direction gives the same result and in particular we can get expressions for the derivative by partially differentiating in the x direction. We can repeatedly differentiate in the x direction to give

$$\begin{aligned} f'(z) = \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x+4y) + i(2y-4x), \\ f''(z) = \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial y^2} = 2-4i. \end{aligned}$$

We can directly verify here that $f'(z)$ and $f''(z)$ are also analytic and in term 2 we will show that this is always the case. As the second derivative is a constant and the third derivative is 0 everywhere the function is a polynomial of degree 2. For the derivatives at 0 we have $f(0) = iA$, $f'(0) = 0$ and $f''(0) = 2 - 4i$. By a finite Maclaurin expansion the polynomial is hence

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 = iA + (1-2i)z^2.$$

The above use of the finite Maclaurin series representation was quite short as a way of expressing in term of z. An even shorter way of getting the result is given next although full justification in all cases of this type depends on something covered in term 2. In this particular case of a polynomial we do actually have full justification now as we explain. If we consider again our analytic function

$$f(x+iy) = (x^2-y^2)(1-2i) + 2xy(2+i) + iA$$

and just let $y = 0$ then we have

$$f(x) = (1-2i)x^2 + iA.$$

We get an analytic function if we just replace x by z, i.e.

$$\phi(z) = (1-2i)z^2 + iA$$

is analytic and the difference $f-\phi$ is analytic with the property that $f(z)-\phi(z) = 0$ when $y = 0$. In this case $f - \phi$ is a polynomial of degree 2 or less and thus can have at most 2 distinct zeros and as it zero on the real axis it is zero everywhere. This is the full justification when an analytic function is a polynomial. More generally in term 2 we show that an analytic function which is not identically zero only has zeros which are isolated. Thus if two analytic function agree on the real axis then they must agree everywhere where they are both defined.

2. Let

$$u(x,y) = \sin x \cosh y.$$

The first partial derivatives are

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \sinh y.$$

If you differentiate again then you get

$$\frac{\partial^2 u}{\partial x^2} = -u, \quad \frac{\partial^2 u}{\partial y^2} = u, \quad \text{and } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as a check that u is harmonic. To get the harmonic conjugate v we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sin x \sinh y.$$

Partially integrating with respect to x gives

$$v = \cos x \sinh y + g(y).$$

To determine $g(y)$ we use the other Cauchy Riemann equation, i.e.

$$\frac{\partial v}{\partial y} = \cos x \cosh y + g'(y) = \frac{\partial u}{\partial x} = \cos x \cosh y.$$

Thus $g'(y) = 0$ and $g(y) = A$ is a constant. The analytic function is

$$f(z) = u + iv = \sin x \cosh y + i \cos x \sinh y + iA.$$

As we will see later this function is $f(z) = \sin(x + iy) + iA = \sin z + iA$.

3.8 The $\partial f/\partial \overline{z} = 0$ form of the Cauchy Riemann equations

Recall that we showed that $f(z) = \overline{z}$ is not analytic anywhere and more generally if $f(z)$ is analytic then $g(z) = f(\overline{z})$ is not complex differentiable at any point for which $f'(\overline{z}) \neq 0$. These observations all suggest that for f to be analytic then it should be possible to express it in a way which does not involve $\overline{z}$. This is indeed the case in the following sense.

From $z = x + iy$ and $\overline{z} = x - iy$ we have

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}$$

and we can write

$$g(z, \overline{z}) = u(x, y) + iv(x, y) = u\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) + iv\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right).$$

For the manipulations we can consider g as independent function of z and $\overline{z}$ and use the chain rule to give

$$\begin{aligned} \frac{\partial g}{\partial \overline{z}} &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \overline{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \overline{z}}\right) \\ &= \left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}\right) + i\left(\frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{1}{2i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \end{aligned}$$

This shows that

$$\frac{\partial g}{\partial \overline{z}} = 0 \quad \text{if and only if} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It is in this sense that if we have a function of z and $\overline{z}$ then it needs to be a function of z alone to be analytic. This does not mean that we can always express an analytic function as a finite combination of standard functions as in all the examples considered so far but, as we will show later in the module, we can always represent such functions in a power series about a point z_0 which just involves powers of $z - z_0$.

Chapter 4

The elementary functions of a complex variable

Most books on functions of a complex variable contain a chapter or a section with a title of the form "the elementary functions". The term elementary functions is usually taken to mean a function of one variable which is built from a finite number of exponential, logarithm, constants, nth roots through composition and combination using the operations of $+$, $-$, $\times$ and $/$ (this is how it is defined in wikipedia). In chapters discussing such functions it is also common to also include polynomials, rational functions and this is what is done here. In all cases considered the functions to be described are functions $f(z)$ of the complex variable $z = x + iy$.

4.1 Polynomials

A polynomial is a function of the form

$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = \sum_{k=0}^{n} a_k z^k$$

where $a_0, a_1, \ldots, a_n$ are complex numbers. If $a_n \neq 0$ then the degree of the polynomial is n. $p_n(z)$ is an entire function, i.e. analytic in $\mathbb{C}$.

4.1.1 The finite Taylor series representation

If we differentiate repeatedly the above relation then we obtain

$$\begin{aligned} p_n'(z) &= a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1}, \\ p_n''(z) &= 2a_2 + 3(2)a_3 z + \cdots + n(n-1)a_n z^{n-2}, \\ \cdots & \quad \cdots \\ p_n^{(k)}(z) &= k! a_k + \cdots + n(n-1)\cdots(n-k+1)a_n z^{n-k}. \end{aligned}$$

Each differentiation gives a polynomial of one degree lower and if we differentiate $n+1$ times then we get the zero function. Also note that the coefficients are given by

$$a_k = \frac{p_n^{(k)}(0)}{k!}, \quad k = 0, 1, \ldots, n$$

so that

$$p_n(z) = \sum_{k=0}^{n} \frac{p_n^{(k)}(0)}{k!} z^k$$

which is a finite Maclaurin expansion. In some applications it is more convenient to expand about a point z_0 instead of 0 and we similarly have the representation

$$p_n(z) = \sum_{k=0}^{n} \frac{p_n^{(k)}(z_0)}{k!} (z - z_0)^k \tag{4.1.1}$$

which is a finite Taylor expansion about z_0. As an example, if we take $p_n(z) = z^3$ then

$$p_n(1) = 1, \quad p_n'(1) = 3, \quad p_n''(1) = 6 \quad \text{and} \quad p_n'''(1) = 6$$

giving

$$z^3 = 1 + 3(z - 1) + 3(z - 1)^2 + (z - 1)^3$$

which can also be deduced by considering the binomial expansion of

$$(1 + (z - 1))^3.$$

If we return to the representation (4.1.1) and we define

$$p_r(z) = \sum_{k=0}^{r} \frac{p_n^{(k)}(z_0)}{k!} (z - z_0)^k, \quad r = 0, 1, \ldots, n$$

and approximate $p_n(z)$ by $p_{m-1}(z)$ with $m \le n$ then the difference is of the form

$$p_n(z) - p_{m-1}(z) = \sum_{k=m}^{n} \frac{p_n^{(k)}(z_0)}{k!} (z - z_0)^k = (z - z_0)^m (\text{ polynomial }).$$

The difference has a zero of multiplicity of at least m at z_0 and the other factor is a polynomial.

4.1.2 A representation in terms of the roots

In other applications it is convenient to factorize the polynomial in the form

$$p_n(z) = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are known as the zeros or roots of $p_n(z)$. This representation is possible as a consequence of the fundamental theorem of algebra (proofs of this theorem will be seen later in the module although they will not be examinable). The n numbers $\alpha_1, \ldots, \alpha_n$ need not be distinct and if we let $\zeta_1, \ldots, \zeta_m$ denote the distinct zeros then

$$p_n(z) = a_n (z - \zeta_1)^{r_1} (z - \zeta_2)^{r_2} \cdots (z - \zeta_m)^{r_m}, \quad r_i \ge 1, \quad r_1 + r_2 + \cdots + r_m = n.$$

Here $p_n(z)$ has a zero at ζ_k of multiplicity $r_k \ge 1$. Another way of stating this is that there exists an analytic function $g(z)$ such that

$$p_n(z) = (z - \zeta_k)^{r_k} g(z), \quad g(\zeta_k) \ne 0, \tag{4.1.2}$$

and in the case of a polynomials of degree n the function $g(z)$ is a polynomial of degree $n - r_k$. If we differentiate (4.1.2) repeatedly then we get

$$\begin{aligned} p_n'(z) &= (z-\zeta_k)^{r_k} g' + r_k (z-\zeta_k)^{r_k-1} g, \\ p_n''(z) &= (z-\zeta_k)^{r_k} g'' + 2r_k (z-\zeta_k)^{r_k-1} g' + r_k(r_k-1)(z-\zeta_k)^{r_k-2} g, \\ \cdots \end{aligned}$$

and if we evaluate the derivatives at ζ_k then we have

$$p_n(\zeta_k) = p_n'(\zeta_k) = \cdots = p_n^{(r_k-1)}(\zeta_k) = 0.$$

As a note, later in the module we consider zeros of analytic functions $f(z)$ more generally and we again have that if ζ is a zero of multiplicity $r \ge 1$ then there exists another analytic function $g(z)$ such that

$$f(z) = (z-\zeta)^r g(z), \quad g(\zeta) \ne 0,$$

from which it follows that $f^{(k)}(\zeta) = 0$, $k = 0, 1, \ldots, r-1$. This result will follow once we have covered Taylor's series in term 2.

4.2 Rational functions

A rational function is a ratio of two polynomials, i.e. it is of the form

$$R_{m,n}(z) = \frac{p_m(z)}{q_n(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}, \quad a_m \ne 0, \quad b_n \ne 0.$$

The properties of such functions are mainly determined by the zeros of the denominator $q_n(z)$ as these are points where generally the function value will tend to infinity as z tends to such points with the precise outcome depending on whether or not $p_m(z)$ also has a zero at the same point. For example, if we note the identity

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$$

and define

$$R(z) = \frac{z-1}{z^n - 1}$$

then $R(z)$ is not defined at any of the n roots of 1 but it has a finite limit as $z \to 1$. The points where $R(z)$ is not defined are said to be singularities although the singularity at $z = 1$ is said to be removable and we can effectively just consider instead

$$\frac{1}{z^{n-1} + z^{n-2} + \cdots + z + 1}.$$

In what follows we will assume that the numerator and denominator have no common zeros.

4.2.1 Partial fraction representation

In previous study you are likely to have considered techniques for integrating rational functions and as first step you would have probably needed to construct a partial fraction representation. When real integrals are considered is it usually necessary to get the complete representation before the integration can be attempted. In the applications involving contour integrals in the complex plane to be considered later in this module it will turn out that we only need part of the representation and thus in the description that follows we consider techniques which enable us to just determine the parts that are needed. The parts that are needed in the contour integral application are the residues which is a term which we define later. Before general comments are given we consider the use of partial fractions for the following functions.

$$\begin{aligned} f_1(z) &= \frac{1}{z^2+1}, \\ f_2(z) &= \frac{z^3}{z^2+1}, \\ f_3(z) &= \frac{4}{(z^2+1)(z-1)^2}. \end{aligned}$$

In all cases we have $z^2+1=(z+i)(z-i)$ and we have singularities at $\pm i$. In the case of $f_3(z)$ we have a singularity at $z=1$ and denominator has a double zero at $z=1$.

The function $f_1(z)$

The partial fraction representation is of the form

$$f_1(z) = \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$$

where A and B are constants. To determine A and B one technique is to write the right hand side as

$$\frac{A(z-i)+B(z+i)}{(z+i)(z-i)}$$

and then equate the numerators to give

$$1 = A(z-i) + B(z+i).$$

We can equate constant terms and the coefficients of z which gives a 2×2 system for A and B but an easier way is to note that as the relation is true for all z we can just set z to specific values which eliminates a term and in particular it is best here to set $z=i$ and also to set $z=-i$ to give

$$1 = B(2i), \quad 1 = A(-2i)$$

giving

$$A = -\frac{1}{2i}, \quad B = \frac{1}{2i}.$$

As an alternative we can multiply $f_1(z)$ by $z-i$ or $z+i$ and note that

$$(z-i)f_1(z) = B + (z-i)(\text{a function analytic at } z=i) \to B \quad \text{as } z\to i$$

and similarly

$$(z+i)f_1(z) \to A \quad \text{as } z \to -i.$$

To actually get the values we can use L'Hopital's rule

$$A = \lim_{z\to -i} \frac{z+i}{z^2+1} = \left.\frac{1}{2z}\right|_{z=-i} = -\frac{1}{2i},$$
$$B = \lim_{z\to i} \frac{z-i}{z^2+1} = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i}.$$

This indicates that we can separately determine A and B. If we actually need the full expression then it is

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

The function $f_2(z)$

The extra complication here is that the numerator (i.e. z^3) is of a higher degree than the denominator (i.e. z^2+1) and we cannot write it in partial fraction form. However we do have

$$z^3 = (\text{degree 1 polynomial})(z^2+1) + (\text{remainder}),$$

where the remainder has degree less than 2, so that we do have a representation of the form

$$f_2(z) = (\text{degree 1 polynomial}) + \frac{A}{z+i} + \frac{B}{z-i}.$$

We can obtain A and B without needing to first determine the polynomial in the brackets by using the technique used in the case of $f_1(z)$.

$$\begin{aligned} A &= \lim_{z\to -i}(z+i)f_2(z) \\ &= \lim_{z\to -i} z^3 \lim_{z\to -i}\frac{(z+i)}{(z^2+1)} = (-i)^3 \frac{\left.\frac{\mathrm{d}}{\mathrm{d}z}(z+i)\right|_{z=-i}}{\left.\frac{\mathrm{d}}{\mathrm{d}z}(z^2+1)\right|_{z=-i}} \\ &= \frac{(-i)^3}{2(-i)} = -\frac{1}{2}. \end{aligned}$$

Similarly

$$B = \lim_{z\to i}(z-i)f_2(z) = \frac{i^3}{2i} = -\frac{1}{2}.$$

To actually get the degree 1 polynomial in the brackets note that

$$z^3 = z(z^2+1) - z$$

so that

$$\frac{z^3}{z^2+1} = z - \frac{z}{z^2+1} = z - \frac{1}{2}\left(\frac{1}{z+i} + \frac{1}{z-i}\right).$$

The function $f_3(z)$

The denominator in $f_3(z)$ has zeros of multiplicity 1 at the points $\pm i$, as in the previous cases, and it has zero of multiplicity 2 at the point $z=1$. This requires a partial fraction representation of the form

$$f_3(z) = \frac{A}{z+i} + \frac{B}{z-i} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}.$$

If we multiply by $z+i$ then we get

$$(z+i)f_3(z) = A + (z+i)(\text{a function analytic at } z=-i).$$

If we multiply by $z-i$ then we get

$$(z+i)f_3(z) = B + (z-i)(\text{a function analytic at } z=i).$$

If we multiply by $(z-1)^2$ then we get

$$(z-1)^2 f_3(z) = C_2 + C_1(z-1) + (z-i)^2(\text{a function analytic at } z=1). \tag{4.2.1}$$

These observations enables us to get A, B and C_2 by taking appropriate limits of these expressions. In the computatations we use properties of limits (specifically that the limit of a product is the product of the limits done separately) and we use L'Hopital's rule. To get A we have the following.

$$\begin{aligned} A &= \lim_{z\to -i}(z+i)f_3(z) = 4\lim_{z\to -i}\left(\frac{z+i}{z^2+1}\right)\frac{1}{(z-1)^2}, \\ &= 4\lim_{z\to -i}\left(\frac{z+i}{z^2+1}\right)\lim_{z\to -i}\frac{1}{(z-1)^2} = 4\frac{1}{2(-i)}\frac{1}{(-i-1)^2} \\ &= 4\left(\frac{1}{-2i}\right)\left(\frac{1}{2i}\right) = 1. \end{aligned}$$

To get B we have the following.

$$\begin{aligned} B &= \lim_{z\to i}(z-i)f_3(z) = 4\lim_{z\to i}\left(\frac{z-i}{z^2+1}\right)\frac{1}{(z-1)^2}, \\ &= 4\lim_{z\to i}\left(\frac{z-i}{z^2+1}\right)\lim_{z\to i}\frac{1}{(z-1)^2} = 4\frac{1}{2i}\frac{1}{(i-1)^2} \\ &= 4\left(\frac{1}{2i}\right)\left(\frac{1}{-2i}\right) = 1. \end{aligned}$$

To get C_2 we have the following.

$$C_2 = \lim_{z\to 1}(z-1)^2 f_3(z) = 4\lim_{z\to 1}\left(\frac{1}{z^2+1}\right) = 2.$$

Although it may not be the shortest technique to get C_1 once all the other coefficients have been found but it is worth mentioning that we can also get C_1 using limits. If we differentiate what is given in (4.2.1) then we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((z-1)^2 f_3(z)\right) = C_1 + (z-1)(\text{some function analytic at } z=1).$$

Hence

$$C_1 = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left((z-1)^2 f_3(z)\right).$$

Fortunately in this example the differentiation does not involve too much work as we have

$$(z-1)^2 f_3(z) = \frac{4}{z^2+1}, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left((z-1)^2 f_3(z)\right) = \frac{-8z}{(z^2+1)^2}$$

and

$$\left.\frac{-8z}{(z^2+1)^2}\right|_{z=1} = -2.$$

To summarise we have

$$f_3(z) = \frac{4}{(z^2+1)(z-1)^2} = \frac{1}{z+i} + \frac{1}{z-1} - \frac{2}{z-1} + \frac{2}{(z-1)^2}.$$

What this example has illustrated is that all the coefficients to find can be given in terms of limits. In this particular case the amount of workings needed could have been a bit shorter by instead first putting the following.

$$\begin{aligned} &\frac{A}{z+i} + \frac{B}{z-i} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2} \\ = \;& \frac{A(z-i)(z-1)^2 + B(z+i)(z-1)^2 + C_1(z-1)(z^2+1) + C_2(z^2+1)}{(z^2+1)(z-1)^2}. \end{aligned}$$

To be the same as $f_3(z)$ we need that for all z

$$1 = A(z-i)(z-1)^2 + B(z+i)(z-1)^2 + C_1(z-1)(z^2+1) + C_2(z^2+1).$$

Letting $z=-i$ then $z=i$ and then $z=1$ quickly gets the values A, B and C_2. Equating the coefficient of z^3 then gives us

$$A + B + C_1 = 0$$

from which we get C_1 as A and B have already been found. In fact this relation is fairly clear from the start when you consider the magnitude of the terms as $|z|$ gets large. When $|z|$ is large we must have that $|f_3(z)|$ is decaying like $4/|z|^4$ whereas the 4 terms in the representation are much larger with a decay of $\mathcal{O}(1/|z|)$ or $\mathcal{O}(1/|z|^2)$. Thus the coefficients of all the $\mathcal{O}(1/|z|)$ terms must be such that the terms nearly cancel for large $|z|$.

Summary of the f_1, f_2 and f_3 examples and some terminology

In the case of $f_1(z)$ we have

$$f_1(z) = \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}.$$

The function has **simple poles** at $\pm i$ and A and B are known as the **residues**. We can get the residues using

$$A = \lim_{z\to -i}(z+i)f_1(z), \quad B = \lim_{z\to i}(z-i)f_1(z).$$

In the case of $f_2(z)$ we can use a similar technique to just get the residues even though the expression is not in the form that we can directly construct a partial fraction representation.

In the case of $f_3(z)$ the situation is more complicated as the function has a **double pole** at the point 1 and the representation needs to be

$$f_3(z) = \frac{A}{z+i} + \frac{B}{z-i} + \frac{C_1}{z-1} + \frac{C_2}{(z-1)^2}.$$

The residues in this case are the coefficient of the simple pole terms, i.e. A, B and C_1 and we can get these from

$$\begin{aligned} A &= \lim_{z\to -i}(z+i)f_3(z), \\ B &= \lim_{z\to i}(z-i)f_3(z), \\ C_1 &= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-1)^2 f_3(z)\right). \end{aligned}$$

As a comment on the terminology, a function $f(z)$ is said to have a **pole** at a point z_0 if $1/f(z)$ is analytic at z_0 and has a zero at that point. A simple zero of $1/f(z)$ (i.e. a zero of multiplicity 1) gives a simple pole of $f(z)$ and similarly a double zero of $1/f(z)$ (i.e. a zero of multiplicity 2) gives a double pole of $f(z)$.

4.2.2 Any rational function with just simple poles

The previous examples indicate what to do in more general cases and thus suppose that the rational function is

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z-z_1)(z-z_2)\cdots(z-z_n), \tag{4.2.2}$$

where $p(z)$ is a polynomial. We are assuming here that $z_1, \ldots, z_n$ are distinct so that the function has simple poles at $z_1, \ldots, z_n$. As we have not given any restriction on the degree of $p(z)$ the representation is of the form

$$R(z) = (\text{polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z-z_k}$$

where A_k is the residue at z_k. To obtain A_k we have

$$A_k = \lim_{z\to z_k}(z-z_k)R(z) = \lim_{z\to z_k}\frac{(z-z_k)p(z)}{q(z)} = p(z_k)\lim_{z\to z_k}\frac{(z-z_k)}{q(z)} = \frac{p(z_k)}{q'(z_k)}.$$

The case when $p(z_k) = 0$ corresponds to $A_k = 0$ and this is the removable singularity case. In the partial fraction case the numerator needs to have a lower degree than the denominator, i.e. $\deg p(z) \le n-1$, and when this is the case we have

$$R(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n}\frac{A_k}{z-z_k}, \quad A_k = \frac{p(z_k)}{q'(z_k)}. \tag{4.2.3}$$

If you wish to fully justify that (4.2.3) is correct for all z then this can be done by the following argument. Consider the difference which we denote by

$$g(z) = \frac{p(z)}{q(z)} - \sum_{k=1}^{n} \frac{p(z_k)/q'(z_k)}{z - z_k},$$

and multiply by $q(z)$ and consider

$$q(z)g(z) = p(z) - \sum_{k=1}^{n} \frac{p(z_k)}{q'(z_k)} \left(\frac{q(z)}{z - z_k} \right). \tag{4.2.4}$$

From (4.2.2) the term $q(z)/(z-z_k)$ is just a product of $n-1$ factors of the form $z-z_i$ with limiting value of $q'(z_k)$ at z_k. As $\deg p(z) \le n-1$ the right hand side is a linear combination of polynomials of degree $n-1$ or less and thus the right hand side a polynomial of degree $n-1$ or less. When we consider this polynomial at the points $z_1, \ldots, z_n$ we get

$$\lim_{z \to z_k} q(z)g(z) = p(z_k) - \frac{p(z_k)}{q'(z_k)} q'(z_k) = 0.$$

Thus the right hand side of (4.2.4) is a polynomial of degree less than or equal to $n-1$ and it is 0 at the n distinct points $z_1, \ldots, z_n$ which implies that it is 0 everywhere.

4.2.3 When there is just 1 multiple pole

As the example with $f_3(z)$ showed, when there are multiple poles there is usually more effort to obtain the residues. However, if there is just one multiple pole then the derivation is much shorter as we can just write a Taylor polynomial for the numerator $p(z)$ with the expansion about the pole. If $p(z)$ has degree n and we have a pole of order r at z_1 then we have the following.

$$\frac{p(z)}{(z - z_1)^r} = \frac{p(z_1) + p'(z_1)(z - z_1) + \cdots + p^{(n)}(z_1)/n!(z - z_1)^n}{(z - z_1)^r}.$$

The residue at z_1 is

$$\frac{p^{(r-1)}(z_1)}{(r-1)!}.$$

4.2.4 The most general case

In the example involving the function $f_3(z)$ there was a double pole. In the most general case we can have a rational function of the form

$$R(z) = \frac{p(z)}{(z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}} \tag{4.2.5}$$

where $r_k \ge 1$ and where we assume that $p(z)$ is a polynomial which is not zero at any of the points z_k. The form of the representation about z_k needs poles of orders $1, 2, \ldots, r_k$ and we have

$$R(z) \;=\; (\text{poly}) + \left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}} \right) + \cdots + \left(\frac{A_{1,n}}{z - z_n} + \cdots + \frac{A_{r_n,n}}{(z - z_n)^{r_n}} \right).$$

To just get the residue at z_k we multiply by $(z-z_k)^{r_k}$ to get a function which is analytic at z_k, and to shorten the notation let $r=r_k$, $\zeta=z_k$ and let $B_1,\ldots,B_r$ denote $A_{1,k},\ldots,A_{r_k,k}$. With this notation we have

$$(z-\zeta)^r R(z) = B_r + B_{r-1}(z-\zeta) + \cdots + B_1(z-\zeta)^{r-1} + (z-\zeta)^r(\text{a function analytic at } z).$$

To extract the residue B_1 we differentiate $r-1$ times to eliminate the terms involving $B_r, B_{r-1},\ldots,B_2$ and take the limit as $z\to\zeta$ to give

$$\lim_{z\to\zeta} \frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}}\left((z-\zeta)^r R(z)\right) = (r-1)!B_1.$$

With similar workings we can get all the coefficients as

$$\lim_{z\to\zeta} \frac{\mathrm{d}^{r-j}}{\mathrm{d}z^{r-j}}\left((z-\zeta)^r R(z)\right) = (r-j)!B_j, \quad j=1,2,\ldots,r.$$

As this reasoning shows, to get the residue when there is a multiple pole you need to know the order of the pole (i.e. r in this case) and then do the above. In the case of $f_3(z)$ we just had $r=2$ and in the calculation we first simplified the expression to differentiate before we actually did the differentiation.

Fortunately, in this module most of the examples will just involve simple poles and poles of higher order will be not appear very often.

The following is not examinable, and the details are close to two pages, but if you wish to prove that the general partial fraction representation of $R(z)$ is actually the same as $R(z)$ there are a number of steps. At the moment we have just shown that if a representation is of the form given then we have given an expression for each coefficient. We wish to now show that the partial fraction representation and the original expression are the same for all z and to do this we can proceed as follows. Firstly, to be a partial fraction representation we need the degree of $p(z)$ to be less that the degree of $q(z)$. Let

$$\tilde{R}(z) = \left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}}{z-z_n} + \cdots + \frac{A_{r_n,n}}{(z-z_n)^{r_n}}\right)$$

denote the candidate for the partial fraction representation with the coefficients given as indicated above. As $R(z)$ and $\tilde{R}(z)$ are both rational functions the difference $R(z)-\tilde{R}(z)$ is also a rational function and the only possible places where the difference can have poles are the points $z_1,\ldots,z_n$. Now because of the degrees of the polynomials involved (the numerator has a lower degree than the denominator) we have that

$$R(z)-\tilde{R}(z)\to 0 \quad \text{as } |z|\to\infty. \tag{4.2.6}$$

Hence if we can show that $R(z)-\tilde{R}(z)$ has no poles then this implies that it is a polynomial and the observation (4.2.6) further implies that $R(z)-\tilde{R}(z)=0$. This last step is because non-constant polynomials are unbounded and if it is constant then the only possibility for the constant is 0.

We consider next the properties of $R(z)-\tilde{R}(z)$ near $z=z_1$ which gives us

$$R(z)-\tilde{R}(z) = R(z) - \left(\frac{A_{1,1}}{z-z_1} + \cdots + \frac{A_{r_1,1}}{(z-z_1)^{r_1}}\right) + (\text{a function analytic at } z_1). \tag{4.2.7}$$

Let

$$Q(z) = (z - z_2)^{r_2} \cdots (z - z_n)^{r_n}$$

so that the denominator can be written as

$$q(z) = (z - z_1)^{r_1} Q(z).$$

For part of (4.2.7) we have

$$
\begin{aligned}
& R(z) - \left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}} \right) && (4.2.8)\\
= & \frac{p(z)}{(z - z_1)^{r_1} Q(z)} - \left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}} \right) && (4.2.9)\\
= & \frac{1}{(z - z_1)^{r_1}} \left(\frac{p(z)}{Q(z)} - \left(A_{r_1,1} + A_{r_1-1,1}(z - z_1) + \cdots + A_{1,1}(z - z_1)^{r_1-1} \right) \right) && (4.2.10)\\
= & \frac{1}{(z - z_1)^{r_1}} \left(\frac{p(z)}{Q(z)} - T_1(z) \right) && (4.2.11)\\
= & \frac{1}{(z - z_1)^{r_1}} \left(\frac{p(z) - Q(z) T_1(z)}{Q(z)} \right) && (4.2.12)
\end{aligned}
$$

where

$$T_1(z) = A_{r_1,1} + A_{r_1-1,1}(z - z_1) + \cdots + A_{1,1}(z - z_1)^{r_1-1}.$$

The formula for the coefficients of $T_1(z)$ show that this is the Taylor polynomial of $p(z)/Q(z)$ about the point z_1, i.e.

$$\left(\frac{p}{Q} \right)^{(k)} (z_1) = T_1^{(k)}(z_1), \quad \text{for } k = 0, 1, \ldots, r_1 - 1.$$

Now by writing

$$p - T_1 Q = Q \left(\frac{p}{Q} - T_1 \right)$$

and differentiating once, twice and further we have

$$
\begin{aligned}
(p - T_1 Q)' &= Q \left(\frac{p}{Q} - T_1 \right)' + Q' \left(\frac{p}{Q} - T_1 \right)\\
(p - T_1 Q)'' &= Q \left(\frac{p}{Q} - T_1 \right)'' + 2Q' \left(\frac{p}{Q} - T_1 \right)' Q'' \left(\frac{p}{Q} - T_1 \right)\\
\cdots &
\end{aligned}
$$

The kth derivative of the polynomial $p - T_1 Q$ is a combination of $(p/Q - T_1)^{(j)}$, $j = 0, 1, \ldots, k$ and as these are all zero at z_1 it follows that

$$(p - Q T_1)^{(k)}(z_1) = 0 \quad \text{for } k = 0, 1, \ldots, r_1 - 1.$$

(This can be deduced more easily once the material on Taylor series has been covered but the above is a direct verification.) As $p(z) - Q(z) T_1(z)$ is a polynomial this implies that we can write

$$p(z) - Q(z) T_1(z) = (z - z_1)^{r_1} h(z),$$

where $h(z)$ is a polynomial. Using this in (4.2.12) and considering again (4.2.7) shows that the difference has no pole at the point z_1. Similar reasoning applies to all the points z_k, $k = 1, \ldots, n$ and we conclude that $R(z) - \tilde{R}(z)$ is a polynomial and (4.2.6) implies that it is the zero polynomial.

4.3 The exponential function $\mathrm{e}^z \equiv \exp(z)$

With $z = x + iy$ recall the definition

$$\mathrm{e}^z \equiv \exp(z) := \mathrm{e}^x(\cos y + i \sin y) = \mathrm{e}^x \mathrm{e}^{iy}. \tag{4.3.1}$$

We now consider some of the properties which quickly follow. From the definition we have

$$\mathrm{e}^{-z} = \mathrm{e}^{-x}(\cos\, y - i \sin\, y) = \frac{1}{\mathrm{e}^x\, \mathrm{e}^{iy}} = \frac{1}{\mathrm{e}^z}. \tag{4.3.2}$$

We have already shown that

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z. \tag{4.3.3}$$

For arbitrary points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$, we have

$$\mathrm{e}^{x_1+x_2} = \mathrm{e}^{x_1}\mathrm{e}^{x_2} \quad \text{and} \quad \mathrm{e}^{i(y_1+y_2)} = \mathrm{e}^{iy_1}\mathrm{e}^{iy_2}$$

and it follows that

$$\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}. \tag{4.3.4}$$

The results (4.3.2), (4.3.3) and (4.3.4) have the same form as the corresponding results when the argument is real.

We consider now a few things specific to having a complex argument. We have the periodic property

$$\mathrm{e}^{z+2\pi i} = \mathrm{e}^z,$$

i.e. the function is periodic with period $2\pi i$. If we let

$$w = \mathrm{e}^z$$

then

$$|w| = \mathrm{e}^x > 0.$$

There is no value $z \in \mathbb{C}$ such that we get 0 but the function does tend to 0 as $x \to -\infty$. Every other value of w is however possible and because of the periodic property every other value is attained infinitely many times and this is discussed in a moment when we discuss the logarithm function. To consider a specific value we take $w = 1$ and note that

$$\mathrm{e}^z = 1 \quad \text{if and only if} \quad |\mathrm{e}^z| = \mathrm{e}^x = 1, \quad \text{and} \quad \mathrm{e}^{iy} = 1.$$

Thus

$$\mathrm{e}^z = 1 \quad \text{if and only if} \quad z = 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots$$

An immediate consequence of this is that if

$$\mathrm{e}^{z_1} = \mathrm{e}^{z_2} \quad \text{then we have} \quad \frac{\mathrm{e}^{z_1}}{\mathrm{e}^{z_2}} = \mathrm{e}^{z_1-z_2} = 1$$

and thus

$$\mathrm{e}^{z_1} = \mathrm{e}^{z_2} \quad \text{implies that} \quad z_1 - z_2 = 2k\pi i \ \text{ for some integer } k.$$

In particular this implies that the function is one-to-one on the strip

$$G = \{x + iy : \ -\infty < x < \infty,\ -\pi < y \le \pi\}.$$

If we take $z \in G$ then

$$w = e^z = e^x e^{iy} \quad \text{gives} \quad |w| = e^x \quad \text{and} \quad \operatorname{Arg} w = \operatorname{Arg} e^{iy} = y,$$

i.e. we have

$$x = \ln |w|, \quad \text{and} \quad y = \operatorname{Arg} w.$$

The inverse of e^z on this strip is the principal valued logarithm function

$$\operatorname{Log} w = \ln |w| + i\operatorname{Arg} w.$$

To attempt to plot the complex exponential one possibility is to plot a rectangular grid in the z-plane and show the image plot in the w-plane and this is done in figure 4.1. Lines with x fixed (i.e. lines parallel to the y-axis) map to circles and lines with y fixed (i.e. lines parallel to the x-axis) map to radial lines.

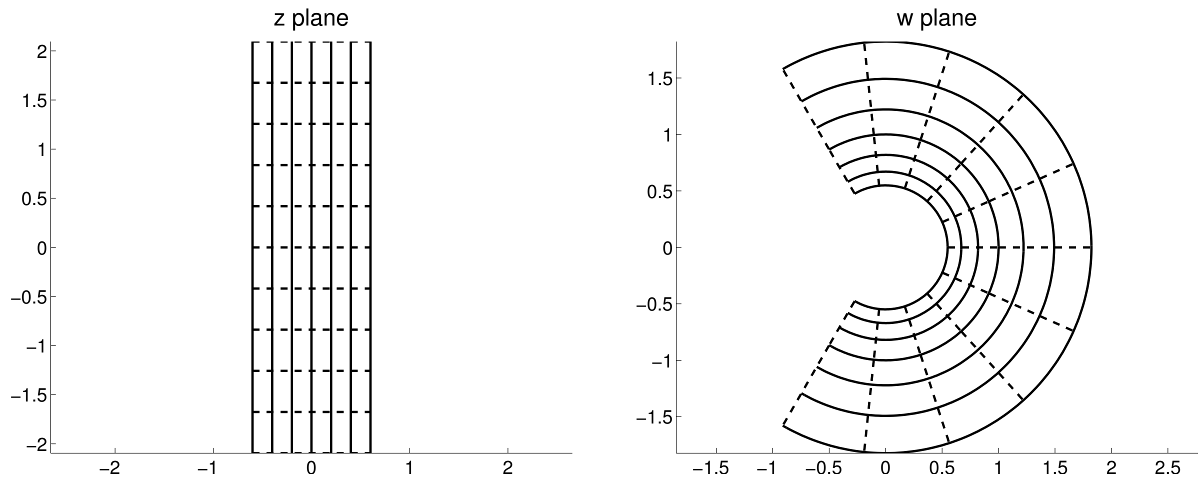


Figure 4.1: $w = f(z) = \exp(z)$, $|x| \le 0.6$, $|y| \le 2\pi/3$.

4.4 sin z, cos z, sinh z, cosh z

When we have real valued functions we have the definitions

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \quad \text{and} \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

and when e^{ix} was introduced we have

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and} \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right).$$

The definition of each of these functions for a general complex variable $z = x + iy$ is just to take

$$\begin{aligned} \cosh z &= \frac{1}{2}\left(e^z + e^{-z}\right), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right), \\ \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \end{aligned}$$

From the result about combining analytic functions it follows that all of these functions are entire functions, i.e. they are all analytic in the whole complex plane $\mathbb{C}$. The derivatives in the complex sense correspond to the real case as we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\cosh z &= \frac{1}{2}\left(\mathrm{e}^{z}-\mathrm{e}^{-z}\right) = \sinh z,\\
\frac{\mathrm{d}}{\mathrm{d}z}\sinh z &= \frac{1}{2}\left(\mathrm{e}^{z}+\mathrm{e}^{-z}\right) = \cosh z,\\
\frac{\mathrm{d}}{\mathrm{d}z}\cos z &= \frac{1}{2}\left(i\mathrm{e}^{iz}-i\mathrm{e}^{-iz}\right) = -\frac{1}{2i}\left(\mathrm{e}^{iz}-\mathrm{e}^{-iz}\right) = -\sin z,\\
\frac{\mathrm{d}}{\mathrm{d}z}\sin z &= \frac{1}{2i}\left(i\mathrm{e}^{iz}+i\mathrm{e}^{-iz}\right) = \frac{1}{2}\left(\mathrm{e}^{iz}-\mathrm{e}^{-iz}\right) = \cos z.
\end{aligned}$$

All these functions are periodic with the hyperbolic functions having period $2\pi i$ (the same as the exponential function) and with sine and cosine having period 2π as in the real case, i.e.

$$\cosh(z+2\pi i) = \cosh z, \quad \sinh(z+2\pi i) = \sinh z,$$

and

$$\cos(z+2\pi) = \cos z, \quad \sin(z+2\pi) = \sin z.$$

The exponential function as defined in (4.3.1) involves an expression involving x and y and we can similarly write each of $\cos(x+iy)$, $\sin(x+iy)$, $\cosh(x+iy)$ and $\sinh(x+iy)$ in terms of x and y as follows. From

$$z = x+iy, \quad -z = -x-iy, \quad iz = -y+ix, \quad -iz = y-ix,$$

and

$$\begin{aligned}
\mathrm{e}^{z} &= \mathrm{e}^{x}(\cos y + i\sin y),\\
\mathrm{e}^{-z} &= \mathrm{e}^{-x}(\cos y - i\sin y),\\
\mathrm{e}^{iz} &= \mathrm{e}^{-y}(\cos x + i\sin x),\\
\mathrm{e}^{-iz} &= \mathrm{e}^{y}(\cos x - i\sin x).
\end{aligned}$$

Thus

$$\begin{aligned}
\cosh z &= \frac{\mathrm{e}^{z}+\mathrm{e}^{-z}}{2} = \left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\cos y + i\left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\sin y\\
&= \cosh x\cos y + i\sinh x\sin y,\\
\sinh z &= \frac{\mathrm{e}^{z}-\mathrm{e}^{-z}}{2} = \left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\cos y + i\left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\sin y\\
&= \sinh x\cos y + i\cosh x\sin y,\\
\cos z &= \frac{\mathrm{e}^{iz}+\mathrm{e}^{-iz}}{2} = \left(\frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\right)\cos x + i\left(\frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2}\right)\sin x\\
&= \cos x\cosh y - i\sin x\sinh y,\\
\sin z &= \frac{\mathrm{e}^{iz}-\mathrm{e}^{-iz}}{2i} = \left(\frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2i}\right)\cos x + i\left(\frac{\mathrm{e}^{-y}+\mathrm{e}^{y}}{2i}\right)\sin x\\
&= \sin x\cosh y + i\cos x\sinh y.
\end{aligned}$$

It is not expected that you remember all of these but hopefully you can work them out from other definitions. It is interesting that every combination of one of the hyperbolic functions and one of the trigonometric functions gives a harmonic function.

Other properties to note are the following.

1. $\cosh z$ and $\cos z$ are both even functions, i.e.
$$\cosh(-z) = \cosh(z), \quad \cos(-z) = \cos(z)$$
and $\sinh z$ and $\sin z$ are both odd functions, i.e.
$$\sinh(-z) = -\sinh(z), \quad \sin(-z) = -\sin(z),$$
as in the real case.

2. From the definitions the hyperbolic and trigonometric functions are related by
$$\cosh(iz) = \cos z \quad \text{and} \quad \cosh z = \cos(iz),$$
$$\begin{aligned} \sinh(iz) &= \frac{e^{iz} - e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = i \sin z, \\ \sin(iz) &= \frac{e^{-z} - e^{z}}{2i} = \frac{1}{i}\left(\frac{e^{-z} - e^{z}}{2}\right) = (-i)(-\sinh z) = i \sinh z. \end{aligned}$$

3. For all z we have
$$\cos^2 z + \sin^2 z = 1$$
as in the real case. This follows directly from the definition, i.e.
$$\begin{aligned} \cos^2 z + \sin^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\ &= \frac{(e^{2iz} + 2 + e^{-2iz}) - (e^{2iz} - 2 + e^{-2iz})}{4} = 1. \end{aligned}$$
We similarly have
$$\cosh^2 z - \sinh^2 z = 1$$
as in the real case.

4. As $e^{i\pi/2} = i$ and $e^{-i\pi/2} = -i$ we have
$$\begin{aligned} \sin(z + \pi/2) &= \frac{e^{iz+i\pi/2} - e^{-(iz+i\pi/2)}}{2i} \\ &= \frac{ie^{iz} - (-i)e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z, \\ \cos(z + \pi/2) &= \frac{e^{iz+i\pi/2} + e^{-(iz+i\pi/2)}}{2} \\ &= \frac{ie^{iz} + (-i)e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{-2i} = -\sin z. \end{aligned}$$
All of these correspond to what we have in the real case.
Similarly as $e^{\pm i\pi} = -1$ using the definitions gives
$$\begin{aligned} \sin(z + \pi) &= -\sin(z), \\ \cos(z + \pi) &= -\cos(z). \end{aligned}$$

5. From the real case we have that $\sin x = 0$ if $x = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. These are also the only points in the complex plane where $\sin z = 0$ because from the definition
$$\sin z = 0 \quad \text{if and only if} \quad \mathrm{e}^{iz} = \mathrm{e}^{-iz}, \quad \text{i.e. } \mathrm{e}^{2iz} = 1.$$
The result follows from the result about the exponential function.

   We similarly have that the zeros of the cosine function are all on the real axis at the points $\pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

6. The formulas for $\sin(x_1 \pm x_2)$ and $\cos(x_1 \pm x_2)$ also have similar complex versions and we give these without any proofs.
$$\begin{aligned} \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \\ \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \end{aligned}$$
and in the case $z_1 = z_2 = z$ we get the double angle formulas
$$\begin{aligned} \sin(2z) &= 2 \sin z \cos z, \\ \cos(2z) &= \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z. \end{aligned}$$

7. When we just consider the function $\sin x$ with a real variable x we know that it increases from -1 to $+1$ as x increases from $-\pi/2$ to $+\pi/2$ and it is invertible if we restrict to $[-\pi/2, \pi/2]$. We consider next the situation with $\sin z$ when $z \in G$ with G denoting the strip
$$G = \{z = x + iy : \ -\pi/2 \le x \le \pi/2, \ -\infty < y < \infty\}$$
and recall that we have shown that
$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$
On the line $x = -\pi/2$ we have $\cos x = 0$ and $\sin x = -1$ and thus
$$\sin(-\pi/2 + iy) = -\cosh y \le -1.$$
Similarly on the line $x = \pi/2$ we have $\cos x = 0$ and $\sin x = 1$ and thus
$$\sin(\pi/2 + iy) = \cosh y \ge 1.$$
On both these lines we get a real value with magnitude of at least 1. In both these cases if we replace y by $-y$ then we get the same value (as $\cosh y$ is even) and hence the function is not one-to-one on G. Hence we consider instead the semi-infinite strip
$$G' = \{z = x + iy : \ -\pi/2 \le x \le \pi/2, \ 0 \le y < \infty\}$$
and note that the boundary part with $x = -\pi/2$, $y \ge 0$ maps to $(-\infty, -1]$, the real segment $[-\pi/2, \pi/2]$ maps to $[-1, 1]$ and the boundary part with $x = \pi/2$, $y \ge 0$ maps to $[1, \infty)$. The image of the boundary of G' under the map $\sin z$ is hence the real axis. To determine the image of the interior of G' we note that $\mathrm{Im}(\sin z) = \cos x \sinh y \ge 0$ when x and y are such that $\cos x \ge 0$ and $\sinh y \ge 0$

and this is the case when $|x| \le \pi/2$ and $y \ge 0$. Thus we only get points with non-negative imaginary part and from this we conclude that the interior of G' maps to the upper half plane and this is illustrated in figure 4.2.

With a bit more effort, which is not done fully here, it can shown that the function $w = \sin z$ is one-to-one on the domain G' and hence has an inverse and for the expression for the inverse first note that

$$\mathrm{e}^{iz} = \cos z + i \sin z.$$

If $w = \sin z$ is a point in the upper half plane then it can be shown that

$$\cos z = \sqrt{1 - \sin^2(z)} = \sqrt{1 - w^2},$$

where the square root is defined by

$$\zeta = \rho \mathrm{e}^{i\phi}, \quad \phi = \mathrm{Arg}\zeta, \quad \sqrt{\zeta} = \sqrt{\rho}\mathrm{e}^{i\phi/2}.$$

(This is known as the principal valued square root.) Hence

$$z = \sin^{-1} w = \frac{1}{i}\mathrm{Log}(iw + \sqrt{1 - w^2})$$

where, as before, Log denotes the principal valued logarithm. The function $\sin^{-1} w$ is analytic at all points except $w = \pm 1$ which correspond to $z = \pm\pi/2$ at which the derivative of $\sin z$ has a simple zero in each case.

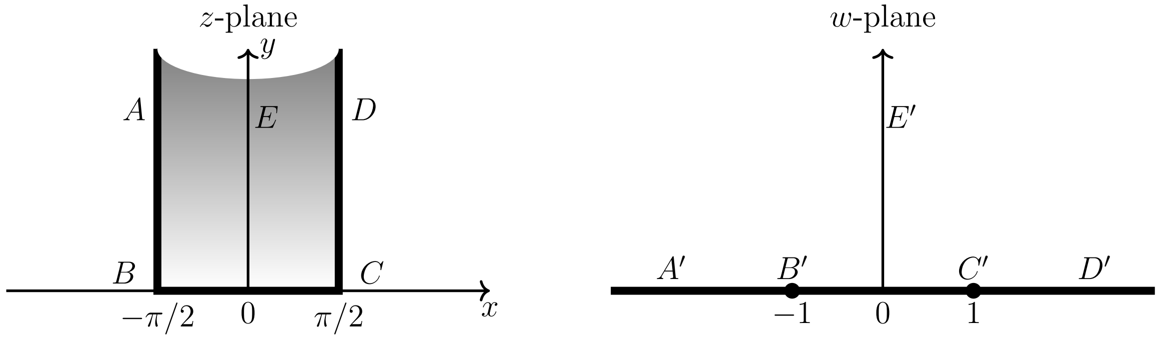


Figure 4.2: The image of the semi-infinite strip G' under the mapping $w = \sin z$, $z = x + iy$.

4.5 tan z, cot z, $\cdots$, tanh z, $\cdots$

With $\cos z$ and $\sin z$ defined we can similarly define

$$\tan z = \frac{\sin z}{\cos z},$$

and also

$$\sec z = \frac{1}{\cos z}, \quad \mathrm{cosec}\, z = \frac{1}{\sin z} \quad \text{and} \quad \cot z = \frac{\cos z}{\sin z} = \frac{1}{\tan z}.$$

There are also hyperbolic versions of all of these but we only give the hyperbolic tangent which is

$$\tanh\, z = \frac{\sinh\, z}{\cosh\, z}.$$

In all cases the functions are analytic for all $z \in \mathbb{C}$ except at the points at which the denominator in these expressions is 0. As $\cos\, z$, $\sin\, z$, $\cosh\, z$ and $\sinh\, z$ all have zeros it thus follows that none of the functions $\tan\, z$, $\sec\, z$, $\operatorname{cosec} z$, $\cot\, z$ and $\tanh\, z$ are entire functions. In what follows we just consider $\tan\, z$ and $\cot\, z$. Both these functions are periodic and by using the results

$$\begin{aligned} \sin(z+2\pi) &= \sin\, z, \quad \sin(z+\pi) = -\sin\, z, \\ \cos(z+2\pi) &= \cos\, z, \quad \cos(z+\pi) = -\cos\, z \end{aligned}$$

it follows that the period is π. The function $\tan\, z$ has simple poles at the zeros of $\cos\, z$ which are the points $\pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ and the function $\cot\, z$ has simple poles at the zeros of $\sin\, z$ which are the points $k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. For the residues of the poles of $\tan\, z$ and $\cot\, z$ we can use the same technique as used with rational functions although this is not fully justified until term 2 when series is considered. If for the moment this accepted as correct then we have the following. For the residues of the poles of $\tan\, z$ we have when $z_k = \pi/2 + k\pi$ that

$$(z - z_k)\tan\, z = \left(\frac{z - z_k}{\cos\, z}\right)\sin\, z \to \left.\left(\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}\cos\, z}\right)\right|_{z=z_k} \sin\, z_k = -1 \quad \text{as } z \to z_k.$$

Similarly for the residues of the poles of $\cot\, z$ we have when $z_k = k\pi$ that

$$(z - z_k)\cot\, z = \left(\frac{z - z_k}{\sin\, z}\right)\cos\, z \to \left.\left(\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}\sin\, z}\right)\right|_{z=z_k} \cos\, z_k = 1 \quad \text{as } z \to z_k.$$

These results may be used later in the module in a technique to sum certain infinite series.

4.6 The logarithm function

We have already mentioned the principal valued logarithm function as

$$\operatorname{Log} z = \ln |z| + i\operatorname{Arg} z,$$

provided $z \neq 0$, and because $\operatorname{Arg} z$ has a jump discontinuity across the negative real axis the function is not analytic everywhere. The domain in which $\operatorname{Log} z$ is analytic is the slit domain

$$G = \left\{ z = r\mathrm{e}^{i\theta} : \; r > 0, \; -\pi < \theta < \pi \right\}.$$

A so called "multi-valued function" is also defined by

$$w = \log\, z$$

(lower case letters are used to distinguish log and Log) which is taken to mean all values of w which satisfy

$$z = \mathrm{e}^w.$$

The principal value logarithm function provides one of the values and the complete set is

$$\begin{aligned} \log z &:= \ln|z| + i \arg z \\ &= \ln|z| + i\mathrm{Arg}\, z + i2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

Note that a "multi-valued function" is not a function as defined earlier as a function which maps to $\mathbb{C}$ must only have one value but this is the terminology that is used. If we just take one specific value for k and define

$$f_k(z) = \mathrm{Log}\, z + i2k\pi \tag{4.6.1}$$

then we do have a function which just differs from the principal valued logarithm function by a constant and, from the section about the Cauchy Riemann equations in polar form, we have

$$f_k'(z) = \frac{1}{z}.$$

The result about the derivative also follows by using the property

$$z = \mathrm{e}^{f_k(z)}.$$

If we differentiate with respect to z then we obtain

$$1 = \mathrm{e}^{f_k(z)} f_k'(z) = z f_k'(z), \quad \text{i.e. } f_k'(z) = \frac{1}{z}.$$

For some terminology, $f_0(z)$ defined in (4.6.1) is the principal valued logarithm function and is one specific **branch** of the function. The other functions $f_k(z)$ define other branches. In all cases the half line where the function has a jump discontinuity is known as the **branch cut** and the point at which the branch cut starts is known as the **branch point** which is the point $z = 0$ for all the functions. These are illustrated in figure 4.3.

4.7 Complex powers

The exponential function e^z is concerned with raising the real number e to any complex power z and when we consider z^n, $n \in \mathbb{Z}$, we are considering any complex number z raised to an integer power. We now consider more generally how to raise any complex number z to any complex power α.

To start, recall that the multi-valued logarithm function $w = \log z$ means all solutions of

$$\mathrm{e}^w = z,$$

and thus

$$z = \mathrm{e}^{\log z},$$

i.e. all the different values for log z all lead to just one value when combined with the exponential in this way. By the properties of the exponential the integer powers of z can be represented by

$$z^n = \mathrm{e}^{n \log z}, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{4.7.1}$$

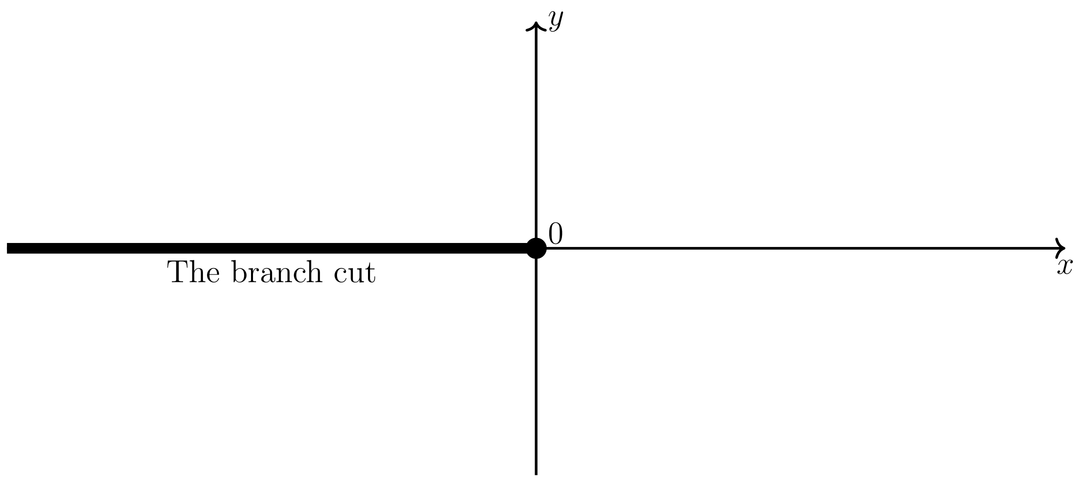


Figure 4.3: The branch cut of the principal logarithm function

This motivates the following definition for any power

$$z^\alpha := \mathrm{e}^{\alpha \log z} \tag{4.7.2}$$

which will usually be multi-valued. The principal value is taken to be

$$\mathrm{e}^{\alpha \mathrm{Log}\, z}.$$

This principal value version defines a function which is at least analytic at all points where $\mathrm{Log}\, z$ is analytic.

We consider next how many different values we get in the multi-valued case for different cases of α.

When α is an integer

When $\alpha = n$ is an integer the definition (4.7.2) corresponds to (4.7.1) and we just have one value. We have no need to use the exponential and logarithm functions to just evaluate terms such as 1, z, z^2, …, $1/z$, $1/z^2$, ….

When $\alpha = 1/n$ where n is an non-zero integer

Observe that with

$$w = z^{1/n} = \exp((\log\, z)/n), \quad w^n = \exp(\log\, z) = z$$

and we have the situation considered in the introduction chapter about finding the nth roots of z. Hence in this case $z^{1/n}$ means the n roots of z and to see how this is consistent with what was done previously we do the following. With $z = r\mathrm{e}^{i\theta}$ and $\theta = \mathrm{Arg}\, z$

$$\mathrm{Log}\, z = \ln\, r + i\theta$$

and the principal value is

$$\exp\left(\frac{\ln\, r + i\theta}{n}\right) = \exp\left(\frac{\ln\, r}{n}\right) \exp\left(\frac{i\theta}{n}\right) = r^{1/n}\mathrm{e}^{i\theta/n}.$$

All n values are described by

$$\exp\left(\frac{\ln r + i\theta + i2k\pi}{n}\right) = r^{1/n} e^{i\theta/n} \omega^k, \quad k = 0, 1, \ldots, n-1,$$

where $\omega = \exp(i2\pi/n)$ is one of the n roots of unity.

When $\alpha = p/q$ is a rational number

When $\alpha = p/q$ is a rational number with p and q having no common factor we have

$$z^{1/q} = \exp((\log z)/q), \tag{4.7.3}$$

which means q different values, and

$$w = z^{p/q} = \exp\left(\frac{p}{q} \log z\right) = \left(z^{1/q}\right)^p$$

also involves q values with each of the values being the pth power of what we get in (4.7.3). The principal value is just the pth power of the principal value of $z^{1/q}$.

When α is a irrational

In this case we get infinitely many values described by

$$z^\alpha = \exp(\alpha \mathrm{Log}\, z + \alpha(2k\pi i)), \quad k = 0, \pm 1, \pm 2, \ldots.$$

All the values have the same absolute value. The case $k = 0$ is the principal value.

When $\alpha \in \mathbb{C}$ is not real

Again we have infinitely many values. We just consider the specific case of $z = i$ and $\alpha = i$, which is commonly done in text books on complex variable theory. By using the definition we have

$$i^i = \exp(i \mathrm{Log}\, i + i(2k\pi i)), \quad k = 0, \pm 1, \pm 2, \ldots.$$

Now

$$\mathrm{Log}\, i = \ln |i| + i \mathrm{Arg}\, i = i\frac{\pi}{2}, \quad \text{and} \quad i \mathrm{Log}\, i = -\frac{\pi}{2}.$$

The case $k = 0$ gives the principal value which is

$$\exp\left(-\frac{\pi}{2}\right)$$

and all the values of i^i are

$$\exp\left(-\frac{\pi}{2} - 2k\pi\right), \quad k = 0, \pm 1, \pm 2, \ldots.$$

An interesting feature of this case is that all the values of i^i are real.

Chapter 5

Contour integrals, the Cauchy integral theorem and examples involving trigonometric integrals

5.1 Introductory remarks

In this chapter we consider what is meant by contour integrals which we write as

$$\int_\Gamma f(z)\,\mathrm{d}z$$

for a "contour" Γ in the complex plane and for a function $f(z)$ defined on Γ and in particular we consider results when $f(z)$ is analytic. Before we define precisely what a contour is and what the integral along a contour means we remark that it coincides with the real integral case when Γ is just part of the real axis such as an interval (a, b), i.e. when $\Gamma = (a, b)$ then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(x)\,\mathrm{d}x.$$

Also, it is worth remarking that if $f(x)$ is complex valued with $f = u + iv$, where u and v are real, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b u(x)\,\mathrm{d}x + i\int_a^b v(x)\,\mathrm{d}x.$$

There has already been an example on an exercise sheet which made use of this involving the indefinite integral

$$\int \mathrm{e}^{kx}\,\mathrm{d}x, \quad k = p + iq, \quad \text{with } p, q \in \mathbb{R}$$

which corresponds to a definite integral of the form

$$\int_a^x \mathrm{e}^{ks}\,\mathrm{d}s = \left[\frac{\mathrm{e}^{ks}}{k}\right]_a^x = \frac{1}{k}\left(\mathrm{e}^{kx} - \mathrm{e}^{ka}\right). \tag{5.1.1}$$

As

$$\mathrm{e}^{kx} = \mathrm{e}^{px}\mathrm{e}^{iqx} = \mathrm{e}^{px}\left(\cos(qx) + i\sin(qx)\right)$$

this enables expressions for

$$\int_a^x e^{ps}\cos(qs)\,ds \quad \text{and} \quad \int_a^x e^{ps}\sin(qs)\,ds$$

to be obtained by just taking the real and imaginary parts of the right hand side of (5.1.1).

5.2 A recap of results about real integrals

5.2.1 The area under a curve

In the case of a continuous function

$$f : (a, b) \to \mathbb{R}$$

the number

$$\int_a^b f(x)\,dx$$

means the "area under the curve of $f(x)$ from $x = a$ to $x = b$". To make this precise we can divide the interval (a, b) by points $a = x_0 < x_1 < \cdots < x_m = b$ and approximate the area by the composite mid-point rule involving

$$A_m = \sum_{i=1}^{m} h_i f(x_{i-1/2}), \quad h_i = x_i - x_{i-1}, \quad x_{i-1/2} = \frac{x_{i-1} + x_i}{2}. \tag{5.2.1}$$

If we consider the limit as $m \to \infty$ in such a way that $\max_i h_i \to 0$ (i.e. the width of all the strips tends to 0) then

$$\int_a^b f(x)\,dx = \lim_{\substack{m\to\infty \\ \max_i h_i \to 0}} A_m \tag{5.2.2}$$

as is illustrated in figure 5.1. Actually, we can do this a bit more rigorously by considering what is known as a Riemann integral but for the purpose of this module it is sufficient to just appreciate that the real integral means the area under the curve and this limit in (5.2.2) exists when $f(x)$ is continuous on (a, b). It also exists when $f(x)$ has a finite number of jump discontinuities.

5.2.2 Integration is the reverse of differentiation

To actually give expressions for integrals one thing that you learn early in a study of integration is that when an anti-derivative $F(x)$ exists, i.e.

$$F'(x) = f(x)$$

then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

This is part of the fundamental theorem of calculus in that the integral of a derivative F' on (a, b) just involves function values at the end points of the interval, i.e. $F(b)-F(a)$. In

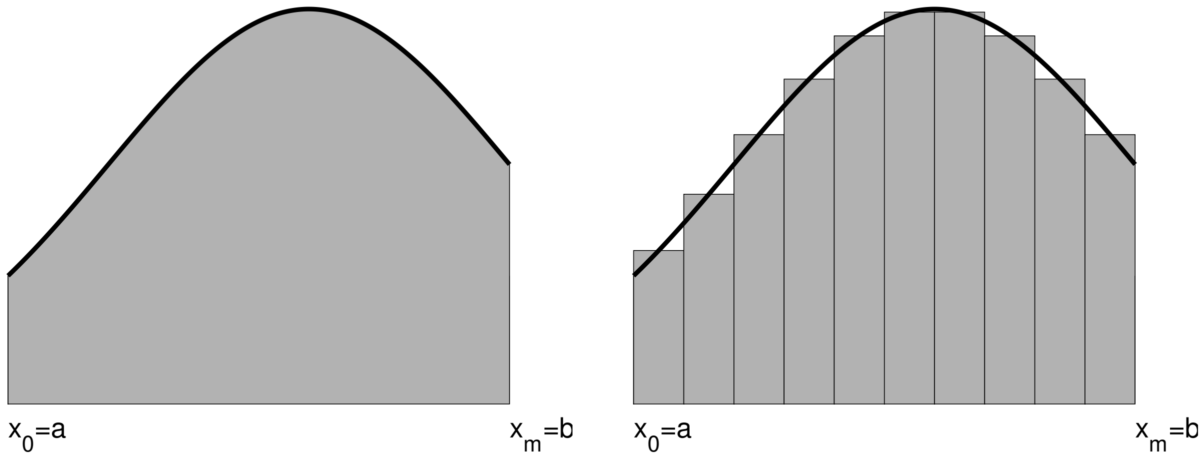


Figure 5.1: Area under the curve (left hand side) approximated by the area of rectangular strips (right hand side).

this sense integration is the reverse of differentiation. The other part of the fundamental theorem of calculus is that when f is continuous

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(s)\,\mathrm{d}s = f(x),$$

i.e. if we differentiate an integral of the type $\int_a^x$ then we get the integrand value at the upper limit.

5.2.3 Remarks about the direction of the integration

As a relatively small point to note at this stage, in the definition of the integral in (5.2.1) and (5.2.2) we had an interval (a,b) which implies that we were assuming that $a<b$ and the integral is from $x=a$ to $x=b$, i.e. there is a direction involved. If we consider instead the integral from b to a then this is defined as

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

The reason for noting this here is that when contour integrals are considered in the next section we need to know the direction involved.

5.3 Integrals along smooth arcs and along contours

In examples in this module a smooth arc will usually be either a straight line segment or a circular arc but when general results are given the results will hold for any smooth arcs and thus we need to define what this means.

Definition 5.3.1 Smooth arc. *A set $\gamma\subset\mathbb{C}$ is a smooth arc if the set can be described in the form*

$$\{z(t):\ a\le t\le b\}$$

where $z(t)$ is continuously differentiable on $[a, b]$, $z'(t) \ne 0$ on $[a, b]$ and the function $z(t)$ is one-to-one on $[a, b]$.

Definition 5.3.2 Smooth closed curve. *A set $\gamma \subset \mathbb{C}$ is a smooth closed curve if the set can be described in the form as in the smooth arc case and the starting point $z(a)$ and the end point $z(b)$ are the same but now we just require the one-to-one property to hold on $[a, b)$ and for the smoothness at the end points we require*

$$z'(b) = z'(a).$$

A straight line segment or part of a circle are examples of smooth arcs which are not closed whereas a complete circle is an example of a smooth arc which is closed. To emphasise one of the requirements in the above definition, the one-to-one conditions means that if a closed curve such as a circle is being considered then the curve is only traversed once which is something you cannot spot if you just look a plot of such a curve.

Definition 5.3.3 Directed smooth arc. *A smooth arc with a specific ordering of the points is known as a directed smooth arc.*

In the case of the notation defining a smooth arc there are two possibilities for the directed smooth arc involving tracing the path by moving from $t = a$ to $t = b$ or alternatively tracing the path from $t = b$ to $t = a$. In either case we have a starting point and an end point.

Definition 5.3.4 Contour (or piecewise smooth curve) *A contour Γ is either just a point or it is a finite sequence of directed smooth arcs γ_1, γ_2, ..., γ_n such that the end point of γ_k is the starting point of γ_{k+1} for $k = 1, 2, \ldots, n-1$.*

Examples of contours

In Figure 5.2 we give some examples of contours. Just to make clear, when we have more than one piece in our contour the parts join to give something which is continuous but the slope is usually not continuous at the join points.

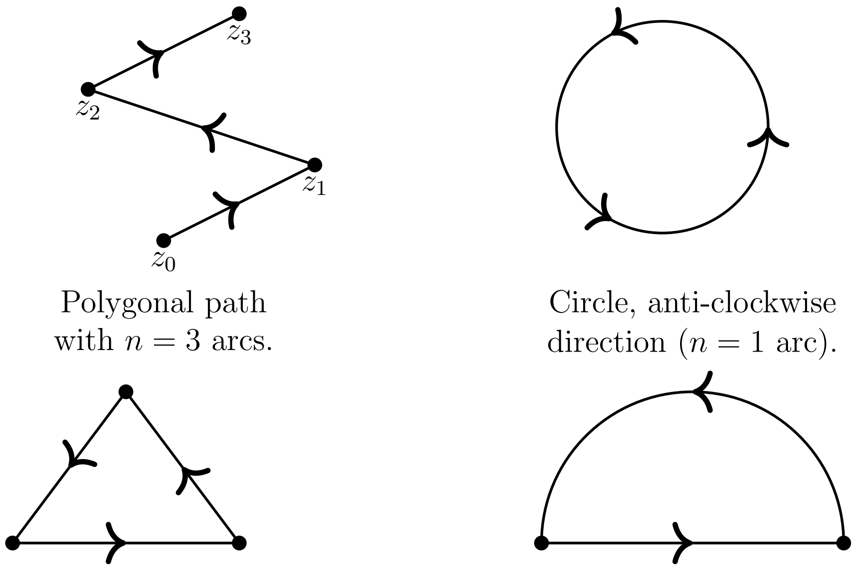
5.3.1 The length of an arc and the length of a contour

If a smooth arc γ is described in the form $\{z(t) = x(t) + iy(t) : \; a \le t \le b\}$, with $x(t)$ and $y(t)$ being real, then an approximation to the length is given by summing the lengths of straight line segments approximating the arc as follows. With $a = t_0 < t_1 < \cdots < t_m = b$ the length of the arc is approximately

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \cdots + |z(t_m) - z(t_{m-1})| = \sum_{i=1}^{m} |z(t_i) - z(t_{i-1})|.$$

(As a straight line segment gives the route of the shortest path between two points this is actually an under estimate of the length.) Now when $t_i - t_{i-1}$ is small

$$z(t_i) - z(t_{i-1}) \approx z'(t_{i-1/2})(t_i - t_{i-1}), \quad t_{i-1/2} := \frac{t_i + t_{i-1}}{2}.$$



Polygonal path with $n = 3$ arcs.

Circle, anti-clockwise direction ($n = 1$ arc).

Closed polygonal path with $n = 3$ arcs.

Closed path, $n = 2$ arcs.

Figure 5.2: Some examples of contours.

By considering the limit as $m \to \infty$ with $\max_i(t_i - t_{i-1}) \to 0$ we get

$$l(\gamma) = \text{length of } \gamma = \int_a^b |z'(t)| \, \mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t.$$

When we have a contour Γ involving arcs $\gamma_1, \ldots, \gamma_n$ the length of Γ is defined to be

$$l(\gamma_1) + l(\gamma_2) + \cdots + l(\gamma_n).$$

5.3.2 Contour integrals

The integral along a directed smooth arc can be defined as an appropriate limit involving sums and to correspond to what was given at the start of the chapter each sum can be taken to be of the form

$$\sum_{i=1}^m h_i f(z(t_{i-1/2})), \quad h_i = z(t_i) - z(t_{i-1}), \tag{5.3.1}$$

where, as before, $a = t_0 < t_1 < \cdots < t_m = b$ and $t_{i-1/2} = (t_i + t_{i-1})/2$. Using a similar derivation to the case of deriving the expression for the length of the curve we now have

$$\begin{aligned} h_i f(z(t_{i-1/2})) &= (z(t_i) - z(t_{i-1})) f(z(t_{i-1/2})) \\ &\approx f(z(t_{i-1/2})) z'(t_{i-1/2})(t_i - t_{i-1}). \end{aligned}$$

By considering the limit as $m \to \infty$ with $\max_i(t_i - t_{i-1}) \to 0$ we get

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(z(t)) z'(t) \, \mathrm{d}t. \tag{5.3.2}$$

For the purpose of this module we take this as defining what is meant by the integral along the directed arc γ. As the derivation applies to every possible parameterization of γ the value given by (5.3.2) is the same for each parameterization.

As notation, if γ means a directed arc then $-\gamma$ means the directed arc in the opposite direction and

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

In the case of a contour Γ involving arcs $\gamma_1, \ldots, \gamma_n$ the integral along Γ means

$$\int_{\Gamma} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\,\mathrm{d}z.$$

5.3.3 The ML inequality, a bound of the size of the integral

Before we consider examples there is one result that we will need in a few places concerning the magnitude of a contour integral. If we let

$$M = \max_{z\in\Gamma} |f(z)|$$

then by the triangle inequality we can bound the sum used in (5.3.1) to give

$$\left|\sum_{i=1}^{m} h_i f(z(t_{i-1/2}))\right| \le \sum_{i=1}^{m} |h_i|\,|f(z(t_{i-1/2}))| \le M \sum_{i=1}^{m} |h_i|.$$

Now

$$\sum_{i=1}^{m} |h_i| = \sum_{i=1}^{m} |z(t_i) - z(t_{i-1})| \le \text{length of } \Gamma.$$

Although we have not rigorously considered the limiting process throughout we have done enough for this module to justify that

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \le ML, \quad \text{where } M = \max_{z\in\gamma} |f(z)| \quad \text{and } L = \text{length of } \gamma.$$

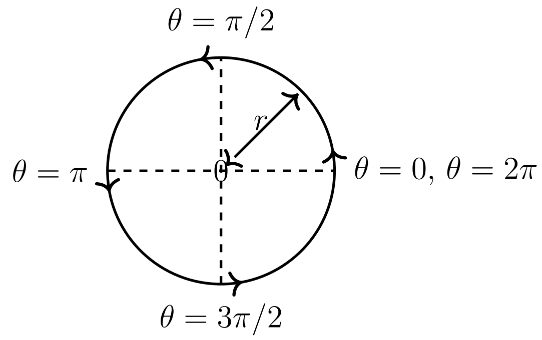
This result is often known as the ML inequality due to the letters commonly used to bound $f(z)$ and for the length of Γ. The result is used at various times later in the module often as part of an argument to explain why certain integrals are 0, e.g. there will be cases when we can arrange to consider contours which are arbitrarily short in length for functions which are bounded and we use it in the next chapter when we consider the Cauchy integral formula.

5.3.4 Examples of contour integrals

1. Let

$$C_r = \left\{ z = z(\theta) = r\mathrm{e}^{i\theta} :\ 0 \le \theta \le 2\pi \right\}$$

denote a circle of radius r with increasing θ corresponding to the anti-clockwise direction. We will sometimes use the terminology that the circle is traversed once in the anti-clockwise direction.



Observe that

$$z'(\theta) = ire^{i\theta}.$$

Consider $f(z) = z^n$, $n \in \mathbb{Z}$, so that we have

$$\begin{aligned} I_n^r := \int_{C_r} f(z)\,dz &= \int_0^{2\pi} f(z(\theta))z'(\theta)\,d\theta \\ &= \int_0^{2\pi} (re^{i\theta})^n ire^{i\theta}\,d\theta \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta. \end{aligned}$$

Now if $n = -1$ then we have $r^{n+1} = r^0 = 1$ and we get

$$I_{-1}^r = i\int_0^{2\pi} d\theta = 2\pi i.$$

If n is any other integer than -1 then

$$I_n = ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = 0$$

as $e^{i(n+1)2\pi} = e^0 = 1$.

As a point about notation, when we have a loop it is common to indicate this with the notation

$$I_n^r := \oint_{C_r} f(z)\,dz \quad \text{instead of just writing} \quad I_n := \int_{C_r} f(z)\,dz.$$

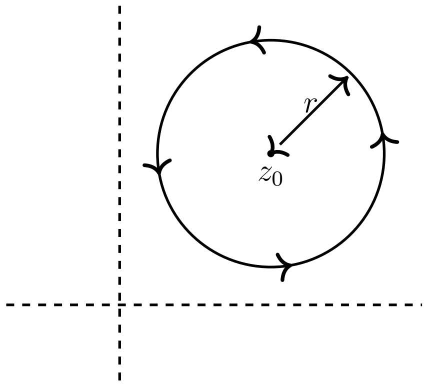
The notation further emphasises that the contour is a loop. We use this notation in what follows when we have a loop.

2. As a minor generalization of the previous example we can shift the centre of the circle to z_0 so that now

$$C_r = \left\{z = z(\theta) = z_0 + re^{i\theta} :\ 0 \le \theta \le 2\pi\right\}$$

and consider the integral of $(z - z_0)^n$ when n is an integer to similarly get

$$\oint_{C_r} (z - z_0)^n\,dz = \begin{cases} 2\pi i, & \text{if } n = -1, \\ 0, & n \in \mathbb{Z} \text{ with } n \ne -1. \end{cases}$$



Note that the answer does not depend on z_0 or r and the value is only non-zero when $n = -1$ when we are just considering integers for the power.

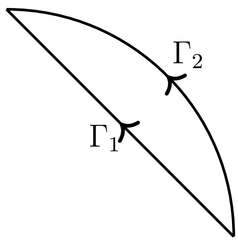
3. In the previous two examples we had closed loops (i.e. circles) and we now consider contours which are not loops and consider two different paths joining the points 1 and i involving the straight line segment

$$\Gamma_1 = \{z_1 + t(z_2 - z_1) : \ 0 \le t \le 1\}, \quad z_1 = 1, \quad z_2 = i,$$

and the circular arc

$$\Gamma_2 = \left\{ \mathrm{e}^{it} : \ 0 \le t \le \pi/2 \right\}$$

and take $f(z) = z^n$ where n is an integer.



We start by considering the case $n \ne -1$. In the case of Γ_1 we have

$$z(t) = z_1 + t(z_2 - z_1), \quad z'(t) = z_2 - z_1$$

giving

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_0^1 (z_1 + t(z_2 - z_1))^n (z_2 - z_1)\,\mathrm{d}t.$$

This can be done easily by noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}(z_1 + t(z_2 - z_1))^{n+1} = (n+1)(z_1 + t(z_2 - z_1))^n (z_2 - z_1).$$

Hence

$$\begin{aligned} \int_{\Gamma_1} f(z)\,\mathrm{d}z &= \left[\frac{(z_1 + t(z_2 - z_1))^{n+1}}{n+1} \right]_0^1 \\ &= \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) = \frac{1}{n+1}(i^{n+1} - 1). \end{aligned}$$

In the case of Γ_2 we have

$$z(t) = \mathrm{e}^{it}, \quad z'(t) = i\,\mathrm{e}^{it}$$

giving

$$\begin{aligned} \int_{\Gamma_2} f(z)\,\mathrm{d}z &= \int_0^{\pi/2} \left(\mathrm{e}^{it}\right)^n i\mathrm{e}^{it}\,\mathrm{d}t = i\int_0^{\pi/2} \mathrm{e}^{i(n+1)t}\,\mathrm{d}t \\ &= \frac{1}{n+1}\left[\mathrm{e}^{i(n+1)t}\right]_0^{\pi/2} = \frac{1}{n+1}(i^{n+1}-1). \end{aligned}$$

In both cases we get the same answer and thus for this function $f(z)$ this suggests that the complex number obtained may not depend on the path taken between $z_1 = 1$ and $z_2 = i$ although at this stage only two different paths have been considered. We show in the next section that the result is indeed path independent.

We now consider the case when $n = -1$. In the case of Γ_1 we have

$$\begin{aligned} \int_{\Gamma_1} f(z)\,\mathrm{d}z &= \int_0^1 (z_1 + t(z_2 - z_1))^{-1}(z_2 - z_1)\,\mathrm{d}t \\ &= \left[\mathrm{Log}(z_1 + t(z_2 - z_1))\right]_0^1 = \mathrm{Log} z_2 - \mathrm{Log} z_1 = i\frac{\pi}{2}. \end{aligned}$$

This is valid as the principal valued logarithm function is analytic on Γ_1. In the case of the circular are Γ_2 we more easily get

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_0^{\pi/2} i\,\mathrm{d}t = i\frac{\pi}{2}.$$

The path independent observation also holds in this case.

Remarks

Observe that in the examples we obtained the result by being able to integrate some of the "elementary functions" from knowledge of "standard derivatives", i.e.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{e}^{i(n+1)\theta}}{i(n+1)}\right) &= \mathrm{e}^{i(n+1)\theta}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(z_1 + t(z_2 - z_1))^{n+1}}{(n+1)}\right) &= (z_1 + t(z_2 - z_1))^n\,(z_2 - z_1), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Log}(z_1 + t(z_2 - z_1))\right) &= \frac{z_2 - z_1}{z_1 + t(z_2 - z_1)}. \end{aligned}$$

The result involving the principal valued logarithm is valid provided the segment from z_1 to z_2 does not cross the branch cut of the logarithm. It is the knowledge of the anti-derivative in each case which gave us the results which do not depend on the path between z_1 and z_2 and we consider this next.

5.4 Independence of path in contour integrals

The key to being able to do the integrals in the examples was in knowing that the integrand was the derivative of something and the fundamental theorem of calculus tells us how to integrate derivatives. In each case the integrand considered was a function of the parameter used to describe the contour. To consider this generally we suppose that when we wish to integrate $f(z)$ there exists an analytic function $F(z)$ such that

$$f(z) = F'(z)$$

on our contour Γ, i.e. in the previous jargon the function f has an anti-derivative F on Γ. We suppose that our contour Γ involves n arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ and to make the notation easy we note that it is always possible to arrange for the parameterization of γ_k to be described in the form

$$\gamma_k = \{z(t) : \ \tau_{k-1} \le t \le \tau_k\}, \quad k = 1, 2, \ldots, n.$$

Then

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{k=1}^n \int_{\gamma_k} f(z)\,\mathrm{d}z = \sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_k} f(z(t)) z'(t)\,\mathrm{d}t.$$

Now by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t))\, z'(z) = f(z(t))\, z'(t).$$

Thus the integrand is the derivative of $F(z(t))$ and by the fundamental theorem of calculus we get

$$\begin{aligned}\int_\Gamma f(z)\,\mathrm{d}z &= (F(z(\tau_1)) - F(z(\tau_0)) + (F(z(\tau_2)) - F(z(\tau_1)) \\ &\quad + \cdots + (F(z(\tau_n)) - F(z(\tau_{n-1})).\end{aligned}$$

The sum is what is known as a telescoping sum in that all the intermediate terms cancel and we obtain

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z(\tau_n)) - F(z(\tau_0)).$$

The result just depends on $F(z)$ evaluated at the final point $z(\tau_n)$ and the initial point $z(\tau_0)$ and thus in particular it does not depend on the path between the two points and we state this as a theorem as follows.

Theorem 5.4.1 *Suppose that the function $f(z)$ is continuous in a domain D and has an anti-derivative $F(z)$ throughout D. Then for any contour Γ contained in D with initial point z_I and an end point z_E we have*

$$\int_\Gamma f(z)\, dz = F(z_E) - F(z_I). \tag{5.4.1}$$

Examples again – evaluated using path independent property

We consider again the examples considered in section 5.3.4.

For the function $f(z) = z^n$ with n as an integer and $n \neq -1$ we have

$$F(z) = \frac{z^{n+1}}{n+1}, \quad F'(z) = f(z).$$

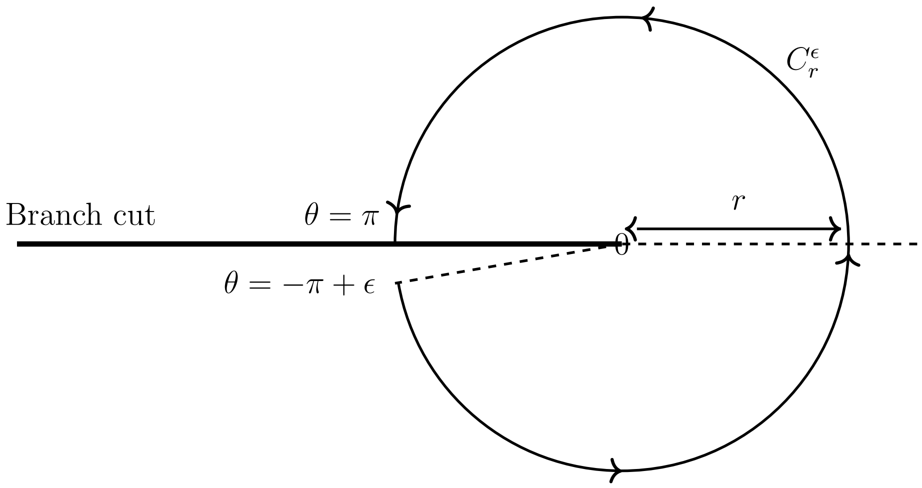
When the contour is the circle $C_r = \left\{ r e^{i\theta} : \ 0 \le \theta \le 2\pi \right\}$, $r > 0$, the points corresponding to $\theta = 0$ and to $\theta = 2\pi$ are the same and thus $F(z(2\pi)) = F(z(0))$ and we can immediately deduce that

$$\oint_{C_r} f(z)\,\mathrm{d}z = F(z(2\pi)) - F(z(0)) = 0.$$

When $n = -1$, i.e. when $f(z) = 1/z$, we cannot proceed in exactly the same way as the candidate for the anti-derivative is $\operatorname{Log} z$ which is not continuous on all parts of the contour due to the jump discontinuity across the negative real axis. However we can still use the path independent property if we start the contour on one side of the branch cut and finish at the corresponding point on the other side of the branch cut. That is for $\epsilon > 0$ being small we let

$$f(z) = \frac{1}{z}, \quad F(z) = \operatorname{Log} z, \quad \text{and we let } C_r^\epsilon = \left\{ r e^{i\theta} : \ -\pi + \epsilon \le \theta \le \pi \right\}.$$

The contour C_r^ϵ is illustrated below.



Note that the contour tends to the complete circle as $\epsilon \to 0$. For the integral we have

$$\int_{C_r^\epsilon} \frac{\mathrm{d}z}{z} = \operatorname{Log}(r e^{i\pi}) - \operatorname{Log}(r e^{-i\pi + i\epsilon}).$$

Now

$$\begin{aligned} \operatorname{Log}(r e^{i\pi}) &= \operatorname{Log} r + i\pi, \\ \operatorname{Log}(r e^{-i\pi + i\epsilon}) &= \operatorname{Log} r - i\pi + i\epsilon, \end{aligned}$$

and thus

$$\int_{C_r^\epsilon} \frac{\mathrm{d}z}{z} = 2\pi i - i\epsilon \to 2\pi i \quad \text{as } \epsilon \to 0.$$

Observe that we have obtained the same result as before and although the details have been slightly longer here the result of (5.4.1) does enable us to deduce that if we take any closed loop Γ which starts at one side of the branch cut and ends at the corresponding point on the other side and has 0 inside the loop and we go anti-clockwise around the loop then

$$\oint_\Gamma \frac{\mathrm{d}z}{z} = 2\pi i$$

and

$$\oint_\Gamma z^n \,\mathrm{d}z = 0, \quad n \text{ an integer and } n \neq -1.$$

We get 0 in all cases (i.e. for all integers n) if the loop does not have the point $z = 0$ as an interior point. This statement also applies to any complex power z^α, $\alpha \in \mathbb{C}$, when $z = 0$ is not an interior point.

We can generalise the above slightly by shifting the origin to give us the following. If z_0 is any point and if Γ is a contour which gives a closed loop in the anti-clockwise direction then

$$\oint_\Gamma \frac{\mathrm{d}z}{z - z_0} = \begin{cases} 2\pi i, & \text{if } z_0 \text{ is inside the loop,} \\ 0, & \text{if } z_0 \text{ is outside the loop,} \end{cases}$$

and

$$\oint_\Gamma (z - z_0)^n \,\mathrm{d}z = 0, \quad n \text{ an integer and } n \neq -1.$$

In the case with the non-zero answer it needs to be understood that the path starts on one side of the branch cut and ends on the other side and does not cross the branch cut at any intermediate point. It is also worth repeating a previous point that the loop around the point only goes around the point once and this will sometimes be expressed as we "wind" around the point once in the anti-clockwise direction.

The examples above involving powers of $(z - z_0)$ for which the contour integral is 0 when the contour is a loop can be generalised to any function $f(z)$ which has an anti-derivative and we state this next as a corollary to theorem 5.4.1.

Corollary 5.4.1 *If f is continuous in a domain D and there exists an analytic function F such that $F'(z) = f(z)$ throughout D then*

$$\oint_\Gamma f(z)\, dz = 0$$

for all loops Γ lying in D.

Further results and remarks about the independence of path

A key result so far of this chapter is that if there exists an anti-derivative $F(z)$ such that $f(z) = F'(z)$ then the contour integrals involving $f(z)$ can easily be determined as we just need to evaluate $F(z)$ at both ends of the contour. In particular, as mathematical statements, we have shown the following when f has an anti-derivative.

(i) Every loop integral vanishes, i.e. when Γ is a loop then

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 0. \tag{5.4.2}$$

(ii) If Γ_1 and Γ_2 have the same initial and final points then

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z. \tag{5.4.3}$$

In fact (i) and (ii) are equivalent. An obvious question to ask is how can we determine when $f(z)$ has an anti-derivative. This will be answered properly in the next section of this chapter when the Cauchy theorem is covered but a partial answer can be given already. Specifically, if we have a continuous function $f(z)$ and we know that contour integrals involving $f(z)$ are path independent then we can deduce the existence of $F(z)$ by giving an explicit expression for it as follows.

Let D be the domain of $f(z)$, let z_0 be any point in D and recall that a domain is a open connected set. For any $z \in D$ there is a path in D from z_0 to z, which we call $\Gamma(z)$, and we can define

$$F(z) = \int_{\Gamma(z)} f(\zeta)\,\mathrm{d}\zeta. \tag{5.4.4}$$

This is well defined as we are assuming that the integral does not depend on the path between z_0 and z. If we take Δz to be sufficiently small in magnitude then $z + \Delta z \in D$ and for the path from z_0 to $z+\Delta z$ we can take the path $\Gamma(z)$ followed by the line segment γ from z to $z + \Delta z$ so that

$$F(z+\Delta z) - F(z) = \int_{\gamma} f(\zeta)\,\mathrm{d}\zeta = \int_0^1 f(z + t\Delta z)\Delta z\,\mathrm{d}t.$$

Then

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \int_0^1 f(z+t\Delta z)\,\mathrm{d}t - f(z) = \int_0^1 (f(z+t\Delta z) - f(z))\,\mathrm{d}t.$$

To show that the right hand side tends to 0 at $\Delta z \to 0$ we can first bound the integral as

$$\left|\int_0^1 (f(z+t\Delta z) - f(z))\,\mathrm{d}t\right| \le \max_{0\le t\le 1} |f(z+t\Delta z) - f(z)| \to 0 \quad \text{as } \Delta z \to 0$$

because f is continuous. Thus the derivative of the function $F(z)$ defined in (5.4.4) is the function $f(z)$. Hence the path independent property implies the existence of F and earlier we noted that the existence of F implies the path independent property, i.e. results (i) and (ii) are equivalent (as already stated above) and these are in turn equivalent to the existence of an anti-derivative of $f(z)$.

5.5 Cauchy's integral theorem

In this section we show one of the most important theorems of this module which is when we have a loop Γ with $f(z)$ being continuous on Γ and analytic inside Γ then

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 0. \tag{5.5.1}$$

In the literature this is reported as being proved by Cauchy in the early part of the nineteenth century under the assumption that not only does $f'(z)$ exist inside Γ but also that $f'(z)$ is continuous. In the latter part of the nineteenth century Goursat showed that the result can be proved without the need to assume that $f'(z)$ is continuous and this version of the theorem is known as the Cauchy-Goursat theorem. In subsection 5.5.1 we comment on a proof when $f'(z)$ is assumed to be continuous as is done in most text books at this level. In subsection 5.5.2 we also comment on the other proof given in the book by Saff and Snider which involves continuously deforming a loop. In subsection 5.5.3 we also comment a longer proof closer to what Goursat shows. The proofs are not examinable and nothing in this section is directly examinable although knowledge of (5.5.1) is needed to give full explanation in some examples when we need to explain why the value of certain given loop integrals are zero.

5.5.1 A proof using Green's theorem

In a study of vector calculus, which has not been part of your course, a result that is covered is Stoke's theorem in three space dimensions and a two dimensional version of this is known as Green's theorem. For continuously differentiable functions $V_1(x,y)$ and $V_2(x,y)$ Green's theorem is the following where S is a region in $\mathbb{R}^2$ with boundary ∂S.

$$\iint_S \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial S} V_1 \mathrm{d}x + V_2 \mathrm{d}y.$$

The theorem relates the double integral over S to a line integral around the boundary ∂S of S.

We now connect the above with the contour integrals of a function $f(z)$ in the complex plane. If Γ corresponds to ∂S, a loop in $\mathbb{C}$, then we defined the integral in the form

$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t,$$

with $z(t)$, $a \le t \le b$ being a parametric description of Γ. (Γ may actually involve several directed smooth arcs to form the loop.) The interior of Γ gives us a simply connected domain (i.e. it has no holes). For the contour integral parts we have

$$\begin{aligned} \mathrm{d}z &= \mathrm{d}x + i\mathrm{d}y, \\ f(z)\mathrm{d}z &= (u+iv)(\mathrm{d}x + i\mathrm{d}y) \\ &= (u\mathrm{d}x - v\mathrm{d}y) + i(v\mathrm{d}x + u\mathrm{d}y). \end{aligned}$$

The complex contour integral hence corresponds to two real integrals, i.e.

$$\oint_\Gamma u\mathrm{d}x - v\mathrm{d}y \quad \text{and} \quad \oint_\Gamma v\mathrm{d}x + u\mathrm{d}y.$$

If we assume that $f'(z)$ exists and is continuous then by Green's theorem

$$\oint_\Gamma u\mathrm{d}x - v\mathrm{d}y = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\mathrm{d}y \qquad (\text{with } V_1 = u,\ V_2 = -v),$$

and

$$\oint_\Gamma v\mathrm{d}x + u\mathrm{d}y = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y \qquad (\text{with } V_1 = v,\, V_2 = u).$$

$f'(z)$ existing throughout S means that $f(z)$ is analytic and thus the Cauchy Riemann equations hold and hence the integrands in the area integrals above are zero and consequently

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

This completes the proof.

At the start of the section was stated that Goursat proved the result without the need to assume that $f'(z)$ is continuous inside Γ and we now state this famous theorem.

Theorem 5.5.1 *(***Cauchy-Goursat theorem***) If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then*

$$\oint_\Gamma f(z)\,dz = 0.$$

Now from section 5.4 we have already had a result involving loop integrals in (5.4.2) which is equivalent to (5.4.3) and which is also equivalent to $f(z)$ having an anti-derivative. Hence when $f(z)$ is analytic in a simply connected domain we now know that all the above hold and we state this in the following theorem.

Theorem 5.5.2 *In a simply connected domain, an analytic function has an anti-derivative, its contour integrals are independent of the path and its loop integrals vanish.*

5.5.2 An overview on a proof by continuously deforming a smooth contour

As already stated, none of the proofs given of the Cauchy Goursat theorem are examinable. In the book by Saff and Snider a proof involving deforming contours is given. Without going into too much detail we just give the main formula involved and assume that everything is sufficiently smooth when needed to avoid too much detail.

Suppose that we a function $z(s,t)$ from which we can give parametric descriptions of two smooth loops

$$\begin{aligned}\Gamma_1 &= \{z(0,t) :\ 0 \le t \le 1\},\\ \Gamma_2 &= \{z(1,t) :\ 0 \le t \le 1\}\end{aligned}$$

and for all $s \in (0,1)$ the following describes an intermediate smooth loop

$$\{z(s,t) :\ 0 \le t \le 1\}.$$

For any of these loops we define

$$I(s) = \int_0^1 f(z(s,t))\frac{\partial z}{\partial t}(s,t)\,\mathrm{d}t.$$

We wish to show that this does not vary with s.

Assuming that to get the expression for the derivative it is valid to partially differentiate the integrand with respect to s we have, using the chain rule and the product rule,

$$I'(s) = \int_0^1 \left(f'(z(s,t)) \frac{\partial z}{\partial s} \frac{\partial z}{\partial t} + f(z(s,t)) \frac{\partial^2 z}{\partial s \partial t} \right) \mathrm{d}t.$$

Now as mixed partial derivatives can be done in any order we have

$$\frac{\partial}{\partial t} \left(f(z(s,t)) \frac{\partial z}{\partial s} \right) = f'(z(s,t)) \frac{\partial z}{\partial t} \frac{\partial z}{\partial s} + f(z(s,t)) \frac{\partial^2 z}{\partial t \partial s} = \text{integrand above.}$$

Hence by the fundamental theorem of calculus

$$I'(s) = \int_0^1 \frac{\partial}{\partial t} \left(f(z(s,t) \frac{\partial z}{\partial s} \right) \mathrm{d}t = f(z(s,1)) \frac{\partial z}{\partial s}(s,1) - f(z(s,0)) \frac{\partial z}{\partial s}(s,0).$$

For our smooth loops we have $z(s,0) = z(s,1)$ and also for the partial derivatives and thus $I'(s) = 0$ and $I(s)$ is a constant. Now when we have simply connected domain we can continuously deform a loop to a point and the value of the integral when the loop is just one point is zero (this follows by the ML inequality as $L = 0$) and thus the constant is zero.

5.5.3 Comments on a proof given in Spiegel, p103–105

A proof is given in Spiegel which covers 2.5 full pages and has several parts. One of the key parts is to show that the result holds when the boundary is a triangle. The other parts then extend the result of being true for a triangle to being true for any closed polygon as any closed polygon can be partitioned as a union of non-overlapping triangles. Finally, as any smooth curve then be approximated arbitrarily closely by a polygon it is deduced that the result is true for any loop.

In the case of the loop being a triangle the proof involves creating a sequence of nested triangles Γ, $\Gamma^1_{k_1}, \ldots, \Gamma^n_{k_n}$ each of which is similar to the previous triangle and has 1/4 of the area. We illustrate this in figure 5.3 in the case of an equilateral triangle uniformly refined two times. Γ^1_i, $i = 1, 2, 3, 4$ are the 4 triangles after the first refinement and Γ^2_i, $i = 1, \ldots, 4^2 = 16$ are the 16 triangles after the second refinement. $1 \le k_1 \le 4$ and $1 \le k_2 \le 16$ is such that $\Gamma^2_{k_2}$ is inside $\Gamma^1_{k_1}$. The point about listing these is that the loop integral around Γ can be written as

$$\oint_\Gamma f(z) \, \mathrm{d}z = \sum_{i=1}^4 \oint_{\Gamma^1_i} f(z) \, \mathrm{d}z = \sum_{i=1}^{16} \oint_{\Gamma^2_i} f(z) \, \mathrm{d}z = \cdots$$

This follows as each internal edge involves one direction on one of the triangles and the opposite direction on the other triangle which shares the edge and as a consequence due to the internal edges is zero. Now by using the triangle inequality

$$\left| \oint_\Gamma f(z) \, \mathrm{d}z \right| \le \sum_{i=1}^4 \left| \oint_{\Gamma^1_i} f(z) \, \mathrm{d}z \right| \le \sum_{i=1}^{16} \left| \oint_{\Gamma^2_i} f(z) \, \mathrm{d}z \right| \le \cdots$$

k_1 is chosen so that it corresponds to the largest contribution of the 4 choices, k_2 is chosen so that it corresponds to the largest contribution of the 4 choices from the triangles inside $\Gamma^1_{k_1}$. Continuing this argument leads to the bound

$$\left|\oint_\Gamma f(z)\,\mathrm{d}z\right| \le 4^n \left|\oint_{\Gamma^n_{k_n}} f(z)\,\mathrm{d}z\right|.$$

If the length of Γ is L then the length of $\Gamma^n_{k_n}$ is $L/2^n$. The nested sequence of triangles have a common point which we label here as z_0. The detail then uses a relation given just after the term analytic is defined earlier in the notes about a consequence of $f(z)$ being analytic at z_0. Specifically, if $f(z)$ is analytic at z_0 then we can define a a continuous function $\lambda(z)$ by

$$\lambda(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0), & z \ne z_0, \\ 0, & z = z_0 \end{cases}.$$

Rearranging this gives

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \lambda(z)(z-z_0).$$

Now as $f(z_0) + f'(z_0)(z-z_0)$ is a polynomial it has an anti-derivative and thus

$$\oint_{\Gamma^n_{k_n}} \left(f(z_0) + f'(z_0)(z-z_0)\right)\,\mathrm{d}z = 0.$$

Hence

$$\oint_{\Gamma^n_{k_n}} f(z)\,\mathrm{d}z = \oint_{\Gamma^n_{k_n}} \lambda(z)(z-z_0)\,\mathrm{d}z.$$

To repeat, the length of $\Gamma^n_{k_n}$ is $L/2^n$ and for all z on the triangle $|z-z_0| < L/2^n$. Now if take and $\epsilon > 0$ the the continuity of $\lambda(z)$ at $z = z_0$ implies that there exists a $\delta > 0$ such that whenever $|z-z_0| < \delta$ we have $|\lambda(z)| < \epsilon$. For sufficiently large n the triangle $\Gamma^n_{k_n}$ will be in this region and ML-inequality gives the bound

$$\left|\oint_{\Gamma^n_{k_n}} \lambda(z)(z-z_0)\,\mathrm{d}z\right| \le \left(\frac{L}{2^n}\right)\left(\frac{L}{2^n}\right)\epsilon = \frac{L^2}{4^n}\epsilon.$$

Thus

$$\left|\oint_\Gamma f(z)\,\mathrm{d}z\right| \le 4^n \left|\oint_{\Gamma^n_{k_n}} \lambda(z)(z-z_0)\mathrm{d}z\right| \le L^2\epsilon.$$

As this is true for all $\epsilon > 0$ it follows that the value of our loop integral is smaller than any positive number and as a consequence our loop integral is 0.

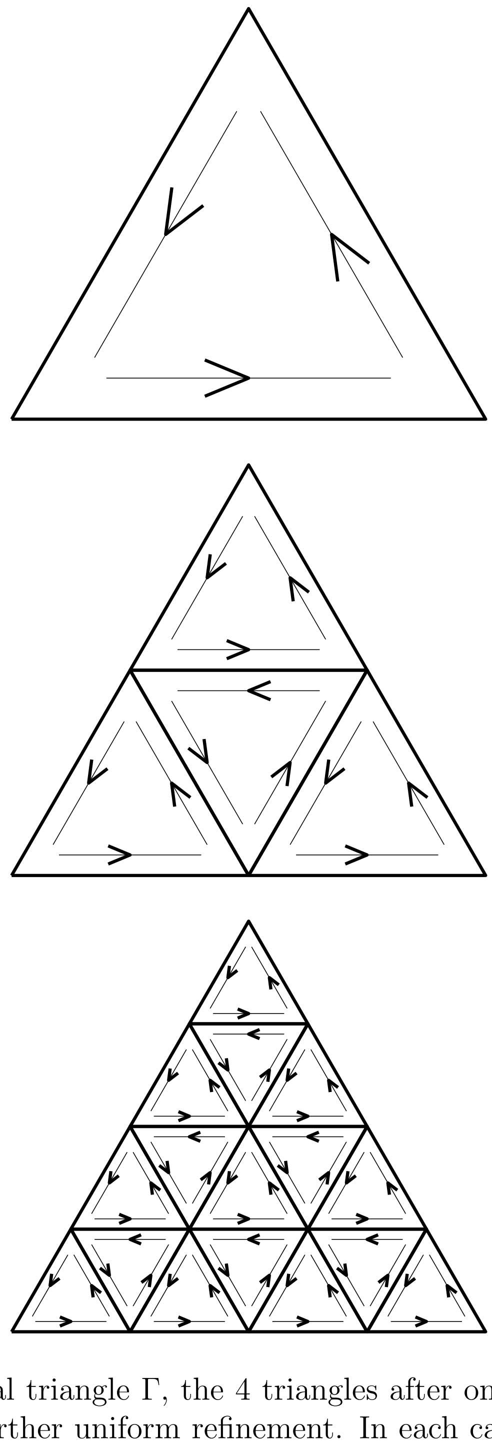


Figure 5.3: The equilateral triangle Γ, the 4 triangles after one uniform refinement, and the 16 triangles after a further uniform refinement. In each case the arrows indicate the direction of the integration for each triangle being considered.

5.5.4 Domains which are not simply-connected and deforming contours

The path independent results in section 5.4 required only that the domain being considered is connected whereas the Cauchy theorem considered here has required that the domain is simply connected. In particular the results of section 5.4 involving anti-derivatives enabled us to be able to integrate functions $f(z)$ with poles and in particular we showed that when we have a loop Γ in the anti-clockwise direction, which winds once around points,

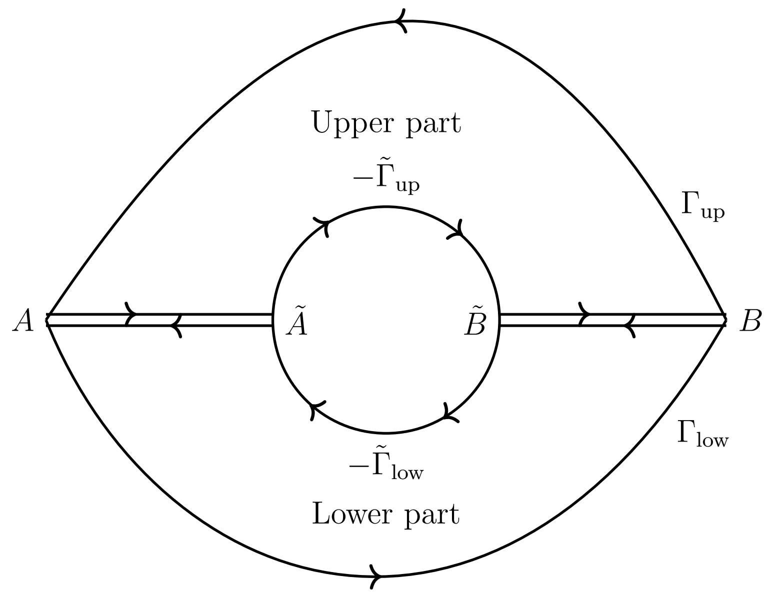
$$\oint_\Gamma \frac{\mathrm{d}z}{z-z_0} = \begin{cases} 2\pi i, & \text{if } z_0 \text{ is inside } \Gamma, \\ 0, & \text{if } z_0 \text{ is outside } \Gamma, \end{cases}$$

and when z_0 is inside Γ we also have

$$\oint_\Gamma \frac{\mathrm{d}z}{(z-z_0)^n} = 0, \quad \text{for } n = 2, 3, \ldots$$

In all cases when z_0 is not inside Γ and we have an analytic function inside the loop the Cauchy integral theorem tells us that we get 0. The Cauchy theorem does not apply in the cases when z_0 is inside the loop as the function is not analytic at all points inside the loop. However we can use the Cauchy theorem to "deform" the contour involved as follows.

Suppose that $f(z)$ is analytic between the loop Γ and the loop $\tilde{\Gamma}$, as illustrated below, and we want to compute the integral around Γ in the anti-clockwise sense.



We let A and B be points on Γ and similarly let $\tilde{A}$ and $\tilde{B}$ be points on $\tilde{\Gamma}$. Referring to the diagram the doubly connected domain between Γ and $\tilde{\Gamma}$ can de divided into two simply connected domains by considering the upper part above A to $\tilde{A}$ and B to $\tilde{B}$ and the lower part which is below these line segments. The Cauchy theorem applies to the simply connected domains. Now let $\Gamma_{\rm up}$ and $\tilde{\Gamma}_{\rm up}$ be the parts of the loops in the upper domain (with both considered in the anti-clockwise sense) as shown in the figure. Note that the contour around the edge of the upper domain in the anti-clockwise sense involves going around $\Gamma_{\rm up}$ in the anti-clockwise sense and with going around $\tilde{\Gamma}_{\rm up}$ in the clockwise sense (this is why it is shown as $-\tilde{\Gamma}_{\rm up}$ in the figure). We have a similar situation in the lower domain and if we let $\Gamma_{\rm low}$ and $\tilde{\Gamma}_{\rm low}$ be the parts of the loops in the lower domain (with both considered in the anti-clockwise sense) then the contour around the edge of the lower domain in the anti-clockwise sense involves going around $\Gamma_{\rm low}$ in the anti-clockwise sense and with going around $\tilde{\Gamma}_{\rm low}$ in the clockwise sense (this is why it is shown as $-\tilde{\Gamma}_{\rm low}$ in the figure). The paths along the line segments from A to $\tilde{A}$ and from B to $\tilde{B}$ are in the opposite sense in the lower domain compared with the upper domain. In terms of the integrals the Cauchy theorem gives for the upper and lower simply connected domains respectively the following.

$$\begin{aligned}
\int_{\Gamma_{\rm up}} f(z)\,{\rm d}z - \int_{\tilde{\Gamma}_{\rm up}} f(z)\,{\rm d}z + \int_A^{\tilde{A}} f(z)\,{\rm d}z + \int_{\tilde{B}}^{B} f(z)\,{\rm d}z &= 0, \\
\int_{\Gamma_{\rm low}} f(z)\,{\rm d}z - \int_{\tilde{\Gamma}_{\rm low}} f(z)\,{\rm d}z + \int_{\tilde{A}}^{A} f(z)\,{\rm d}z + \int_{B}^{\tilde{B}} f(z)\,{\rm d}z &= 0.
\end{aligned}$$

Adding these last two results, dealing with the cancellations along the lines joining the loops, and collecting the parts involving Γ and $\tilde{\Gamma}$ gives

$$\int_{\Gamma_{\rm up}} f(z)\,{\rm d}z + \int_{\Gamma_{\rm low}} f(z)\,{\rm d}z = \int_{\tilde{\Gamma}_{\rm up}} f(z)\,{\rm d}z + \int_{\tilde{\Gamma}_{\rm low}} f(z)\,{\rm d}z$$

which is equivalent to

$$\oint_{\Gamma} f(z)\,{\rm d}z = \oint_{\tilde{\Gamma}} f(z)\,{\rm d}z.$$

It is in this sense that we can deform Γ to $\tilde{\Gamma}$ without changing the value of the integral when $f(z)$ is analytic between Γ and $\tilde{\Gamma}$. In particular, if $f(z)$ just has an isolated singularity at z_0 which is inside Γ then we can deform Γ to a circle (with a choice of the radius) with z_0 as the centre and attempt to directly determine the integral and we have already directly computed integrals of the type $1/(z-z_0)^n$, $n = 1, 2, \ldots$ when we have circles centred at z_0.

5.6 Examples of integrating rational functions

Later in the module the residue theorem will be covered for evaluating integrals involving loops for general functions $f(z)$ which are analytic except at isolated pole singularities. In the case of rational functions which we can express, or partially express, in terms of partial fractions we can do this already as follows. If we just restrict to the case

$$R(z) = \frac{p(z)}{q(z)}, \quad q(z) = (z-z_1)(z-z_2)\cdots(z-z_n),$$

where $p(z)$ is a polynomial and $z_1, \dots, z_n$ are distinct then we have already shown that

$$R(z) = (\text{polynomial}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}$$

with

$$A_k = \frac{p(z_k)}{q'(z_k)}, \quad k = 1, \dots, n. \tag{5.6.1}$$

More generally, if instead the denominator is of the form

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}, \quad r_k \ge 1$$

then we still have a relation of the form

$$R(z) = (\text{a function with an anti-deriv}) + \sum_{k=1}^{n} \frac{A_k}{z - z_k}.$$

It is convenient at this stage to introduce some notation which we will use later for a residue which is

$$\text{Res}(R,\, z_k) = \text{Residue of the function } R \text{ at the point } z_k.$$

If we have a loop Γ in the anticlockwise sense, and for ease of labelling we have $z_1, \dots, z_m$ inside the loop and the other points outside of the loop, then

$$\begin{aligned}
\oint_\Gamma R(z)\,\mathrm{d}z &= \oint_\Gamma (\text{a function with an anti-deriv})\,\mathrm{d}z + \sum_{k=1}^{n} A_k \oint_\Gamma \frac{\mathrm{d}z}{z - z_k} && (5.6.2) \\
&= \sum_{k=1}^{m} A_k \oint_\Gamma \frac{\mathrm{d}z}{z - z_k} && (5.6.3) \\
&= 2\pi i \sum_{k=1}^{m} A_k = 2\pi i \sum_{k=1}^{m} \text{Res}(R,\, z_k), && (5.6.4)
\end{aligned}$$

where (5.6.3) follows from (5.6.2) as the loop integral of a function with an anti-derivative is zero and we have already directly obtained integrals involving terms such as $1/(z - z_k)$. The final answer just depends on the residues at the points $z_1, \dots, z_m$ which are inside the loop and the residue A_k is given explicitly in (5.6.1) in the case of a simple pole. This result is a particular case of the Residue theorem which is covered later in the module.

5.6.1 Application to trigonometric integrals over $[0, 2\pi]$ — using the the unit circle

We consider here integrals involving $\cos\theta$ and $\sin\theta$ over a 2π-range of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta$$

where $R(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$. Thus as examples, we can have the following.

1.
$$\int_{-\pi}^{\pi} \frac{4\mathrm{d}\theta}{5+2\cos\theta}.$$

2.
$$\int_{0}^{2\pi} (\cos\theta)^{2n}\,\mathrm{d}\theta.$$

3.
$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{1+\sin^2\theta}.$$

4.
$$\int_{0}^{2\pi} \frac{\sin^2\theta}{5+4\cos^2\theta}.$$

Note that in each case the integrand is real and we hence get a real value for the integral.

The technique to determine each of these is by using the substitution

$$z = \mathrm{e}^{i\theta}, \quad \text{and note that } \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz},$$

and as θ varies from $-\pi$ to π or from 0 to 2π this corresponds to z traversing the unit circle once in the anti-clockwise direction. To write all parts in terms of z we have for the trig. functions

$$\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}\right) = \frac{1}{2}\left(z+\frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}\right) = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Let C denote the unit circle. In the general case this gives

$$\int_{0}^{2\pi} R(\cos\theta,\sin\theta)\,\mathrm{d}\theta = \oint_C R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{\mathrm{d}z}{iz}.$$

Let

$$F(z) = \frac{1}{z}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right).$$

As $R(.,.)$ is a rational function of its arguments it follows that $F(z)$ is a rational function of z and hence $F(z)$ has isolated singularities which are poles. If the poles of $F(z)$ which are inside the unit circle are at $z_1, z_2, \ldots, z_m$ then by the Residue theorem result that we already have

$$\int_{0}^{2\pi} R(\cos\theta,\sin\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i}F(z)\,\mathrm{d}z = 2\pi\sum_{k=1}^{m}\mathrm{Res}(F,\, z_k).$$

We illustrate this using the examples listed earlier.

1. Let
$$I = \int_{-\pi}^{\pi} \frac{4\mathrm{d}\theta}{5+2\cos\theta}.$$

With the substitution $z = e^{i\theta}$ this leads to

$$F(z) = \frac{1}{z}\left(\frac{4}{5 + \left(z + \frac{1}{z}\right)}\right) = \frac{4}{z^2 + 5z + 1}.$$

The poles of $F(z)$ are at the zeros of the denominator which are

$$z_1 = \frac{-5 + \sqrt{21}}{2} \quad \text{and} \quad z_2 = \frac{-5 - \sqrt{21}}{2}.$$

The roots are real and the product of the roots is $z_1 z_2 = 1$ and by inspection $|z_1| < 1$ and $|z_2| > 1$. Hence we only have one simple pole inside the unit circle and by the Residue theorem result

$$I = 2\pi \mathrm{Res}(F,\, z_1).$$

To get the residue we have

$$\mathrm{Res}(F,\, z_1) = \lim_{z \to z_1} (z - z_1)F(z) = \lim_{z \to z_1} \frac{4(z - z_1)}{z^2 + 5z + 1} = \frac{4}{2z_1 + 5} = \frac{4}{\sqrt{21}}.$$

Hence

$$I = \frac{8\pi}{\sqrt{21}}.$$

2. Let now

$$I = \int_0^{2\pi} (\cos\,\theta)^{2n}\, \mathrm{d}\theta.$$

With the substitution $z = e^{i\theta}$ this leads to

$$F(z) = \frac{1}{z}\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^{2n} = \frac{1}{2^{2n}}\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}.$$

This has a pole at $z = 0$ of order $2n + 1$. In this case we can get the residue by considering the binomial expansion

$$\left(z + \frac{1}{z}\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^k \left(\frac{1}{z}\right)^{2n-k}.$$

We need the constant term from this part to get the residue and this corresponds to $k = n$ in the sum and we obtain

$$\mathrm{Res}(F,\, 0) = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}.$$

Thus

$$I = \frac{2\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}.$$

The cases when $n = 0$ and $n = 1$ can be quickly checked and we get 2π and π respectively.

3. Let now
$$I = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{1+\sin^2\theta}.$$
With the substitution $z = \mathrm{e}^{i\theta}$ this leads to
$$\begin{aligned} F(z) &= \frac{1}{z}\left(\frac{1}{1-\dfrac{1}{4}\left(z-\dfrac{1}{z}\right)^2}\right) \\ &= \frac{4z}{4z^2-(z^2-1)^2} = \frac{4z}{-z^4+6z^2-1}. \end{aligned}$$
The denominator is a quadratic in z^2 and at a root
$$z^2 = \frac{-6 \pm \sqrt{36-4}}{-2} = 3 \pm \sqrt{8}.$$
Let $\pm z_1$ be such that $z_1^2 = 3-\sqrt{8}$ which are the positions of the poles of $F(z)$ inside the unit circle. (If $\pm z_2$ denote the two points such that $z_2^2 = 3+\sqrt{8}$ then $|z_2| > 1$, i.e. the other poles of $F(z)$ are outside the unit circle.) For the residue at z_1 we have
$$\begin{aligned} \mathrm{Res}(F,\, z_1) &= \lim_{z\to z_1}(z-z_1)F(z) \\ &= 4z_1 \lim_{z\to z_1}\left(\frac{z-z_1}{-z^4+6z^2-1}\right) \\ &= 4z_1\left(\frac{1}{-4z^3+12z_1}\right) \\ &= \left(\frac{1}{-z_1^2+3}\right) = \frac{1}{\sqrt{8}}. \end{aligned}$$
Similarly
$$\mathrm{Res}(F,\, -z_1) = \frac{1}{\sqrt{8}}.$$
Hence
$$I = 2\pi\left(\mathrm{Res}(F,\, z_1) + \mathrm{Res}(F,\, -z_1)\right) = \frac{4\pi}{\sqrt{8}} = (\sqrt{2})\pi.$$
In this particular example the amount of detail when using this residue technique could have been shortened by noting that $\sin^2\theta$ has period of π as we can write
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$
Hence
$$1+\sin^2\theta = \frac{3-2\cos(2\theta)}{2} \quad \text{and} \quad \frac{1}{1+\sin^2\theta} = \frac{2}{3-2\cos(2\theta)}.$$
Then
$$I = 2\int_0^{\pi} \frac{\mathrm{d}\theta}{1+\sin^2\theta} = 4\int_0^{\pi} \frac{\mathrm{d}\theta}{3-2\cos(2\theta)}.$$

Now make the substitution $t = 2\theta$ and note that

$$\theta = \frac{t}{2}, \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}$$

we get

$$I = 2\int_0^{2\pi} \frac{\mathrm{d}t}{3 - 2\cos(t)}.$$

If we apply the technique to this version with $z = \mathrm{e}^{it}$ then we get

$$\frac{\mathrm{d}t}{\mathrm{d}z}\left(\frac{2}{3 - 2\cos(t)}\right) = \frac{1}{i}F(z)$$

where now

$$F(z) = \frac{1}{z}\left(\frac{2}{3 - (z + 1/z)/2}\right) = \frac{4}{6z - z^2 - 1}.$$

$$z^2 - 6z + 1 = 0 \quad \text{when } z = 3 \pm \sqrt{8}$$

and the root which is inside the unit circle is at $z_0 = 3 - \sqrt{8}$.

$$I = \oint_C \frac{1}{i}F(z)\,\mathrm{d}z = 2\pi\mathrm{Res}(F,\, z_0).$$

For the residue we have

$$\mathrm{Res}(F,\, z_0) = \lim_{z\to z_0}(z - z_0)F(z) = \frac{4}{6 - 2z_0} = \frac{2}{3 - z_0} = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}}.$$

We again get

$$I = \sqrt{2}\pi.$$

4. Let now

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos^2\theta}\,\mathrm{d}\theta.$$

We can just apply the usual technique with the substitution $z = \mathrm{e}^{i\theta}$ with the expression in its current form but the details are on the long side. To shorten things a little we can note what was done in the second version of the previous example in that the integrand in terms of θ as written has period π and we can express both $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos(2\theta)$.

$$\frac{\sin^2\theta}{5 + 4\cos^2\theta} = \frac{(1 - \cos(2\theta))/2}{5 + 2(1 + \cos(2\theta))} = \frac{1 - \cos(2\theta)}{14 + 4\cos(2\theta)}.$$

To further simplify before integration is attempted we note that

$$1 - \cos(2\theta) = -\frac{1}{4}(14 + 4\cos(2\theta)) + \frac{18}{4}$$

and thus

$$\frac{1 - \cos(2\theta)}{14 + 4\cos(2\theta)} = -\frac{1}{4} + \frac{9/4}{7 + 2\cos(2\theta)}$$

giving

$$I = -\frac{\pi}{2} + \frac{9}{4}\int_0^{2\pi} \frac{\mathrm{d}\theta}{7 + 2\cos(2\theta)}.$$

As the contribution from $(0, \pi)$ and $(\pi, 2\pi)$ are the same we have

$$I = -\frac{\pi}{2} + \frac{9}{2}\int_0^{\pi} \frac{\mathrm{d}\theta}{7 + 2\cos(2\theta)}.$$

If we let $t = 2\theta$ then $\mathrm{d}\theta/\mathrm{d}t = 1/2$ and we have

$$I = -\frac{\pi}{2} + \frac{9}{4}\int_0^{2\pi} \frac{\mathrm{d}t}{7 + 2\cos(t)}.$$

We now make the substitution $z = \mathrm{e}^{it}$ and have

$$\frac{\mathrm{d}t}{\mathrm{d}z}\left(\frac{\mathrm{d}t}{7 + 2\cos(t)}\right) = \frac{1}{i}F(z) \quad \text{with } F(z) = \frac{1}{z}\left(\frac{1}{7 + z + \dfrac{1}{z}}\right) = \frac{1}{z^2 + 7z + 1}.$$

$F(z)$ has two simple poles at points given by

$$z_1 = \frac{-7 + \sqrt{45}}{2} \quad \text{and} \quad z_2 = \frac{-7 - \sqrt{45}}{2}$$

with the point z_1 inside the unit circle. For the residue we have

$$\mathrm{Res}(F,\, z_1) = \lim_{z \to z_1} \frac{z - z_1}{z^2 + 7z + 1} = \frac{1}{2z_1 + 7} = \frac{1}{\sqrt{45}}.$$

$$I = -\frac{\pi}{2} + \frac{9\pi}{2\sqrt{45}} = -\frac{\pi}{2} + \frac{3\pi}{2\sqrt{5}}.$$

Chapter 6

Cauchy's integral formula, consequences and bounds

The latter parts of this chapter, specifically the parts after Liouville's theorem on page 6-13, are not examinable.

6.1 A recap of some properties of analytic functions

We start by collecting together the definition of an analytic function together with some of the key results about such functions encountered so far.

1. $f(z)$ is complex differentiable at z_0 if

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, i.e. the limit is independent of how $h \to 0$.

2. A function f is analytic at z_0 if f is differentiable at all points in some neighbourhood of z_0.

3. Let $f = u + iv$ with $u(x, y)$ and $v(x, y)$ being real valued. The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

When u and v have continuous partial derivatives in a domain D the function $f = u + iv$ is analytic on D if and only if the Cauchy Riemann equations are satisfied throughout D.

4. Cauchy-Goursat theorem: If f is analytic in a simply connected domain D and Γ is any loop (i.e. a closed contour) in D then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

5. The Cauchy-Goursat theorem implies the existence of an anti-derivative F (which is analytic) such that $f(z) = F'(z)$ when $f(z)$ is analytic in a simply-connected domain.

6. If f is analytic between loops Γ_1 and Γ_2 then

$$\oint_{\Gamma_1} f(z)\,\mathrm{d}z = \oint_{\Gamma_2} f(z)\,\mathrm{d}z.$$

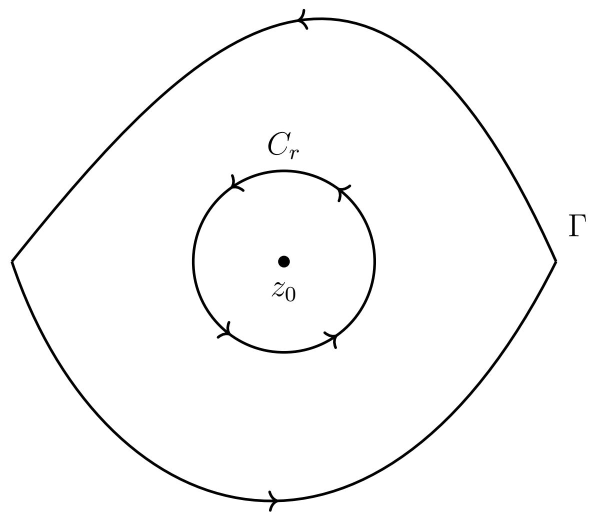
7. For a particular loop integral which gives a non-zero value we have

$$\oint_\Gamma \frac{\mathrm{d}z}{z-z_0} = \begin{cases} 2\pi i, & \text{if } z_0 \text{ is inside } \Gamma, \\ 0, & \text{if } z_0 \text{ is outside } \Gamma. \end{cases}$$

In the remainder of this chapter we show that an analytic function can be represented by an integral (the Cauchy integral formula) and we will then go on to justify the previous claim that being able to differentiate once in the complex sense actually implies that we must be able to differentiate infinitely many times.

6.2 The Cauchy integral formula

To derive the integral formula let Γ denote a closed positively orientated loop traversed once and suppose that $f(z)$ is analytic in a simply connected domain D which contains Γ and further let z_0 be a point inside Γ. Also let C_r be a circle centred at z_0 with radius r where r is sufficiently small such that the circle is inside Γ. A set-up of this type is shown below.



Now if we consider the integral

$$\oint_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

then although $f(z)/(z-z_0)$ is not analytic at all points inside Γ it is analytic between Γ and C_r and thus

$$\oint_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

and we can take r to be arbitrarily small. To obtain the value we write

$$f(z) = f(z_0) + (f(z) - f(z_0))$$

so that

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + \frac{f(z)-f(z_0)}{z-z_0}.$$

We can integrate the first term and hence

$$\oint_{C_r} \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i f(z_0) + \oint_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z.$$

We explain next that the last integral in the above is 0 and the key to showing this is to get a suitable bound on the magnitude of the integrand valid for all $r>0$.

To get a bound let

$$M(r) = \max\{|f(z)-f(z_0)| : z \in C_r\}$$

and note that $M(r)\to 0$ as $r\to 0$ as $f(z)$ is continuous in D. On the circle we have $|z-z_0| = r$ and thus on the circle the integrand is bounded by

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le \frac{M(r)}{r}.$$

As the length of C_r is $2\pi r$ the use of the ML inequality gives the bound

$$\left|\oint_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\right| \le \frac{M(r)}{r}(2\pi r) = 2\pi M(r) \to 0 \quad \text{as } r\to 0.$$

As the value of the integral does not depend on r it follows that it must be less than $2\pi M(r)$ for all r and this is only possible if the value is 0.

To summarize, we have shown that when z_0 is inside a positively orientated loop Γ traversed once and $f(z)$ is analytic on and inside Γ we have

$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z. \tag{6.2.1}$$

This is known as the Cauchy integral formula.

6.2.1 Representations for $f(z)$, $f'(z)$, ..., $f^{(n)}(z)$, ...

The result in (6.2.1) tells us that when f is known to be analytic on Γ and inside Γ then the value of $f(z)$ at any point z inside Γ is completely determined by the values of the function on Γ. That is if we first change the integration variable in (6.2.1) from z to ζ

$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta$$

and then replace z_0 by z we have the representation

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

As we next show, we can differentiate the expression on the right hand side to get an expression for $f'(z)$ which will in turn show that $f'(z)$ is also analytic and we can continue this process to establish that the nth derivative $f^{(n)}(z)$ is analytic and in each case we have a representation as an integral.

The expressions for the derivatives can be obtained relatively easily if we just accept that that we can differentiate through the integral. With the expression being considered as a function of ζ and z in the meaning of the following partial derivatives we have

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right) &= \frac{1}{(\zeta - z)^2}, \\
\frac{\partial^2}{\partial z^2}\left(\frac{1}{\zeta - z}\right) &= \frac{2}{(\zeta - z)^3}, \\
\cdots & \quad \cdots \\
\frac{\partial^n}{\partial z^n}\left(\frac{1}{\zeta - z}\right) &= \frac{n!}{(\zeta - z)^{n+1}},
\end{aligned}$$

and the result is that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

To properly justify this it is necessary to justify that we can interchange the operations of taking the derivative and of doing the integral (which both involve taking limits) and we consider this next just in the case of $f'(z)$ and to this end we consider the following. Let

$$I = \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta.$$

Our aim is to show that this tends to 0 as $h \to 0$. We first substitute the integral representations for $f(z+h)$ and $f(z)$ and combine the integrands to give

$$I = \frac{1}{2\pi i} \oint_\Gamma f(\zeta) \left(\frac{1}{h}\left(\frac{1}{\zeta - (z+h)} - \frac{1}{\zeta - z} \right) - \frac{1}{(\zeta - z)^2} \right) \mathrm{d}\zeta.$$

Now

$$\frac{1}{h}\left(\frac{1}{\zeta - (z+h)} - \frac{1}{\zeta - z} \right) = \frac{1}{(\zeta - (z+h))(\zeta - z)}$$

and

$$\frac{1}{(\zeta - (z+h))(\zeta - z)} - \frac{1}{(\zeta - z)^2} = \frac{h}{(\zeta - z)^2(\zeta - (z+h))}$$

and our expression for I is

$$I = \frac{h}{2\pi i} \oint_\Gamma f(\zeta) \frac{1}{(\zeta - z)^2(\zeta - (z+h))}\, \mathrm{d}\zeta.$$

To justify that this tends to 0 as $h \to 0$ it is sufficient to show that the integral remains bounded as $h \to 0$ and we can achieve this by proving that the integrand is bounded for all ζ on Γ. Let

$$M = \max\{|f(\zeta)| : \zeta \in \Gamma\}$$

and for z inside Γ let d be the distance of z from Γ, i.e.

$$d = \min_{\zeta \in \Gamma} |\zeta - z|.$$

Now for some values of h the point $z + h$ may be closer to Γ than is z but when h is sufficiently small we can guarantee that it is not too much closer with in particular

$$|\zeta - (z + h)| \le \frac{d}{2}, \quad \text{for all } \zeta \in \Gamma,$$

and thus for $\zeta \in \Gamma$

$$\left| \frac{f(\zeta)}{(\zeta - z)^2(\zeta - (z + h))} \right| \le \frac{M}{d^3/2}.$$

Hence by the ML inequality

$$|I| \le \frac{h}{2\pi} \left(\frac{M}{d^3/2} \right) \text{length}(\Gamma) \to 0 \quad \text{as } h \to 0.$$

Thus we have shown that

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta,$$

i.e. we have shown that

$$f'(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta.$$

It is the fact that we have this representation which enables us to deduce that $f'(z)$ is also analytic and we can similarly get a representation for $f''(z)$ and continue the process for all the derivatives.

To summarize, if f is analytic in a domain D then f', f'', ... exist and are also analytic and if Γ is any positively orientated loop which is traversed once and which contains z_0 then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z. \qquad (6.2.2)$$

The representation (6.2.2) is known as the generalised Cauchy integral formula. In the next chapter we will consider the Taylor series representation of an analytic function and it is worth noting here that we have a representation for the Taylor coefficients of the form

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z. \qquad (6.2.3)$$

6.2.2 A comment about representation of functions by integrals

If Γ denotes a closed loop traversed once in the anti-clockwise direction and if $g(\zeta)$ is any continuous function defined on Γ (but possibly not defined anywhere else) then we can define a function of the form

$$G(z) = \oint_\Gamma \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

By the same technique as was used in the previous section it can be shown that this defines an analytic function for z inside Γ and it also defines an analytic function for z outside Γ. We will meet an expression of this type to represent the remainder in a Taylor expansion in this next chapter and we consider a particular case of this here.

Consider the case when we have a function f which is analytic on Γ and inside Γ and we let

$$g(\zeta) = \frac{1}{2\pi i}\left(\frac{f(\zeta)}{\zeta - z_0}\right),$$

where z_0 is a point inside Γ, and hence

$$G(z) = \oint_\Gamma \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)(\zeta - z)}\,\mathrm{d}\zeta.$$

When $z = z_0$ we have the generalised Cauchy integral formula case and

$$G(z_0) = f'(z_0).$$

When $z \ne z_0$ we can use partial fractions to re-write part of the integrand, i.e.

$$\frac{1}{(\zeta - z_0)(\zeta - z)} = \frac{A}{\zeta - z_0} + \frac{B}{\zeta - z}$$

and for all ζ

$$1 = A(\zeta - z) + B(\zeta - z_0) \quad \text{giving } B = -A = \frac{1}{z - z_0}.$$

Thus

$$G(z) = \left(\frac{1}{z - z_0}\right)\frac{1}{2\pi i}\left(-\oint_\Gamma \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta + \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right) = \frac{f(z) - f(z_0)}{z - z_0}$$

where the last equality is from noting that each integral is directly of the form where the Cauchy integral formula applies. Thus to summarize, we have shown that the function

$$G(z) = \begin{cases} f'(z_0), & \text{if } z = z_0, \\ \dfrac{f(z) - f(z_0)}{z - z_0}, & \text{if } z \ne z_0, \end{cases}$$

is analytic inside Γ as well as having an integral representation.

Just for information at this stage, the generalisation that we show in the next chapter is that

$$G(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)}\,\mathrm{d}\zeta = \begin{cases} \dfrac{f^{(n+1)}(z_0)}{(n+1)!}, & \text{if } z = z_0, \\ \dfrac{f(z) - \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k}{(z - z_0)^{n+1}}, & \text{if } z \ne z_0. \end{cases}$$

6.2.3 Morera's theorem characterising the analytic property

As another immediate consequence of the previous sections and the results about anti-derivatives we obtain another way of determining whether or not a given function f is analytic in a domain D which has the name of **Morera's theorem** which we state next.

Theorem 6.2.1 *If f is continuous in a domain D and if*

$$\oint_\Gamma f(z)\,dz = 0$$

for every closed contour Γ in D then f is analytic in D.

Proof: The properties that f is continuous and that all loop integrals are 0 implies that there exists an anti-derivative F, i.e. there exists an analytic function F such that $F' = f$. (This was shown in the last part of section 5.4.) The result of the last section implies that F', F'' etc. are also analytic and thus in particular $f = F'$ is analytic. □

This theorem can be considered as the reverse of the Cauchy integral theorem, i.e. if f is analytic then all loop integrals of f are 0 and if all loop integrals of f are 0 then f is analytic.

We have not encountered a situation yet when this result would have helped to determine that a given function f is analytic as in all cases we have had an explicit expression for $f(z)$ and we could determine the analyticity more directly. However the result is needed later in the module when series are considered and our knowledge of $f(z)$ is only as the limit of a uniformly convergent series involving analytic functions and Morera's theorem is the result that will be used to explain why the limit function is also analytic.

6.2.4 The differentiability of u and v when $f = u + iv$ is analytic

If $f = u + iv$ is analytic in a domain D, with u and v being real valued, then by the Cauchy Riemann equations we can represent $f'(z)$ in the following ways

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

As f' is also analytic all of the 4 partial derivatives above are also continuously differentiable with respect to x and y, i.e. u and v are twice continuously differentiable, and we can represent f'' in several ways, i.e.

$$\begin{aligned} f''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial x \partial y} \\ &= \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}. \end{aligned}$$

This process can be repeated indefinitely and we deduce that all partial derivatives of u and v exist are are continuous in a domain D.

A particular outcome of the result about the differentiability of u and v is that when f is analytic the functions u and v always have continuous partial derivatives of all orders to be able to deduce that both u and v are harmonic functions, i.e.

$$\nabla^2 u = 0 \quad \text{and} \quad \nabla^2 v = 0.$$

6.2.5 The existence of the harmonic conjugate in simply connected domains

When we have an analytic function we immediately have two related harmonic functions and in term 1 several examples were considered in which a harmonic function u was given and a harmonic conjugate v was constructed using the Cauchy Riemann equations. If you revisit that section then there is a comment that this is always possible in a domain which is simply connected and we are now in a position to justify that comment as follows.

Suppose that D is a simply connected domain and suppose that ϕ is harmonic in D. Now if ϕ is the real part of an analytic function then the derivative of that analytic function is given by

$$g(z) = u_1 + iv_1 = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}. \tag{6.2.4}$$

Hence we start by defining g in this way and consider its properties, i.e. is $g(z)$ always analytic and does it have an anti-derivative in D. Now by considering the Cauchy Riemann equations applied to $u_1 + iv_1$ we have

$$\begin{aligned} \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{as } \phi \text{ is harmonic,} \\ \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} &= \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} = 0, \end{aligned}$$

as mixed partial derivatives can be done in any order, which shows that g is analytic. The property that D is simply connected implies that there exists an anti-derivative $G = u+iv$, with

$$g = G' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{6.2.5}$$

Now if we compare (6.2.4) and (6.2.5) it follows that

$$\frac{\partial (u - \phi)}{\partial x} = \frac{\partial (u - \phi)}{\partial y} = 0$$

and u and ϕ only differ by a constant, i.e.

$$\phi = u + c,$$

where c is a constant, and ϕ is the real part of the analytic function $G(z) + c$ and the existence of a harmonic conjugate $v = \mathrm{Im}(G + c)$ has been established.

To illustrate why we cannot always get another function v such that $u + iv$ is analytic throughout D when D is not simply connected we just need to consider the case when D is the annulus

$$D = \{z : \ 0 < r_1 < |z| < r_2\}$$

and we have the function

$$u(x, y) = \ln(r) = \ln |z| = \frac{1}{2} \ln(x^2 + y^2).$$

This function is harmonic in D but no harmonic conjugate v exists such that $u + iv$ is analytic in D. If we remove part of D to create the simply connected domain

$$D' = \{z \in D : \ \mathrm{Arg}\, z \neq \pi\}$$

then the harmonic conjugate

$$v = \operatorname{Arg} z$$

is such that

$$u + iv = \ln|z| + i\operatorname{Arg} z = \operatorname{Log}(z)$$

is analytic in D' but the function is not analytic throughout D due to the branch cut along the negative real axis.

6.2.6 Examples involving evaluating loop integrals

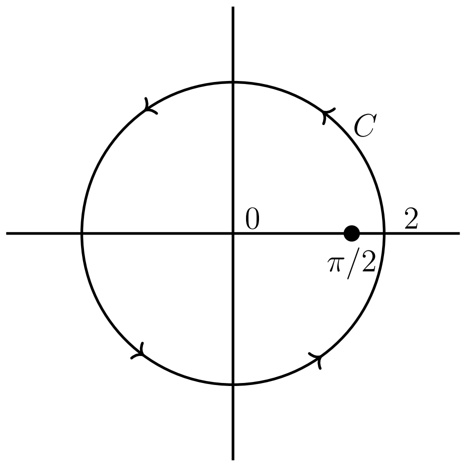
At the end of chapter 5 there were examples involving loop integrals and rational functions. In the following examples we now consider integrands which are more general.

1. Let C denote the contour which is the circle centred at 0 and radius 2 and traversed once in the anti-clockwise direction.

   (a) Evaluate

   $$I = \oint_C \frac{\sin(3z)}{z - \pi/2}\,\mathrm{d}z.$$

   To picture the contour and the pole of the integrand we have the following.

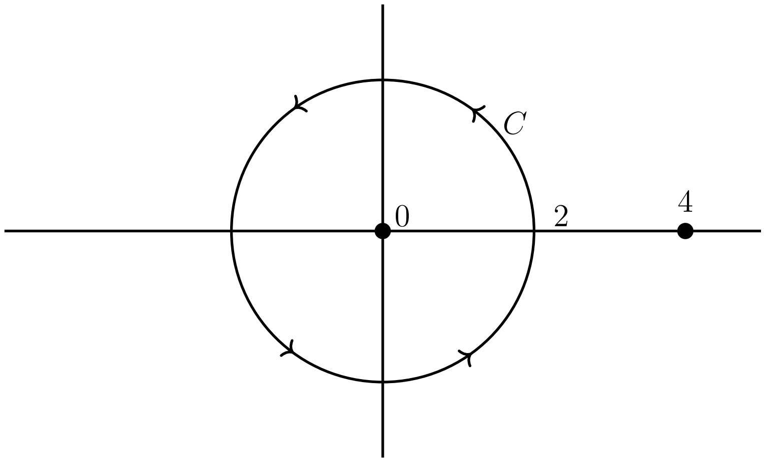
   The simple pole at $z_0 = \pi/2$ is inside C and if we let $f(z) = \sin(3z)$ then the function $f(z)$ is analytic on and inside C and hence the conditions of the Cauchy integral formula apply and we have

   $$I = 2\pi i f(\pi/2) = (2\pi i)\sin(3\pi/2) = -2\pi i.$$

   (b) Evaluate

   $$I = \oint_C \frac{\sin\, z}{z^2(z-4)}\,\mathrm{d}z.$$

   To picture where the poles of the integrand are relative to the contour we have the following.



The pole at $z_0 = 0$ is inside C but the simple pole at 4 is outside the contour. We let

$$f(z) = \frac{\sin\, z}{z-4}$$

which is analytic inside C. By the generalised Cauchy integral formula to represent the first derivative we get

$$I = 2\pi i f'(0).$$

The formula for the first derivative is used here as the integrand is $f(z)/z^2$. Now

$$f'(z) = (\sin\, z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z-4}\right) + (\cos\, z)\left(\frac{1}{z-4}\right) \quad \text{giving } f'(0) = -\frac{1}{4}$$

and thus

$$I = -\frac{\pi}{2} i.$$

As a comment here, the integrand actually has a simple pole at $z = 0$ as $\sin(z)/z$ has removable singularity at $z = 0$ with a limit value of 1 as $z \to 0$. Thus we could have taken

$$g(z) = \frac{\sin(z)}{z(z-4)}.$$

The integral formula just involves the values of this on the boundary and this is analytic inside the circle when the value at $z = 0$ is the limit as $z \to 0$. If we assume that here then

$$g(0) := \lim_{z\to 0} g(z) = \frac{1}{-4}$$

and we get the same result as above, i.e.

$$I = 2\pi i g(0) = -\frac{\pi}{2} i.$$

2. In section 5.6 we established the residue theorem result for any rational function of the form

$$R(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials, although we restricted some of the discussion to a rational function with simple poles. In the general case of poles of any order as a result of the denominator being of the form

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n},$$

with $r_k \ge 1$ for $k = 1, \ldots, n$, it is still true that

$$\oint_\Gamma R(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(R,\, z_k),$$

where $z_1, \ldots z_m$ are inside Γ and $z_{m+1}, \ldots, z_n$ are outside Γ. With the use of partial fractions and the Cauchy integral formulas we are now in a position to cope with functions of the form $f(z)/q(z)$ where now $f(z)$ is any function which is analytic inside the loop Γ but the denominator is still a polynomial as we describe next.

When we only have simple poles we have the partial fraction representation of $1/q(z)$ of the form

$$\frac{1}{q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - z_k} \quad \text{and } A_k = \lim_{z \to z_k} \frac{z - z_k}{q(z)} = \frac{1}{q'(z_k)}.$$

In this case if $f(z)$ is any function which is analytic inside the loop Γ then

$$g(z) = \frac{f(z)}{q(z)} = \sum_{k=1}^{n} A_k \frac{f(z)}{z - z_k}$$

Hence if Γ denotes a positively orientated loop considered once and if $z_1, \ldots, z_m$ are inside Γ and $z_{m+1}, \ldots, z_n$ are outside Γ then

$$\oint_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} A_k f(z_k).$$

The term in the sum is $\mathrm{Res}(g,\, z_k)$ since

$$\begin{aligned} \mathrm{Res}(g,\, z_k) &= \lim_{z \to z_k} (z - z_k) g(z) = \lim_{z \to z_k} f(z) \left(\frac{z - z_k}{q(z)} \right) \\ &= \lim_{z \to z_k} f(z) \lim_{z \to z_k} \left(\frac{z - z_k}{q(z)} \right), \quad \text{by the properties of limits,} \\ &= f(z_k) A_k. \end{aligned}$$

Thus again we have

$$\oint_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(g,\, z_k).$$

If $q(z)$ has multiple zeros then the representation for $1/q(z)$ is of the form

$$\frac{1}{q(z)} = \left(\frac{A_{1,1}}{z - z_1} + \cdots + \frac{A_{r_1,1}}{(z - z_1)^{r_1}} \right) + \cdots + \left(\frac{A_{1,n}}{z - z_n} + \cdots + \frac{A_{r_n,n}}{(z - z_n)^{r_n}} \right)$$

and thus

$$\frac{f(z)}{q(z)} = \left(\frac{A_{1,1}f(z)}{z-z_1} + \cdots + \frac{A_{r_1,1}f(z)}{(z-z_1)^{r_1}}\right) + \cdots + \left(\frac{A_{1,n}f(z)}{z-z_n} + \cdots + \frac{A_{r_n,n}f(z)}{(z-z_n)^{r_n}}\right).$$

The integral in this case needs the use of generalised Cauchy integral formula although it can still be shown to be the case that

$$\oint_\Gamma g(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(g,\ z_k).$$

This will not be pursued further here as in a later chapter we will extend this further to the case of any function which is analytic inside Γ except at a finite number of isolated singularities at points $z_1, \ldots, z_m$ and to achieve the extension to this more general case we will need Laurent series representations of the function about each isolated singularity. Laurent series will be covered after Taylor series in the next chapter.

6.3 Bounds for analytic functions

One of the consequences of the Cauchy integral formula is that if z_0 is inside a positively orientated loop Γ and if f is analytic on and inside Γ then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z, \quad n = 0, 1, 2, \ldots$$

It is convenient now to take as the loop a circle

$$C_R = \left\{z = z_0 + R\mathrm{e}^{it} :\ 0 \le t \le 2\pi\right\}$$

and as the value of the integral does not depend on the loop used it follows that

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i}\oint_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z && (6.3.1)\\ &= \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0+R\mathrm{e}^{it})Ri\mathrm{e}^{it}}{R^{n+1}\mathrm{e}^{i(n+1)t}}\,\mathrm{d}t && (6.3.2)\\ &= \frac{n!}{2\pi R^n}\int_0^{2\pi} f(z_0+R\mathrm{e}^{it})\mathrm{e}^{-int}\,\mathrm{d}t. && (6.3.3)\end{aligned}$$

The case $n = 0$ is known as the mean value property.

For some bounds let

$$M(R) = \max\left\{|f(z)| :\ z \in C_R\right\} = \max\left\{|f(z_0+R\mathrm{e}^{it})| :\ 0 \le t \le 2\pi\right\}.$$

The magnitude of the integrand is bounded by $M(R)$ and thus we obtain

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi R^n}\int_0^{2\pi} M(R)\,\mathrm{d}t = \frac{n!M(R)}{R^n}. \qquad (6.3.4)$$

We start with the case of entire functions and what is known as **Liouville's theorem** and recall that an entire function is analytic in the entire complex plane.

Theorem 6.3.1 *The only bounded entire functions are constant functions.*

Proof: We show that $f(z)$ is a constant by showing that $f'(z) = 0$ and thus consider (6.3.4) with $n = 1$.

Now as $|f(z)|$ is assumed to be bounded we have that there exists a constant $K \ge 0$ such that

$$M(R) \le K \quad \text{for all } R.$$

Hence for all points z_0 and for all R we have

$$|f'(z_0)| \le \frac{K}{R} \to 0 \quad \text{as } R \to \infty.$$

As $|f'(z_0)|$ is less than every positive number we have $f'(z_0) = 0$ at every point $z_0 \in \mathbb{C}$ and thus $f(z)$ is a constant.

□

A polynomial is an entire function and Liouville's theorem is a key step in proving that when the degree of the polynomial is at least 1 it must have at least one zero in the complex plane and this is known as the **fundamental theorem of algebra**. The proof of this is a bit technical and it is not examinable but the result should be known. Thus this is where the non-examinable material in this chapter starts.

Theorem 6.3.2 *Every non-constant polynomial with complex coefficients has at least one zero.*

Proof: Let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0, \quad n \ge 1, \quad a_n \ne 0$$

denote the polynomial and let

$$f(z) = \frac{1}{p(z)}.$$

We use a proof by contradiction to show that $p(z)$ must have a zero and hence we start by assuming that $p(z)$ has no zeros in $\mathbb{C}$. An immediate consequence of this is that the function $f(z)$ is an entire function and we will get a contradiction if we can show that $f(z)$ is bounded as Liouville's theorem would then imply that it is a constant function which contradicts the degree n being at least 1.

To get a bound we first consider what happens for large $|z|$. Now

$$\frac{p(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \to a_n \ne 0 \quad \text{as } |z| \to \infty.$$

This in turn implies that there exists $R > 0$ such that for $|z| \ge R$ we have

$$\left|\frac{p(z)}{z^n}\right| \ge \frac{|a_n|}{2} \quad \text{and equivalently} \quad \left|\frac{z^n}{p(z)}\right| \le \frac{2}{|a_n|}$$

from which it follows that

$$|f(z)| = \left|\frac{1}{p(z)}\right| \le \frac{2}{|a_n||z^n|} \le \frac{2}{|a_n|R^n}.$$

Hence $|f(z)|$ is bounded in the region $|z| \ge R$. The region $|z| \le R$ is a closed bounded region and as the entire function $f(z)$ is continuous on the region it is bounded in this region. Thus $|f(z)|$ is bounded in the entire complex plane and, as mentioned above, this implies that $f(z)$ is constant which in turn implies that $p(z)$ is constant. This is the contradiction as we are assuming that $n \ge 1$, $a_n \ne 0$ and $p(z)$ is a non-constant polynomial. □

We finish the chapter by just stating a few results about where the largest value of $|f(z)|$ can occur when $f(z)$ is analytic with the final result being one version of what is known as the maximum modulus theorem which basically states that on a bounded domain the largest value can only occur on the boundary of the domain when the function is not a constant. No proofs are given here but it is worth noting that a named result which we have already established. When we take $n = 0$ in (6.3.3) we get the **mean value property**

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + R\mathrm{e}^{it})\,\mathrm{d}t. \tag{6.3.5}$$

This says that the value at z_0 is the average of the values on the circle of radius R for all values of R such that $f(z)$ is analytic inside the circle of this radius.

Lemma 6.3.1 *If $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D then $f(z)$ is a constant.*

As a note, this was on an exercise sheet when analytic functions were first discussed.

Lemma 6.3.2 *Suppose $f(z)$ is analytic in a disk centred at z_0. If the maximum value of $|f(z)|$ over the disk is the value at the centre, i.e. the value $|f(z_0)|$, then $f(z)$ is constant on the disk.*

The next result is one version of the **maximum modulus principle**.

Theorem 6.3.3 *If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point $z_0 \in D$ then f is a constant in D.*

If we now restrict to a bounded domain D and to functions $f(z)$ which are analytic in D and continuous in D and on its boundary then we know that $|f(z)|$ is bounded and it attains its maximum value. The previous theorem tells us that if $f(z)$ is not constant in D then the maximum value cannot be attained in D and thus it can only be attained on the boundary and this is another version of the maximum modulus principle.

Chapter 7

Taylor series and Laurent series

7.1 Introductory comments

Much of the module MA3614 is about analytic functions $f(z)$ of a complex variable z and in many cases $f(z)$ has just been an extension of a real valued function $f(x)$ of a real variable x to the complex plane. For example, the real valued functions x, x^2, ..., e^x, ... have been generalised to z, z^2, ..., e^z. Much of chapter 4 about the elementary functions was about such functions. We have also considered functions which are not analytic everywhere such as the following

$$\frac{1}{z-z_0}, \quad \frac{1}{(z-z_0)^2}, \dots, \quad \tan z = \frac{\sin z}{\cos z}, \dots, \exp(-1/z^2), \dots$$

which have isolated singularities. In these examples $1/(z-z_0)^m$, $m \ge 1$ being an integer has a pole of order m at z_0, $\tan z$ has infinitely many simple poles at $\pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \dots$ and $\exp(-1/z^2)$ has a singularity at $z = 0$ which, as we will see, is classified as an essential singularity. In addition, functions such as the principal valued logarithm $\operatorname{Log} z$ and the principal valued root function z^α, $\alpha \notin \mathbb{Z}$ have been introduced which have branch cuts starting at the branch point $z = 0$, i,e. $z = 0$ is not an isolated singularity, but the functions are analytic if we restrict to a region which does not include the branch cut.

This chapter is about representing analytic functions by a Taylor series about a point z_0 and conversely if we have a power series then we show that this defines an analytic function within a disk. Thus, as an example, if our function is

$$e^z := e^x(\cos y + i \sin y)$$

then we show that this implies that

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots .$$

Conversely, if

$$g(z) := \sum_{1}^{\infty} \frac{z^n}{n^2}$$

then we show that $g(z)$ defines a function which is analytic in a disk $|z| \le R$ for all $R < 1$ and there is a point on $|z| = 1$ where it is not analytic. (Actually the series is not defined for any z with $|z| > 1$ but it can be shown that the function defined in $|z| < 1$ can be analytically continued across part of the unit circle but not all of it and it is in this sense that there is at least one point on $|z| = 1$ at which the function is not analytic.) Hence, when $f(z)$ is analytic we will be concerned with series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and we generalise to the case when $f(z)$ has an isolated singularity at z_0 to a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

i.e. with negative powers as well. A series of this type is known as a Laurent series.

In this chapter there will not be rigorous proofs for all the results and the full longer proofs that are included, e.g. of the Taylor series representation and of the Laurent series representation, will not be examinable in their complete form. However, some of the techniques in these proofs may correspond, or nearly correspond, to what appears on the exercise sheets.

7.2 A revision of definitions and results about series of numbers

We collect together here some definitions and results about series which you should have done before year 3. Before year 3 you probably only considered series involving real numbers but all of the following also apply to complex numbers as often just the magnitude of numbers are involved and these are non-negative real numbers.

First a reminder of some definitions and results about the convergence of sequences.

Definition 7.2.1 *A sequence z_0, z_1, $z_2, \ldots$* **converges** *to z if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that*

$$|z_n - z| < \epsilon \quad \text{for all } n \ge N.$$

Definition 7.2.2 *A sequence z_0, z_1, $z_2, \ldots$ is a* **Cauchy sequence** *if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that*

$$|z_n - z_m| < \epsilon \quad \text{for all } n \ge N \text{ and } m \ge N.$$

As in the real case we have convergence if and only if a sequence is a Cauchy sequence.

Definition 7.2.3 *Let c_0, c_1, c_2, $\ldots$ denote a sequence. A* **series** *is an expression of the form*

$$c_0 + c_1 + c_2 + \cdots$$

which we write as

$$\sum_{k=0}^{\infty} c_k.$$

The sequence s_0, s_1, s_2, ... with

$$s_n = \sum_{k=0}^{n} c_k$$

is known as **the sequence of partial sums**. *The series* **converges** *if the sequence of partial sums converges and it* **diverges** *if the sequence of partial sums diverges. When we have convergence we say that*

$$s = \sum_{k=0}^{\infty} c_k$$

is **the sum of the series**.

Definition 7.2.4 *If the series $\sum |c_k|$ converges then $\sum c_k$ is said to be* **absolutely convergent**.

For a collection of results about series we have the following.

- If a series $\sum c_k$ converges then $c_n \to 0$ as $n \to \infty$.
- If the series $\sum |c_k|$ converges then $\sum c_k$ converges.

  One way to show this is to use the Cauchy sequence property and the triangle inequality to show that the partial sums of $\sum c_k$ are a Cauchy sequence.
- **Comparison test:** If $|c_k| \le M_k$ for all $k \ge K$ and $\sum M_k$ converges then $\sum c_k$ converges.
- From the identity

  $$(1-c)(1+c+c^2+\ldots+c^n) = 1-c^{n+1}$$

  we have that the **geometric series** diverges if $|c| \ge 1$ and $\sum c^k$ converges if $|c| < 1$ with

  $$\sum_{k=0}^{\infty} c^k = \frac{1}{1-c}.$$
- **Ratio test:** If $|c_{k+1}/c_k| \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.

  This is proved by comparing with a geometric series.
- **Root test:** If $|c_k|^{1/k} \to L$ as $k \to \infty$ then the series converges if $L < 1$ and it diverges if $L > 1$.

  This is proved by comparing with a geometric series.

7.3 Series of functions and uniform convergence

We next discuss definitions and results about series of functions and this will include a new result about the uniform convergence of a sequence of analytic functions.

Let $\sum_0^\infty f_k(z)$ denote a series of functions with each term $f_0(z)$, $f_1(z)$, $\ldots$ in the series being defined on the same domain D. Also let

$$F_n(z) = \sum_{k=0}^{n} f_k(z), \quad n = 0, 1, 2, \ldots$$

be the corresponding sequence of partial sums.

Definition 7.3.1 *The series* **converges pointwise** *if the sequence $(F_n(z))$ converges for each z in the domain D.*

Definition 7.3.2 *The series* **converges uniformly to $F(z)$ on D** *if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n \ge N$*

$$|F_n(z) - F(z)| < \epsilon$$

for all $z \in D$.

In the case of uniform convergence we prove this if we can show that

$$\sup_{z\in D} |F_n(z) - F(z)| \to 0 \quad \text{as } n \to \infty.$$

Here sup is the notation for the least upper bound and in our contexts the value will correspond to the maximum of the expression on the boundary of D. In the case of series a usable test which gives a sufficient condition to prove that a series converges uniformly is the **Weierstrass M-test** which is as follows.

Theorem 7.3.1 *If $|f_k(z)| \le M_k$ for all $z \in D$ and $\sum M_k$ converges then the series converges uniformly in D.*

Thus in practice you rarely need to compute the supremum as a suitable bound is usually sufficient to establish the uniform convergence property.

One important result about uniform convergence that you meet in the case of real valued functions of a real variable which also holds in the complex case with an almost identical proof is the following.

Theorem 7.3.2 *If $F_n(z)$, $n = 0, 1, 2, \ldots$ are continuous in D and $F_n \to F$ uniformly on D then the limit function $F(z)$ is also continuous in D.*

In the case of contour integrals that are considered in this module we have the following which is effectively the same as what is done with real integrals.

Theorem 7.3.3 *If $F_n(z)$, $n = 0, 1, 2, \ldots$ are continuous in D and $F_n \to F$ uniformly in D and Γ is a contour (which has finite length) then*

$$\int_\Gamma F_n(z)\, dz \to \int_\Gamma F(z)\, dz \quad \textit{as } n \to \infty.$$

Proof: Let

$$M_n = \sup_{z\in D} |F_n(z) - F(z)|$$

and note that the uniform convergence means that $M_n \to 0$ as $n \to \infty$. Now by the ML inequality

$$\begin{aligned} \left| \int_\Gamma F_n(z)\,\mathrm{d}z - \int_\Gamma F(z)\,\mathrm{d}z \right| &= \left| \int_\Gamma (F_n(z) - F(z))\,\mathrm{d}z \right| \\ &\le (\text{length of } \Gamma) M_n \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

For a new result relating to analytic functions we have the following.

Theorem 7.3.4 *Let $F_n(z)$ be a sequence of analytic function in a simply connected domain D and converging uniformly to $F(z)$ in D. Then $F(z)$ is analytic in D.*

Proof: Let Γ denote any closed loop in D. As $F_n(z)$ is analytic in D we can use Cauchy's theorem to deduce that

$$\oint_\Gamma F_n(z)\,\mathrm{d}z = 0, \quad n = 0, 1, 2, \ldots.$$

As we are given that $F_n \to F$ uniformly in D the previous theorem gives

$$\oint_\Gamma F(z)\,\mathrm{d}z = \lim_{n\to\infty} \oint_\Gamma F_n(z)\,\mathrm{d}z = 0.$$

As this is true for all loops Morera's theorem (see page 6-7) implies that $F(z)$ is analytic in D. □

We will use the result in theorem 7.3.3 in a few places and in the case of series this means that if

$$F(z) = \sum_{k=0}^{\infty} f_k(z),$$

with the series converging uniformly in D, then

$$\int_\Gamma \sum_{k=0}^{\infty} f_k(z)\,\mathrm{d}z = \sum_{k=0}^{\infty} \int_\Gamma f_k(z)\,\mathrm{d}z.$$

That is, term-by-term integration gives us the integral of a function defined by a series. In this chapter $f_k(z)$ is of the form $(z - z_0)^k$.

7.4 Taylor series for an analytic function

If a function $f(z)$ is analytic in a domain then from the results of chapter 6 a consequence of the Cauchy integral formula is that f', f'', … are also analytic and we can always consider $f(z_0)$, $f'(z_0)$, …, $f^{(n)}(z_0)$, … for all points z_0 in the domain.

Definition 7.4.1 *If $f(z)$ is analytic at z_0 then the series*

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

is called the **Taylor series** *for $f(z)$ around z_0. When $z_0 = 0$ the series is known as the* **Maclaurin series** *for $f(z)$.*

The purpose of this section is to show that this series converges uniformly in a disk centred at z_0 and the sum is $f(z)$. We are also interested in the size of the disk for which we have the representation, i.e. if $\{z : \ |z - z_0| < R\}$ denotes the disk then how large can we take R? As we will see this can be answered quite easily as either $R = \infty$, which is the case with entire functions, or it is such that the circle $|z - z_0| = R$ contains at least one point for which $f(z)$ is not analytic with $f(z)$ being analytic for all z inside the disk. For the uniform convergence we need to restrict to a closed disk of the form $|z - z_0| \le R' < R$. Thus, as we next prove, the series converges in the largest disk for which $f(z)$ is analytic.

Theorem 7.4.1 *If $f(z)$ is analytic in the disk $|z - z_0| < R$ then the Taylor series converges to $f(z)$ for all z in this disk and in any closed disk $|z - z_0| \le R' < R$ the convergence is uniform.*

Proof: The proof involves using the Cauchy integral formula and writing the term $1/(\zeta - z)$ in the integrand in such a way that we get the terms in the series.

First we need to identify where $f(z)$ is analytic, the closed disk where we show that the convergence is uniform and the contour C which we use in the Cauchy integral formula. Let $R' < R$ and let

$$D' = \{z : \ |z - z_0| \le R'\}$$

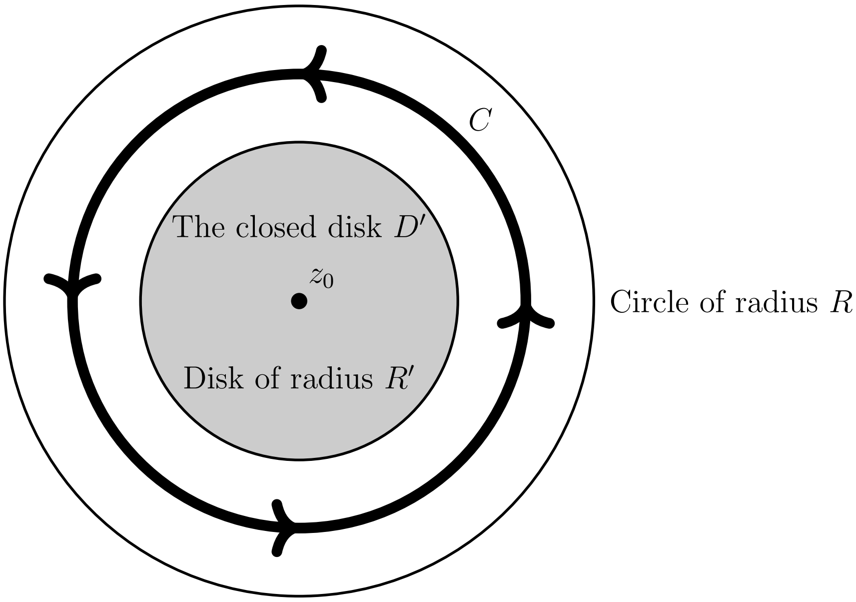
be the closed disk for the uniform convergence part. For the loop we want something which surrounds D' and it needs to be where $f(z)$ is analytic and we choose

$$C = \{\zeta : \ |\zeta - z_0| = R_1\} \quad \text{where } R_1 = \frac{R' + R}{2} < R.$$

With this choice, we have that when $z \in D'$ and $\zeta \in C$ the bound

$$|\zeta - z| \ge R_1 - R' = \frac{R' + R}{2} - R' = \frac{R - R'}{2} > 0.$$

As $R' < R$ can be taken arbitrarily close to R we will establish the convergence for all z with $|z - z_0| < R$ if we can establish the uniform convergence for all $z \in D'$.



The integral formula gives

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

Now we need to introduce z_0 into the expression in the integrand and we can do this by writing

$$\zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0) \left(1 - \left(\frac{z - z_0}{\zeta - z_0} \right) \right).$$

To shorten the writing of the expressions that we need let

$$\alpha = \frac{z - z_0}{\zeta - z_0}, \quad \text{so that } \zeta - z = (\zeta - z_0)(1 - \alpha) \tag{7.4.1}$$

and note that for $z \in D'$ and $\zeta \in C$ we have

$$|\alpha| \le \frac{R'}{R_1} < 1 \quad \text{and } |1 - \alpha| \ge 1 - |\alpha| \ge \frac{R_1 - R'}{R_1} > 0.$$

This is needed as we expand in powers of α the term $(1 - \alpha)^{-1}$ by noting the relation

$$(1 - \alpha)(1 + \alpha + \alpha^2 + \cdots + \alpha^n) = 1 - \alpha^{n+1}$$

giving

$$\frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^n + \left(\frac{\alpha^{n+1}}{1 - \alpha} \right)$$

and the integrand to consider is

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - z_0)(1 - \alpha)} = \frac{f(\zeta)}{\zeta - z_0} \left(1 + \alpha + \alpha^2 + \cdots + \alpha^n + \left(\frac{\alpha^{n+1}}{1 - \alpha} \right) \right).$$

Now

$$\frac{1}{2\pi i}\oint_C \left(\frac{f(\zeta)}{\zeta - z_0}\right)\alpha^k \,\mathrm{d}\zeta = \frac{1}{2\pi i}(z-z_0)^k \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,\mathrm{d}\zeta = \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

by the generalised Cauchy integral formula in the case of the kth derivative. Thus we get

$$f(z) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right) + T_n(z)$$

where the first part is the Taylor polynomial and where $T_n(z)$ denotes the remainder term and to complete the proof we need to show that the remainder term tends to 0 uniformly in D'.

Part of the expression for $T_n(z)$ involves

$$\left(\frac{f(\zeta)}{\zeta - z_0}\right)\left(\frac{\alpha^{n+1}}{1-\alpha}\right)$$

and from (7.4.1) $\zeta - z = (\zeta - z_0)(1-\alpha)$ and thus this is

$$(z-z_0)^{n+1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta - z)}$$

and we get a representation for $T_n(z)$ as

$$T_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta-z)}\,\mathrm{d}\zeta.$$

The uniform convergence of $T_n(z)$ to 0 on D' is because $f(\zeta)$ is bounded on the circle C, $|\zeta - z| > (R-R')/2$ for all $\zeta \in C$ and $z \in D'$ and we have

$$\left|\left(\frac{z-z_0}{\zeta-z_0}\right)^{n+1}\right| = |\alpha^{n+1}| \le \left(\frac{R'}{R_1}\right)^{n+1} \to 0 \quad \text{as } n \to \infty.$$

□

Remark about $T_n(z)$ and R

In the proof we get an integral representation for $T_n(z)$ which is the difference between $f(z)$ and the Taylor polynomial involving terms up to $(z-z_0)^n$ which is valid for all z in the disk. If we restrict z to be sufficiently close to z_0 then

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta-z)}\,\mathrm{d}\zeta \approx \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+2}}\,\mathrm{d}\zeta = \frac{f^{(n+1)}(z_0)}{(n+1)!},$$

by using the formula for $f^{(n+1)}(z_0)$, which gives us the usual estimate

$$T_n(z) \approx \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0)^{n+1}.$$

As stated in the theorem, the Taylor series result holds for the largest possible disk centred at z_0 such that $f(z)$ is analytic for $|z-z_0| < R$. When $f(z)$ is not an entire function, so that $R < \infty$, this implies that if we take R as large as possible then $f(z)$ is not analytic in $|z-z_0| < R + \epsilon$ for any $\epsilon > 0$. Another way of stating this is that there is at least one point on the circle $|z-z_0| = R$ where $f(z)$ is not analytic and we will see this when examples are considered.

The Taylor series for $f'(z)$

When $f(z)$ is analytic the derivatives are also analytic and we can consider the Taylor series for these as well. The Taylor series for $f'(z)$ about z_0 is obtained by just replacing f by f' giving

$$f'(z) = f'(z_0) + f''(z_0)(z - z_0) + \frac{f'''(z_0)}{2!}(z - z_0)^2 + \cdots$$

and as $f'(z)$ is analytic in the same domain as $f(z)$ it follows that the series also converges uniformly in $\{z : |z - z_0| \le R'\}$ for all $R' < R$. Note that this series is exactly the same as what is obtained by differentiating the series for $f(z)$ term-by-term. That is we can get the series for f', f'', … by term-by-term differentiation and we can also get the series for the anti-derivative by integrating term-by-term.

The Maclaurin series for even and odd functions

When $z_0 = 0$ the Taylor series is known as the Maclaurin series and we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad \text{with} \quad \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} f(r\mathrm{e}^{it})\,\mathrm{e}^{-int}\,\mathrm{d}t,$$

where we can take any value of $r > 0$ such that $f(z)$ is analytic in the domain containing the disk $\{z : |z| \le r\}$. We give some examples shortly and in a few of these we have functions which are odd functions or even functions and thus we first prove why we only have odd or even powers in these cases. To do this we consider the part $(-\pi, 0)$ of $(-\pi, \pi)$ and let $s = t + \pi$ so that $t = s - \pi$. With this substitution

$$\mathrm{e}^{it} = \mathrm{e}^{i(s-\pi)} = -\mathrm{e}^{is}, \quad \mathrm{e}^{-int} = (-1)^n \mathrm{e}^{-ins},$$

so that

$$\int_{-\pi}^{0} f(r\mathrm{e}^{it})\,\mathrm{e}^{-int}\,\mathrm{d}t = (-1)^n \int_{0}^{\pi} f(-r\mathrm{e}^{is})\,\mathrm{e}^{-ins}\,\mathrm{d}s$$

giving

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi r^n} \int_{0}^{\pi} \left(f(r\mathrm{e}^{it}) + (-1)^n f(-r\mathrm{e}^{it})\right) \mathrm{e}^{-int}\,\mathrm{d}t.$$

If $f(z)$ is even, i.e. $f(-z) = f(z)$, then the integrand is 0 when n is odd. Thus there cannot be any odd powers in the series of an even function, the series for an even function can only involves even powers. Similarly, if $f(z)$ is odd, i.e. $f(-z) = -f(z)$, then the integrand is 0 when n is even and and thus the series only involves odd powers.

The Taylor series for real valued functions

In practice we are often interested in analytic functions $f(z)$ which are such that when $z = x \in \mathbb{R}$ we have $f(x) \in \mathbb{R}$, i.e. the value is real when the argument is real. In these cases all the coefficients in the Taylor series about a point $z_0 = x_0 \in \mathbb{R}$ are also real. This follows almost immediately by letting $f = u + iv$, with as usual u and v being real valued, and noting that $f(z)$ being real on the real axis implies that for all x

$$v(x, 0) = 0 \quad \text{implying that} \quad \frac{\partial^n}{\partial x^n} v(x, 0) = 0, \quad n = 0, 1, 2, \ldots$$

The analyticity of $f(z)$ means that we get the value of the derivative by differentiating in any direction and thus if we differentiate in the x-direction then we get

$$f^{(n)}(x_0) = \frac{\partial^n}{\partial x^n} u(x, 0) + i \frac{\partial^n}{\partial x^n} v(x, 0) = \frac{\partial^n}{\partial x^n} u(x, 0).$$

Related to this remark that real valued functions have real coefficients when we expand about a point on the real axis we consider again something that was done in chapter 3 when analytic functions were first defined. It was stated that if $f(z)$ is analytic and not a constant then $g(z) = f(\overline{z})$ is not analytic. However it can be quite quickly shown from the definition that if we define

$$h(z) = \overline{g(z)} = \overline{f(\overline{z})} \tag{7.4.2}$$

then $h(z)$ is analytic. For a general function $f(z)$ analytic in a disk $|z| < R$ with the Maclaurin series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a + n z^n + \cdots$$

it follows that

$$f(\overline{z}) = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^n + \cdots$$

and

$$h(z) = \overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_n} z^n + \cdots .$$

The coefficients in the Maclaurin expansion of $h(z)$ are the complex conjugate of the coefficients of the Maclaurin series for $f(z)$. When all the coefficients are real we have

$$f(z) = \overline{f(\overline{z})}.$$

This was mentioned on a exercise sheet when $f(z)$ is a polynomial with real coefficients and this now shows that it is true more generally.

Combining functions

We consider now some elementary operations that can be done with Taylor series. Firstly, if we have the Taylor series for $f(z)$ about z_0 and we consider the function

$$h(z) = c\, f(z), \quad \text{where } c \text{ is a constant,}$$

then

$$h^{(n)}(z_0) = c\, f^{(n)}(z_0)$$

which tells us that we get the Taylor series for $h(z)$ by just multiplying each term by the constant c. Similarly, if $g(z)$ is another function analytic at z_0 and we now define

$$h(z) = f(z) + g(z)$$

then we can add the series for $f(z)$ and $g(z)$ term-by-term as

$$h^{(n)}(z_0) = f^{(n)}(z_0) + g^{(n)}(z_0).$$

Examples of entire functions

1. Let $z = x + iy$ and let

$$f(z) = \mathrm{e}^z = \mathrm{e}^x(\cos\, y + i \sin\, y).$$

With this as the definition of e^z and with $f = u + iv$ and that $f(z)$ is analytic everywhere we can get the derivative by just differentiating in the x direction so that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \mathrm{e}^x\,(\cos\, y + i \sin\, y) = \mathrm{e}^z.$$

All the derivatives are e^z and they all have the value 1 when $z = 0$ and we have

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad (7.4.3)$$

It is in this sense the exponential function which was defined using x and y can be written as a function of z alone. As $f(z)$ is an entire function it follows that the series converges uniformly in every closed disk of the form $\{z :\ |z| \le R\}$.

2.

$$\mathrm{e}^{-z} = 1 - z + \frac{z^2}{2!} + \cdots + (-1)^n\frac{z^n}{n!} + \cdots \qquad (7.4.4)$$

which has been obtained by just replacing z by $-z$ in (7.4.3).

3. Let $f(z) = \cos\, z$. This is an even function and hence only even powers are involved. If we differentiate 2 times then we get $-\cos\, z$ and this is sufficient to deduce that

$$\begin{aligned} f(0) &= f^{(4)}(0) = f^{(8)}(0) = \cdots = f^{(4k)}(0) = 1, \\ f^{(2)}(0) &= f^{(6)}(0) = f^{(10)}(0) = \cdots = f^{(2+4k)}(0) = -1 \end{aligned}$$

and

$$\cos\, z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad (7.4.5)$$

4. Let $f(z) = \sin\, z$. As $f(0) = 0$ and $f'(z) = \cos\, z$ we can get the Taylor series for this by just integrating (7.4.5) term-by-term to give

$$\sin\, z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

5. Let

$$f(z) = \cosh\, z = \frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right).$$

This is an even function and we can get the series by adding (7.4.3) and (7.4.4) term-by-term and then dividing by 2 to give

$$\cosh\, z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \qquad (7.4.6)$$

The terms in the series are the even power terms in the e^z series.

6. Let
$$f(z) = \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right).$$
This is an odd function and it is the derivative of cosh z and hence we can get the terms by differentiating (7.4.6) term-by-term to give
$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \qquad (7.4.7)$$
The terms in the series are the odd power terms in the e^z series.

In all of the above examples there are several techniques to get the Maclaurin series for the function being considered.

Examples of functions which are not entire

1. Let
$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n, \quad \text{which is valid for } |z| < 1. \qquad (7.4.8)$$
This is the geometric series and which of course is also the Maclaurin series of $f(z)$. This function has a simple pole at $z = 1$ and is analytic for $|z| < 1$ which is evident by both considering the left hand side of (7.4.8) and by considering the convergence of the series.

We can also expand about a point other than $z_0 = 0$ and we consider next taking $z_0 = -1$.

Without using Taylor series we can get the series by first writing
$$1 - z = (1 - z_0) - (z - z_0) = 2 - (z + 1) = 2\left(1 - \left(\frac{z+1}{2}\right)\right).$$
By using the geometric series approach we get
$$(1-z)^{-1} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n, \quad \text{which is valid for } \left|\frac{z+1}{2}\right| < 1.$$
The radius of this disk is 2 which is because this is the distance of the position of the simple pole from the point $z_0 = -1$.

If we instead derive this by successively differentiating $f(z)$ then we have
$$\begin{aligned} f(z) &= (1-z)^{-1}, \\ f'(z) &= (1-z)^{-2}, \\ f''(z) &= 2(1-z)^{-3}, \\ \cdots & \quad \cdots \\ f^{(n)}(z) &= n!(1-z)^{-(n+1)}. \end{aligned}$$
Hence
$$f^{(n)}(-1) = n! 2^{-(n+1)}$$

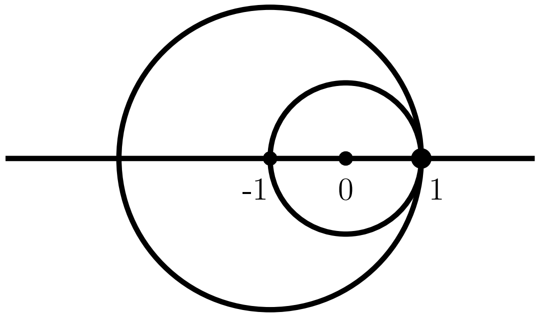


Figure 7.1: The circles of convergence when we expand about -1 and when we expand about 0. The simple pole at $z = 1$ is on both circles.

in agreement with the result obtained by just using geometric series. We illustrate the circles of convergence when we take $z_0 = -1$ and when we take $z_0 = 0$ in figure 7.1.

2. As we have already stated, term-by-term differentiation is valid and hence differentiating the series for $(1-z)^{-1}$ term-by-term gives

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots, \quad |z| < 1.$$

With an absolutely convergent series it is valid to multiply by z term-by-term to deduce that

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots, \quad |z| < 1.$$

This function is known as the Koebe function and it appears when functions which are analytic in the unit disk and normalised by $f(0) = 0$ and $f'(0) = 1$ are considered. It can be shown that this function is one-to-one in the unit disk and in some sense it is maximal in that if a function is of the form

$$z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

is analytic in the unit disk and it is one-to-one in the disk then it can be shown that

$$|a_n| \le n.$$

This was proved by de Branges in about 1984/5 and which had previously been a conjecture by Bierberbach in 1916. It is reported that de Branges' first version of the proof was 350 pages long. The difficult property to deal with when considering functions of this type is that the function must be one-to-one in the unit disk.

3. Let

$$f(z) = \mathrm{Log}\,(1-z)$$

where, as usual, Log denotes the principal valued logarithm. $f(z)$ has a branch point at $z = 1$ and as $1 - z$ is negative when $z = x > 1$ it follows that the branch cut does not cross the unit disk $|z| < 1$. In the unit disk the derivative is

$$f'(z) = -\frac{1}{1-z}$$

and as we have the series for this and as $f(0) = \mathrm{Log}\,1 = 0$ we can get the series for $f(z)$ by term-by-term integration to give

$$-\mathrm{Log}\,(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} + \cdots, \quad \text{which is valid for } |z| < 1.$$

4. Let

$$f(z) = (1+z)^\alpha, \quad \alpha \notin \mathbb{Z},$$

where we mean the principal valued version of this function, i.e.

$$f(z) = \exp(\alpha \mathrm{Log}\,(1+z)).$$

As in the previous example this is a function with a branch point which in this case is at $z = -1$ and the branch cut is along $z = x < -1$. In the unit disk the function is analytic and for the derivatives

$$\begin{aligned} f'(z) &= \alpha(1+z)^{\alpha-1}, \\ f''(z) &= \alpha(\alpha-1)(1+z)^{\alpha-2}, \\ \cdots & \quad \cdots \\ f^{(n)}(z) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n}. \end{aligned}$$

The Maclaurin series for this function is known as the generalised binomial series and is

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n + \cdots$$

and is valid for $|z| < 1$.

Taylor series for a product of functions

Previously we noted that if we have the Taylor series for $f(z)$ then we get the Taylor series for $cf(z)$, where c is a constant, by just multiplying each term by c. We also had an example where we obtained the series for the Koebe function $z/((1-z)^2)$ from $1/((1-z)^2)$ by just multiplying each term by z. We can also deal with the more general case of multiplying the series for $f(z)$ and $g(z)$ to get the Taylor series for $h(z) = f(z)g(z)$ as follows.

Suppose

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \\ g(z) &= b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots \end{aligned}$$

then

$$h(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

where

$$\begin{aligned} c_0 &= a_0 b_0, \\ c_1 &= a_1 b_0 + a_0 b_1, \\ c_2 &= a_2 b_0 + a_1 b_1 + a_0 b_2, \\ \cdots &= \cdots \\ c_n &= a_n b_0 + a_{n-1} b_1 + \cdots + a_1 b_{n-1} + a_0 b_n. \end{aligned}$$

The series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad c_n = a_n b_0 + a_{n-1} b_1 + \cdots + a_1 b_{n-1} + a_0 b_n \tag{7.4.9}$$

is known as the Cauchy product of the two series. This can be shown to be valid by considering the Taylor series for $h(z) = f(z)g(z)$ and using the Leibnitz's formula for the nth derivative of a product.

$$\begin{aligned} h &= fg, \\ h' &= f'g + fg', \\ h'' &= f''g + 2f'g' + fg'', \\ \cdots & \quad \cdots \\ h^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}. \end{aligned} \tag{7.4.10}$$

Now

$$h^{(n)}(z_0) = n! c_n, \quad f^{(k)}(z_0) = k! a_k, \quad g^{(n-k)}(z_0) = (n-k)! b_{n-k}, \tag{7.4.11}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{7.4.12}$$

By using (7.4.10), (7.4.11) and (7.4.12) we verify (7.4.9).

The first few terms for $\tan z$

We typically use the technique of multiplying two series together described above just to get the first few terms as more and more work is needed for each further term in the series. We illustrate this here to get the first few terms of the Maclaurin series of $\tan z$. Now let

$$f(z) = \tan z = \frac{\sin z}{\cos z}. \tag{7.4.13}$$

This function has simple poles at points where $\cos z = 0$ and from this we deduce that the Maclaurin series converges for $|z| < \pi/2$. As $\tan z$ is an odd function we know that the Maclaurin series only involves odd powers and hence we have

$$\tan z = b_1 z + b_3 z^3 + b_5 z^5 + \cdots .$$

The aim here is to get b_1, b_3 and b_5 without differentiating $\tan z$ but instead using the known series for $\sin z$ and $\cos z$ and we use the result about the product of series by re-writing (7.4.13) as

$$\sin z = (\cos z)(\tan z).$$

We substitute in the series to give

$$\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots \right) = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots \right) (b_1 z + b_3 z^3 + b_5 z^5 + \cdots).$$

By equating the coefficients of the powers (which is basically the Cauchy product result) we get the following.

Equating coefficient of z:

$$1 = b_1.$$

Equating coefficient of z^3:

$$-\frac{1}{6} = b_3 - \frac{b_1}{2} = b_3 - \frac{1}{2}, \quad \text{hence } b_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Equating coefficient of z^5:

$$\frac{1}{120} = b_5 - \frac{b_3}{2} + \frac{b_1}{24}, \quad \text{hence } b_5 = \frac{1}{6} - \frac{1}{24} + \frac{1}{120} = \frac{16}{120} = \frac{2}{15}.$$

Thus we have

$$\tan z = z + \frac{z}{3} + \frac{2z^5}{15} + \cdots$$

If you continue this process to get the coefficients of z^7 and z^9 etc. then this gets messy by hand calculation. For information, it can be shown that the coefficient of z^7 is 17/315 and the coefficient of z^9 is 62/2835.

The generalised L'Hopital's rule

Earlier in the module L'Hopital's rule was given to determine a limit of $f(z)/g(z)$ as $z \to z_0$ when $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$. We now use Taylor series to extend the result to cases where $f(z)$ and $g(z)$ have a multiple zero at z_0.

If

$$g(z_0) = g'(z_0) = \cdots = g^{(m-1)}(z_0) = 0 \quad \text{and } g^{(m)}(z_0) \neq 0$$

and if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

then the first few terms of the Taylor series for $f(z)$ and $g(z)$ are 0 and near z_0 we have

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots \\ g(z) &= \frac{g^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{g^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots . \end{aligned}$$

If $z \neq z_0$ and we take out a factor of $(z-z_0)^m/m!$ in both then for the ratio we have

$$\frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0) + (f^{(m+1)}(z_0)/(m+1))(z-z_0) + \cdots}{g^{(m)}(z_0) + (g^{(m+1)}(z_0)/(m+1))(z-z_0) + \cdots} \to \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \quad \text{as } z \to z_0.$$

7.5 Power series

In the previous section we started with an analytic function $f(z)$ and showed that it could be represented by a Taylor series in a disk $\{z : \ |z - z_0| < R\}$ with R being as large as possible such that $f(z)$ is analytic in the disk. Thus in that section the sum of the series was known, i.e. $f(z)$, and we considered the validity and properties of the representation. In this section we start with the reverse of this in that we are given a power series and we want to know if a sum exists (i.e. if it converges), and if the sum exists does it define an analytic function in some region.

Definition 7.5.1 *A series of the form*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is called a **power series** *and the terms* a_0, a_1, $\ldots$ *are the coefficients of the power series.*

The definition gives no indication as to whether or not the series converges but note that it trivially converges when $z = z_0$ with a sum of a_0. There are series which only converge at the point z_0 and you may have met the series

$$\sum_{n=0}^{\infty} n! z^n, \quad \text{i.e. } a_n = n! \text{ and } z_0 = 0,$$

and applied the ratio test using the terms involving z^n and z^{n+1} to give

$$\frac{(n+1)! z^{n+1}}{n! z^n} = (n+1)z$$

which remains bounded as $n \to \infty$ only when $z = 0$. From this we deduce that the series only converges when $z = 0$.

The domain in which a power series converges is a disk and we justify this next although rigorous proofs will not be given in all cases. First suppose that there exists a point $z_1 \neq z_0$ such that the series converges and let

$$r = |z_1 - z_0| > 0.$$

The convergence of the series implies that the terms tend to 0, i.e.

$$a_n (z_1 - z_0)^n \to 0 \quad \text{as } n \to \infty$$

and in particular this implies that the sequence $(a_n(z_1 - z_0)^n)$ is bounded and we let

$$M = \sup \{ |a_n| r^n : \; n = 0, 1, 2, \ldots \}.$$

If we take any z in any disk of the form $\{z : \; |z - z_0| \le r' < r\}$ then the terms in the power series satisfy

$$|a_n (z-z_0)^n| = \left| a_n r^n \left(\frac{z-z_0}{r} \right)^n \right| \le M \left(\frac{r'}{r} \right)^n.$$

Now as $r'/r < 1$ the right hand side is a convergent geometric series and by the Weierstrass M-test it follows that the power series converges uniformly for all z satisfying $|z - z_0| \le r' < r$. Thus to summarise, this reasoning has shown that convergence at z_1 means that the series must converge at all points closer to z_0.

The previous argument started with the statement "suppose that there exists a point $z_1 \neq z_0$ such that ...". The existence of such a point depends on the coefficients a_0, a_1, ... When the root test and/or the ratio test can be used and are such that

$$\sqrt[n]{|a_n|} \to \alpha \in \mathbb{R} \quad \text{as } n \to \infty$$

then the root test applied to the power series gives

$$|a_n(z-z_0)^n|^{1/n} \to |\alpha|\,|z-z_0|$$

and we have convergence if

$$|\alpha|\,|z-z_0| < 1$$

and we have divergence if

$$|\alpha|\,|z-z_0| > 1.$$

When $\alpha = 0$ we have convergence for all z but otherwise if we define

$$R = \frac{1}{\alpha} = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

then we know that the series converges uniformly when $|z-z_0| \le r < R$ and it diverges for $|z-z_0| > R$. R is known as the **radius of convergence** and $\{z \in \mathbb{C} :\ |z-z_0| = R\}$ is known as the **circle of convergence**. In many situations the ratio test is easier to use than the root test. If the sequence (a_n) is such that the ratio test can be used, i.e. $|a_{n+1}/a_n| \to \alpha$ as $n \to \infty$ then it can be shown that $|a_n|^{1/n} \to \alpha$ and we get the same outcome that the series converges when $|z-z_0| \le r < R$ and it diverges when $|z-z_0| > R$.

In all the examples given in these notes the sequence a_0, a_1, ... generating the series are such that the ratio test and/or the root test can be applied. For a rigorous treatment you also need to cope with the situation that the sequence $(|a_n|^{1/n})$ is not a convergent sequence. If this sequence is not bounded then we know that the series only converges at $z = z_0$ which was the case when $a_n = n!$. To cope with the remaining case when $(|a_n|^{1/n})$ is bounded but does not converge we can define

$$b_n = \sup\left\{|a_m|^{1/m} :\ m \ge n\right\}. \tag{7.5.1}$$

As n increases the set being considered in (7.5.1) reduces and hence we conclude that the sequences b_0, b_1, ... is a decreasing sequence of non-negative numbers and by the monotone convergence theorem it converges. There is a name and notation for this limit which is

$$\alpha = \limsup |a_n|^{1/n} = \lim_{n\to\infty} \sup\left\{|a_m|^{1/m} :\ m \ge n\right\} = \lim_{n\to\infty} b_n.$$

This value α is the same as that described above when the sequence $(|a_n|^{1/n})$ converges and all that has been done here is to indicate that a decision about convergence can, in theory, be made for any series. The outcome is that if we define

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

then the power series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$. The understanding here is that if the sequence $(|a_n|^{1/n})$ is unbounded then $R = 0$ and if the denominator is 0 then $R = \infty$. We summarise this in the following theorem.

Theorem 7.5.1 *For any power series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

there exists a real number R between 0 and ∞ inclusive which depends on the coefficients a_0, a_1, ... such that we have the following.

(i) The series converges for $|z - z_0| < R$.

(ii) The series converges uniformly in every closed subdisks $|z - z_0| \le R' < R$.

(iii) The series diverges for $|z - z_0| > R$.

The radius R is called the **radius of convergence** *of the power series.*

If we truncate a power series then we have a polynomial and the result tells us that a sequence of polynomials, which are analytic, converges uniformly in a disk of the form

$$\{z : \ |z - z_0| \le R' < R\}.$$

A sequence of analytic functions which converges uniformly gives us a limit function which is analytic. Let $f(z)$ denote the limit function, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

By using the generalised Cauchy integral formula applied to f with any loop Γ in the disk and with z_0 being inside the loop we have

$$\begin{aligned} \frac{f^{(m)}(z_0)}{m!} &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{m+1}} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \oint_\Gamma \frac{\sum_{n=0}^{\infty} a_n (z - z_0)^n}{(z - z_0)^{m+1}} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n \oint_\Gamma \frac{(z - z_0)^n}{(z - z_0)^{m+1}} \, \mathrm{d}z \\ &= a_m, \end{aligned}$$

where the interchange of the sum with the integral is valid because the sequence converges uniformly and we get a non-zero value only when $n = m$ in the last part. Hence the power series that we started with is the Taylor series of the analytic function $f(z)$ which is the sum of the series.

Concluding remarks about the size of R

In section 7.4 we used the Cauchy integral formula for a given function $f(z)$ to derive the Taylor series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

which, by construction, converges inside a disk which we write here as $\{z : \ |z - z_0| < R_T\}$ whenever $f(z)$ is analytic inside this disk. The largest value for R_T corresponds to a circle which contains at least one point for which $f(z)$ is not analytic. The result of this current subsection about power series tell us us that the series cannot converge for $|z - z_0| > R$ with

$$R = \frac{1}{\limsup |a_n|^{1/n}}, \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}.$$

We must have $R_T = R$. This follows by showing that if $R_T < R$ or if $R_T > R$ then we get a contradiction. If $R_T < R$ then we know that the series defines an analytic function inside the circle of convergence and in particular this would imply that the function is analytic at all points on $|z - z_0| = R_T < R$ which is in contradiction with $f(z)$ being not analytic somewhere on this circle. If $R_T > R$ then by the construction in section 7.4 we have that the series converges at points satisfying $R_T > |z - z_0| > R$. But the theory of this subsection tells us the terms in the series are unbounded when $|z - z_0| > R$ and the series diverges which is our contradiction.

Examples

1. Consider
$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$
Here $a_n = 1/n$ and the ratio test and the root test can both be applied. By using the ratio test
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} = \frac{1}{1 + 1/n} \to 1 \quad \text{as } n \to \infty.$$
Hence the radius of convergence is $R = 1$. This is consistent with what was done earlier as we showed that
$$-\mathrm{Log}\,(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$
and we know that $-\mathrm{Log}\,(1 - z)$ is not analytic at $z = 1$.

   We do not consider what happens on the circle of convergence but it is interesting to note that in this particular case it can be shown that
$$-\mathrm{Log}\,(1 - \mathrm{e}^{i\theta}) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{in\theta}}{n}, \quad 0 < \theta < 2\pi,$$
i.e. the series converges at all points on the unit circle except $z = 1$. You may have considered this in a previous analysis module when $\theta = \pi$ and the point $z = -1$ and used the alternating series test.

2. Consider
$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$
As before the ratio test can be used and we get
$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^2 \to 1 \quad \text{as } n \to \infty$$
which tells us that the radius of convergence is again $R = 1$ and there must be at least one point on $|z| = 1$ where the function is not analytic. (As before, not being analytic is in the sense of the function being analytically continued to a larger region by crossing part of the unit circle.) In this case the series does actually converge

when $|z| = 1$. It is possible here to identify a point on $|z| = 1$ where $f(z)$ is not analytic as follows. Let g be a function such that $g(0) = 1$ and for $z \neq 0$

$$g(z) = -\frac{\operatorname{Log}(1-z)}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots .$$

This function is analytic in the unit disk and the anti-derivative $G(z)$ with $G(0) = 0$ has the series

$$G(z) = z + \frac{z^2}{2^2} + \cdots + \frac{z^n}{n^2} + \cdots$$

which is our function $f(z)$. If $G(z)$ was analytic at $z = 1$ then this would imply that $g(z) = G'(z)$ is also analytic at $z = 1$ but $g(z)$ has a branch point at $z = 1$. Thus although $f(1)$ is bounded the function is not analytic at $z = 1$ or to be more precise we cannot analytically continue f to a function defined in a larger region which contains $z = 1$. (Analytic continuation is not a topic to be covered as part of this module and please note that if a function represented by a power series can be analytically continued outside of the circle of convergence then the continuation is not represented by the power series.)

7.6 Laurent series

Taylor series are concerned with series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which represent analytic functions in a disk centred at z_0, i.e. in a domain of the form $\{z : |z - z_0| < R\}$. In this section we consider representing functions which may have isolated singularities by series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{7.6.1}$$

in an annulus of the form $\{z : r < |z - z_0| < R\}$. As the series can have positive and negative powers this allows for singularities on and inside the inner circle $|z - z_0| = r$. A series of the type (7.6.1) is known as a **Laurent series** and it reduces to a power series when $a_n = 0$ for $n < 0$. When we have such a representation the part with non-negative powers, i.e.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

gives a function analytic inside the outer circle and the part with negative powers, i.e.

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$$

gives a function which is analytic outside of the inner circle.

Before we prove that a function which is analytic in an annulus must have a Laurent series representation we consider some examples of how we can construct the representation in some specific cases.

Examples

1. Let
$$f(z) = \frac{1}{1-z}.$$
This is a one-term Laurent series if we take $z_0 = 1$.

If we take $z_0 = 0$ then the geometric series is the Taylor series for $|z| < 1$ and we have
$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_0^\infty z^n.$$
In the Laurent series terminology this is the Laurent series with $r = 0$ and $R = 1$ and as it is the Taylor series there are no negative powers.

We can use the above to get a series valid for $|z| > 1$ by letting $w = 1/z$, so that $z = 1/w$, and letting
$$\begin{aligned} f(z) &= \frac{1}{1-1/w} = \frac{w}{w-1} = -\left(\frac{w}{1-w}\right) = -w\left(1 + w + w^2 + \cdots\right) \\ &= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) \end{aligned} \tag{7.6.2}$$
which is a Laurent series involving only negative powers. In the notation of Laurent series $r = 1$ and $R = \infty$. In this case we obtained the series for $|z| > 1$ by considering the Maclaurin series for
$$g(w) = \frac{w}{w-1} \quad \text{with } w = \frac{1}{z}.$$

2. Let now
$$f(z) = \frac{1}{(1-z)(2-z)}.$$
We can use partial fractions to write this in the form
$$f(z) = \frac{A}{1-z} + \frac{B}{2-z} \quad \text{with } 1 = A(2-z) + B(1-z) \quad \text{giving } A = 1, \quad B = -1,$$
i.e.
$$f(z) = \frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z}.$$
$f(z)$ has 2 simple poles and is analytic in the annulus $\{z :\ 1 < |z| < 2\}$ and we can represent the function in this domain by a Laurent series by combining the Taylor series for $-1/(2-z)$ valid for $|z| < 2$ with the Laurent series for $1/(1-z)$ valid for $|z| > 1$ which was obtained in the previous example. We can get the Taylor series for $-1/(2-z)$ by using a geometric series by writing
$$2 - z = 2\left(1 - \frac{z}{2}\right)$$
so that
$$(2-z)^{-1} = \frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right). \tag{7.6.3}$$

By combining (7.6.2) and (7.6.3) we get a Laurent series which is valid in $1 < |z| < 2$ given by

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right) - \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right).$$

In this case $r = 1$ and $R = 2$ and we cannot take a smaller value for r due to the simple pole at $z = 1$ and we cannot take a larger value for R due to the simple pole at $z = 2$.

3. Let now

$$f(z) = \frac{\mathrm{e}^z}{z^2},$$

which has a double pole at $z = 0$. If we consider an annulus centred at $z_0 = 0$ then by using the Maclaurin series of e^z we get

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots . \end{aligned}$$

This is a Laurent series with $r = 0$ and $R = \infty$.

In all the examples presented the function being considered is analytic in an annulus and the Laurent series was obtained using Taylor series of part of the term defining the function. As we next show when we have a function which is analytic in an annulus a Laurent series representation is always possible and it is unique and thus if any method is used to generate such a series then it has found the series. Later we will be concerned with the case when $r = 0$ with the function having an isolated singularity at z_0 but note that when $r > 0$, and no smaller value can be used, all we know is that if it is possible to consider $f(z)$ in $|z - z_0| \le r$ then it is not analytic somewhere in this region.

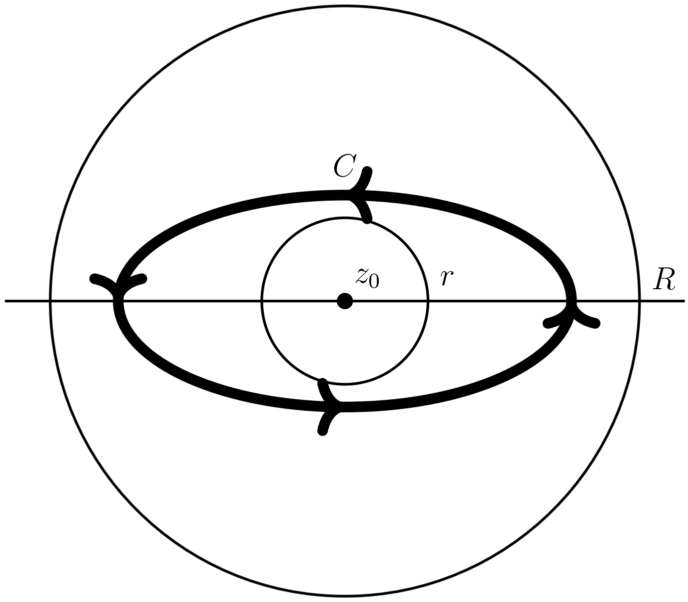
Theorem 7.6.1 *Let $f(z)$ be analytic in an annulus $r < |z - z_0| < R$. Then $f(z)$ can be expressed in this annulus as the sum of two series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

with both series converging in the annulus and converging uniformly in any closed sub-annulus $r < \rho_1 \le |z - z_0| \le \rho_2 < R$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \qquad (7.6.4)$$

where C is any positively orientated simple closed curve lying in the annulus which has z_0 as an interior point as illustrated below.



Proof: The proof is similar to the proof of the Taylor series representation and starts by representing $f(z)$ using the Cauchy integral formula with the extra complexity being that our domain is not simply connected and thus some care is needed in using this formula with the choice of the loop Γ which surrounds the point z.

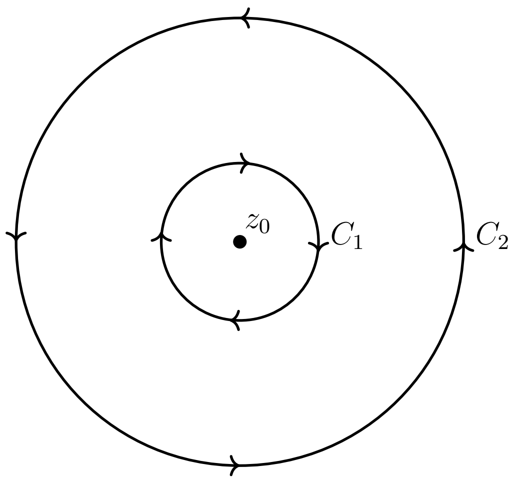
Take a point z satisfying $r < \rho_1 \le |z - z_0| \le \rho_2 < R$ and define

$$R_1 = \frac{r + \rho_1}{2} > r \quad \text{and} \quad R_2 = \frac{R + \rho_2}{2} < R.$$

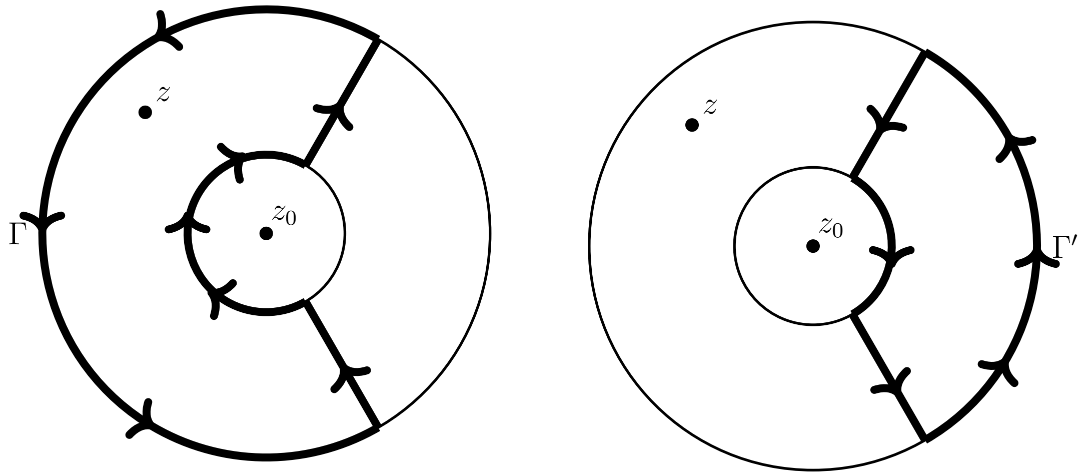
In this way $f(\zeta)$ is analytic on both circles

$$C_1 = \{\zeta : \ |\zeta - z_0| = R_1\} \quad \text{and} \quad C_2 = \{\zeta : \ |\zeta - z_0| = R_2\}$$

and the point z is bounded away from any point on these circles. We will consider contour integrals involving these circles and in these the direction of integration will be anti-clockwise on the outer circle C_2 and clockwise on the inner circle C_1 as illustrated below.



For our given point z we consider loops Γ and Γ' as shown below which are each composed of parts of C_1 and C_2 joined by radial lines.



With this set-up $f(z)$ is analytic inside both Γ and Γ' and we can use the Cauchy integral formula and Cauchy's theorem to deduce that

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \qquad (7.6.5)$$

and

$$\frac{1}{2\pi i}\oint_{\Gamma'} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0 \qquad (7.6.6)$$

as the integrand is analytic inside Γ'. Now the union of Γ and Γ' includes all the points on C_1 and C_2 (considered just once) and also all the points on the radial line segments. When we consider contour integrals the direction of integration on the radial line segments in the case of Γ is opposite to that of Γ' and hence it follows that for any suitable integrand

$$\oint_{\Gamma} \cdot\,\mathrm{d}\zeta + \oint_{\Gamma'} \cdot\,\mathrm{d}\zeta = \oint_{C_1} \cdot\,\mathrm{d}\zeta + \oint_{C_2} \cdot\,\mathrm{d}\zeta$$

as the contributions from the radial line segments cancel. By using this result and (7.6.5) and (7.6.6) we get the representation that we need in this proof which is

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

The term involving C_2 defines a function which is analytic inside C_2 and the term involving C_1 defines a function which is analytic outside C_1 and we show next that both can be represented by series.

The term involving C_2 corresponds to what appeared in the Taylor series proof and thus we have

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,\mathrm{d}\zeta$$

with the series converging uniformly in $|z - z_0| \le \rho_2$.

For the integral involving C_1 we handle this by doing similar manipulations to that involving C_2 except that now we need to note that z is further from z_0 than are points $\zeta \in C_1$ and thus we now write

$$\zeta - z = (\zeta - z_0) - (z - z_0) = (z - z_0)(\beta - 1) = -(z - z_0)(1 - \beta), \quad \text{where } \beta = \frac{\zeta - z_0}{z - z_0}.$$

For all $\zeta \in C_1$ and with $|z - z_0| \ge \rho_1$ we have

$$|\beta| \le \frac{R_1}{\rho_1} < 1.$$

As $|\beta| < 1$ we have, as in the Taylor series proof, that

$$\frac{1}{\zeta - z} = \frac{-1}{z - z_0}\left(1 + \beta + \beta^2 + \cdots + \beta^n + \frac{\beta^{n+1}}{1-\beta}\right).$$

As in the Taylor series proof we consider things term-by-term and consider

$$-\frac{1}{2\pi i}\oint_{C_1} \frac{\beta^{k-1} f(\zeta)}{z_0 - z}\,\mathrm{d}\zeta = \frac{a_{-k}}{(z - z_0)^k} \quad \text{with } a_{-k} = -\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^{k-1}\,\mathrm{d}\zeta.$$

With a similar argument to what was used in the Taylor series proof it can be shown that

$$\max_{\rho_1 \le |z - z_0| \le \rho_2} \left| \oint_{C_1} \frac{\beta^{n+1}}{(z - z_0)(1 - \beta)}\,\mathrm{d}\zeta \right| \to 0 \quad \text{as } n \to \infty.$$

Putting everything together we have in the case of the circle C_1 that

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k}, \quad a_{-k} = -\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^{k-1}\,\mathrm{d}\zeta$$

with the series converging uniformly in $|z - z_0| \ge \rho_1$.

The region where the terms involving C_1 and C_2 both converge uniformly is $\rho_1 \le |z - z_0| \le \rho_2$. The only remaining thing to do in the proof is to explain why we can change the contour in the coefficients to C in each case. In the case of C_2 this is because the integrand is analytic between and C_2 and C and hence

$$a_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C} \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

In the case of C_1 this is because the integrand is analytic between and C_1 and C and for both to correspond to the same direction we need to consider the contour $-C_1$.

$$a_{-n} = -\frac{1}{2\pi i}\oint_{C_1} f(z)(z - z_0)^{n-1}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C} f(z)(z - z_0)^{n-1}\,\mathrm{d}z.$$

□

Further comments about the Laurent series

Uniqueness

A function analytic in an annulus centred at z_0 hence has a Laurent series involving positive and negative powers of $z - z_0$. Conversely if we are given a Laurent series with the part involving positive powers converging for $|z - z_0| < R$ and the one with negative powers converging for $|z - z_0| > r$ and $r < R$ then this can be shown to define a function analytic in the annulus $r < |z - z_0| < R$. We mention this as it follows that once z_0 is specified the Laurent series representation of a given function is unique as the coefficients are given by (7.6.4) in the previous theorem. It is rare in practice to evaluate the integrals to actually get the coefficients as other techniques are usually easier involving using Taylor series for different parts of whatever expression is being considered.

Term-by-term differentiation

Recall again our representation in $r < |z - z_0| < R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

which we may write as

$$f(z) = f_1(z) + f_2(z), \quad f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad f_2(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}.$$

$f_1(z)$ is analytic in the disk $\{z : |z - z_0| < R\}$ and $f_2(z)$ is analytic in the outer region $\{z : |z - z_0| > r\}$. If we let

$$g_2(z) = \begin{cases} f_2(1/(z - z_0)), & 0 < |z - z_0| < 1/r, \\ 0, & z = 0 \end{cases}$$

then $g_2(z)$ is analytic in the disk $\{z : |z - z_0| < 1/r\}$ and has the series representation

$$g_2(z) = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^n.$$

Thus

$$f(z) = f_1(z) + g_2(1/(z - z_0)).$$

In the case of power series and Taylor's series we showed that it is valid to differentiate term-by-term the series for $f(z)$ to get the series for $f'(z)$, $f''(z)$ etc.. It was the uniform convergence of the series which enabled this. We can do the same here with

$$f'(z) = f_1'(z) - \frac{1}{(z - z_0)^2} g_2'(1/(z - z_0)) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1} - \sum_{n=1}^{\infty} \frac{n a_n}{(z - z_0)^{n+1}}.$$

Example

Let

$$f(z) = \frac{1}{e^z - 1}, \quad 0 < |z| < 2\pi.$$

The function has simple poles at points where $e^z = 1$, i.e. at $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots$ and thus the annulus is between the pole at $z = 0$ and the poles at $z = \pm 2\pi i$. We consider how to get the first 3 non-zero terms in the Laurent series. We can adapt the method used to get the first few terms in the Maclaurin series of $\tan z$ to get the Maclaurin series for $f(z)$ as we have the relation

$$f(z)\ (e^z - 1) = 1.$$

Thus if we let

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots$$

then

$$1 = \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) \left(\frac{c_{-1}}{z} + c_0 + c_1 z + \cdots \right).$$

Equating constant term.

$$1 = c_{-1}.$$

Equating coefficient of z.

$$0 = c_0 + \frac{c_{-1}}{2}, \quad c_0 = -\frac{1}{2}.$$

Equating coefficient of z^2.

$$0 = c_1 + \frac{c_0}{2} + \frac{c_{-1}}{6}, \quad c_1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

7.7 Zeros and isolated singularities

When polynomials were considered at the start of chapter 4 about the elementary functions there are only a finite number of terms to consider and we defined what is meant by a zero of multiplicity m of a polynomial $p(z)$ of degree $n \ge m$ and when partial fraction were considered the knowledge of the multiplicity was needed in constructing a partial fraction representation of $1/p(z)$. We now generalise what is meant by a zero of multiplicity m to any analytic function $f(z)$ and classify isolated singularities which involve negative powers as appeared in the Laurent series.

Definition 7.7.1 *Let $f(z)$ be a function analytic at z_0. z_0 is called* **a zero of multiplicity** m *if*

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \textit{and } f^{(m)}(z_0) \ne 0.$$

By considering the Taylor series of $f(z)$ around z_0 this implies that it is of the form

$$f(z) = (z - z_0)^m \left(a_m + a_{m+1}(z - z_0) + \cdots\right), \quad a_m = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

The function

$$g(z) = a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots$$

has the same radius of convergence as the series for $f(z)$ and is such that $g(z_0) = a_m \neq 0$. As $g(z)$ is analytic and non-zero at z_0 it follows by continuity that $g(z) \neq 0$ in some neighbourhood $|z - z_0| < \delta$ of z_0. Thus since

$$f(z) = (z - z_0)^m g(z) \tag{7.7.1}$$

it follows that $f(z)$ is non-zero in $0 < |z - z_0| < \delta$. Hence when we have an analytic function $f(z)$ with $f(z_0) = 0$ then either all the derivatives of $f(z)$ are zero at z_0 and $f(z) = 0$ everywhere or $f(z)$ is of the form (7.7.1) and has no other zero in a neighbourhood of z_0, i.e. the zero at z_0 is **isolated**. Although it was not stated when it was described but this is what we had when the generalised L'Hopital rule was described.

As another case where the the result that analytic functions which are not identically zero can only have isolated singularities consider again (7.4.2) where we had that $f(z)$ being analytic implied that $\overline{f(\overline{z})}$ is also analytic. When f is such that $f(x) \in \mathbb{R}$ when $x \in \mathbb{R}$ it follows that the difference $f(z) - \overline{f(\overline{z})}$ is zero on the real axis from which we deduce that the difference must be zero for all z where $f(z)$ is defined.

We now consider isolated singularities. If we have a function $f(z)$ which is analytic in a domain such as $0 < |z - z_0| < R$ but which is not analytic at z_0 then we say that $f(z)$ has an isolated singularity at z_0. The theory of the previous section implies that such a function has a representation of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R \tag{7.7.2}$$

and we classify the type of singularity by the coefficients which appear in this Laurent series.

Definition 7.7.2 *With the conditions which lead to (7.7.2) we have the following terms.*

1. *If $a_n = 0$ for all $n < 0$ then $f(z)$ has* **a removable singularity at** z_0.
2. *Suppose $m \geq 1$. If $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$ then we have* **a pole of order** m **at** z_0.
3. *If $a_n \neq 0$ for an infinite number of negative index values n then $f(z)$ has* **an essential singularity at** z_0.

Examples

1. Let

$$f(z) = \exp\left(\frac{1}{z}\right) = 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{z^2}\right) + \cdots + \left(\frac{1}{n!}\right)\left(\frac{1}{z^n}\right) + \cdots$$

This has an essential singularity at $z = 0$.

2. Let
$$f(z) = \frac{\sin z}{z}.$$
As specified here this it is not defined at $z = 0$ but we know that
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$
from which it follows that for $z \neq 0$
$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots .$$
The Laurent series has no negative powers and hence $f(z)$ has a removable singularity at $z = 0$.

To show that $f(z)$ has a removable singularity at z_0 it is sufficient to show that there is a finite limit as $z \to z_0$ as in this example. If we then define $f(z_0)$ to correspond to the limiting value then we get a function which is analytic at z_0. That is, if we define
$$g(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0, \\ 1, & \text{when } z = 0, \end{cases}$$
then we have a function with the Maclaurin series
$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$
and this is analytic everywhere, i.e. an entire function.

A few more points about zeros, poles and essential singularities

1. If $f(z)$ has a pole of order $m \geq 1$ at z_0 then from the Laurent series representation
$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots , \quad 0 < |z - z_0| < R$$
and the function
$$g(z) = (z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{-m+n}(z - z_0)^n, \quad 0 < |z - z_0| < R$$
has a removable singularity at z_0 and if we define $g(z_0) = a_{-m}$ then we have an analytic function. Thus a function with a pole of order m can always be represented in the form
$$f(z) = \frac{g(z)}{(z - z_0)^m}$$
where $g(z_0) \neq 0$ and $g(z)$ is analytic in a neighbourhood of z_0,

2. If $f(z)$ has a zero of multiplicity $m \geq 1$ at z_0 then $1/f(z)$ has a pole of order m at z_0.

3. If $f(z)$ has a pole of order $m \geq 1$ at z_0 then $g(z) = 1/f(z)$ has a zero of multiplicity m at z_0 provided we define $g(z_0) = 0$.

4. As already stated, if $f(z)$ is analytic in $0 < |z - z_0| < R$ and there is a finite limit of $f(z)$ as $z \to z_0$ then $f(z)$ has a removable singularity at z_0 and we can define, or re-define, $f(z_0)$ to give us an analytic function. In the case of a pole of order m at z_0 we can similarly do this by multiplying by $(z - z_0)^m$ and in particular this tells us that $|f(z)|$ tends to ∞ as z gets closer and closer to z_0 along any path.

   The situation with essential singularities is more complicated as is illustrated by considering $\exp(1/z)$. If we let $z = r\mathrm{e}^{i\theta}$ then

$$\frac{1}{z} = \frac{1}{r}(\cos\,\theta - i\sin\,\theta)$$

   and as

$$\exp(x + iy) = \exp(x)(\cos\,y + i\sin\,y)$$

   we have

$$\exp\left(\frac{1}{z}\right) = \exp\left(\frac{\cos\,\theta}{r}\right)\left(\cos\left(\frac{\sin\,\theta}{r}\right) - i\sin\left(\frac{\sin\,\theta}{r}\right)\right).$$

   If we consider the behaviour as $r \to 0$ with θ being constant then we get ∞ if $\cos\,\theta > 0$, we get 0 if $\cos\,\theta < 0$ and we have that the magnitude remains at 1 if $\cos\,\theta = 0$.

Chapter 8

Residue theory

8.1 Introduction

In this chapter we return to the problem of evaluating integrals of the form

$$\oint_\Gamma f(z)\,\mathrm{d}z \tag{8.1.1}$$

where $f(z)$ is a simple closed positively orientated contour and where now $f(z)$ can be any function which is analytic on Γ and inside Γ except for a **finite number of isolated singularities**. In this case we show that the value of the integral can be expressed in terms of the residues of $f(z)$ inside Γ and we then use this result to help evaluate certain real integrals. The theory also enables us to estimate the location of zeros of analytic functions.

8.2 A recap of previous material on integrals involving loops

We start by recapping previous material on integrals of the type (8.1.1) inorder to indicate what is new in this chapter and specifically we note the following.

1. If f has an anti-derivative which is continuous in a region containing Γ then

   $$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

   This was covered on page 5-12 and takes care of powers z^n for all integers n except $n=-1$.

2. If $f(z)$ is analytic on and inside Γ, i.e. we have no isolated singularities, then

   $$\oint_\Gamma f(z)\,\mathrm{d}z = 0$$

   by the Cauchy-Goursat theorem which was covered at the end of chapter 5.

3. If the integrand is of the form

$$f(z) = \frac{g(z)}{(z - z_0)^{m+1}},$$

where m is an integer and where $g(z)$ is analytic on and inside Γ, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_\Gamma \frac{g(z)}{(z - z_0)^{m+1}}\,\mathrm{d}z = 2\pi i \frac{g^{(m)}(z_0)}{m!}$$

by the generalised Cauchy integral formula. This was done at the start of chapter 6. This result hence deals with the case of one specific isolated singularity at $z = z_0$ inside Γ.

4. An extension of the previous case was to the case when $f(z)$ was of the form

$$f(z) = \frac{g(z)}{Q(z)}$$

where $Q(z)$ is any polynomial as we can always express $Q(z)$ in a factored form and represent $1/Q(z)$ in partial fraction form. For example, if $Q(z) = (z - z_1)\cdots(z - z_n)$ with $z_1, \ldots, z_n$ being distinct then

$$\frac{1}{Q(z)} = \sum_{k=1}^{n} \frac{A_k}{z - z_k}, \quad A_k = \lim_{z \to z_k} \frac{z - z_k}{Q(z)} = \frac{1}{Q'(z_k)}$$

and

$$\frac{g(z)}{Q(z)} = \sum_{k=1}^{n} \left(\frac{1}{Q'(z_k)}\right) \frac{g(z)}{z - z_k}$$

so that

$$\oint_\Gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \left(\frac{1}{Q'(z_k)}\right) \oint_\Gamma \frac{g(z)}{z - z_k}\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \frac{g(z_k)}{Q'(z_k)} = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f,\, z_k),$$

where $z_1, \ldots, z_m$ are the points which are inside Γ. Polynomials $Q(z)$ with multiple roots can also be dealt with in a similar way with the details being slightly longer and with the value also depending on derivatives of $g(z)$ at the roots of $Q(z)$.

The extension in this chapter to what has already been done is that $f(z)$ can now have any isolated singularity, i.e. we are no longer restricted to cases where the singularity arises by dividing by polynomials, and the tool that enables us to handle any isolated singularity is the knowledge of the existence of a Laurent series in the vicinity of each isolated singularity and recall that Laurent series was considered in section 7.6.

8.3 The residue theorem

Before the general case of a finite number of isolated singularities is covered we consider first the case when there is just one isolated singularity at a point z_0 inside the loop Γ.

As $f(z)$ is analytic inside Γ except at the point z_0 it follows that there is a region of the form

$$0 < |z - z_0| < r$$

in which $f(z)$ has a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

with the series converging uniformly in a region $0 < |z - z_0| \le r' < r$. Let C denote the circle $\{z : \ |z - z_0| = \rho\}$, $\rho < r'$, and note that $f(z)$ is analytic between C and Γ which enables us to write

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_C f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} \oint_C a_n (z - z_0)^n\,\mathrm{d}z = 2\pi i a_{-1}.$$

The value just depends on one of the coefficients in the Laurent series and there is a name and notation associated with this.

Definition 8.3.1 *If $f(z)$ has an isolated singularity at z_0 then the coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series of $f(z)$ around z_0 is called* **the residue of f at z_0** *and is denoted by Res$(f,\, z_0)$.*

Thus with this notation we have

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f,\, z_0).$$

When the function f being considered is clear we sometimes shorten $\mathrm{Res}(f,\, z_0)$ to just $\mathrm{Res}(z_0)$.

As a final point before the case of more than one isolated singularity is considered, recall the reasoning which enables us to replace Γ by C (which was given on page 5-19) as we generalise this in a moment. With reference to the figure 8.1 given on page 8-4 the doubly connected region between Γ and the circle C is divided into two simply connected regions each of which are bounded by parts of Γ and C and straight lines joining the two curves as in the figure. As $f(z)$ is analytic inside the loop containing the upper part and it is also analytic inside the loop containing the lower part the Cauchy theorem tells us that these loop integrals of $f(z)$ are 0 and when we consider the union of the two contours the contributions from the straight line segments cancel and we get all of Γ and all of the circle C. Thus with reference to the labelling in the diagram we have

$$\int_{\Gamma_{\mathrm{up}}} f(z)\,\mathrm{d}z + \int_{\Gamma_{\mathrm{low}}} f(z)\,\mathrm{d}z - \int_{C_{\mathrm{up}}} f(z)\,\mathrm{d}z - \int_{C_{\mathrm{low}}} f(z)\,\mathrm{d}z = 0$$

which leads to the result. One way of interpreting the result is that $\Gamma \cup (-C)$ is the boundary of the region in which $f(z)$ is analytic and thus the integral around the entire boundary is 0.

If we now have n isolated singularities inside Γ at points $z_1, \ldots, z_n$ then as these points are isolated it is possible to construct circles $C_1, \ldots, C_n$ of sufficiently small radius so that the circles do not intersect and if each of the circles is considered in the anti-clockwise

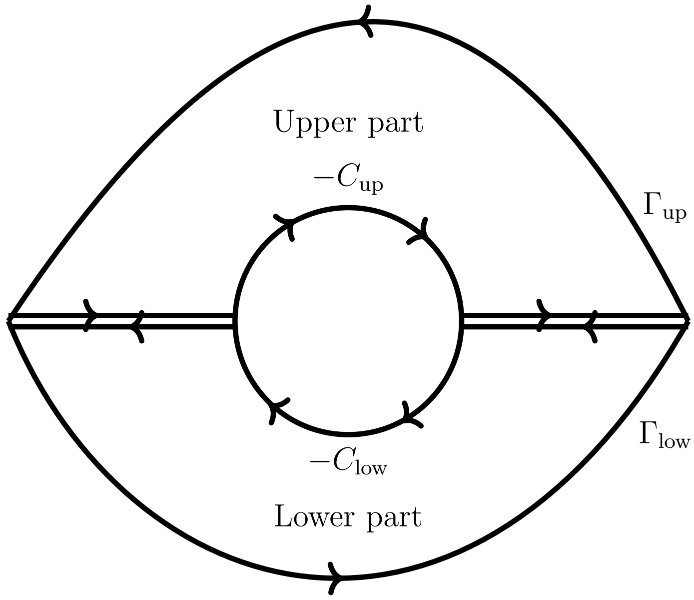


Figure 8.1: The doubly connected region between the inner curve C and the outer curve Γ is divided into two simply connected regions. The boundary of these lower and upper regions gives us two loops and we can use the Cauchy theorem with each loop.

sense then $\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)$ bounds a multi-connected region in which $f(z)$ is analytic. Here it is assumed that z_k is the centre of the circle C_k. A set-up of this type is illustrated in the top left plot in figure 8.2 on page 8-6 in the case $n = 4$. It is always possible to divide the the multi-connected region into two parts by joining the n circles in some way and with connecting to the outer boundary Γ and this is illustrated in the top right plot in figure 8.2. From the division of the multi-connected region into two simply connected regions we can consider the integrals involving the boundary of the simply connected regions and this is illustrated in the bottom two plots of figure 8.2. Observe that part of Γ is part of the boundary of one of regions and the other part is part of the boundary of the other region. We have a similar situation with the circles with part of a circle being part of the boundary of one of the regions and the other part is part of the boundary of the other region. In each case the direction of integration on the circles is in the clockwise sense. In the case of the straight line segments which join the different parts, we have one direction for one of the regions and the opposite direction for the other region. As we account for all of Γ and all of the n circles, Cauchy's theorem applied to both regions leads to

$$\oint_{\Gamma \cup (-C_1) \cup \cdots \cup (-C_n)} f(z)\,\mathrm{d}z = \oint_{\Gamma} f(z)\,\mathrm{d}z + \sum_{k=1}^{n} \oint_{-C_k} f(z)\,\mathrm{d}z = 0$$

and this re-arranges to

$$\oint_\Gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \oint_{C_k} f(z)\,\mathrm{d}z$$

with all integrals being in the anticlockwise sense. As there is just one isolated singularity inside C_k we have, as before,

$$\oint_{C_k} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f,\, z_k).$$

and we have established the **Cauchy Residue Theorem** which we state below.

Theorem 8.3.1 *If Γ is a simple closed positively orientated contour and f is analytic inside and on Γ except at points $z_1, \ldots, z_n$ inside Γ, then*

$$\oint_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} Res(f,\, z_k).$$

Example

Let $C = \{z :\ |z| = 2\pi\}$ with the circle traversed once in the anti-clockwise sense and consider

$$\oint_C \tan\, z\,\mathrm{d}z.$$

To help with the description let

$$f(z) = \tan\, z = \frac{\sin\, z}{\cos\, z}.$$

The singularities of $\tan\, z$ are at the points where $\cos\, z = 0$ and such points which satisfy $|z| < 2\pi$ are $\pm\pi/2$ and $\pm 3\pi/2$. By the residue theorem we thus have

$$\oint_C \tan\, z\,\mathrm{d}z = 2\pi i \left(\mathrm{Res}(f,\, -3\pi/2) + \mathrm{Res}(f,\, -\pi/2) + \mathrm{Res}(f,\, \pi/2) + \mathrm{Res}(f,\, 3\pi/2)\right).$$

To determine the residues note that $\cos\, z$ only has simple zeros and hence $\tan\, z$ only has simple poles and the Laurent series about $z = \pi/2$ is of the form

$$\tan\, z = \frac{a_{-1}}{z - \pi/2} + a_0 + a_1(z - \pi/2) + \cdots$$

so that

$$(z - \pi/2)\tan\, z = a_{-1} + a_0(z - \pi/2) + \cdots + a_1(z - \pi/2)^2 + \cdots$$

and we can get the residue by taking the limit in the right hand side as $z \to \pi/2$. In this case

$$\begin{aligned} \mathrm{Res}(f,\, \pi/2) &= \lim_{z\to\pi/2} \frac{(z - \pi/2)\sin\, z}{\cos\, z} \\ &= \left(\lim_{z\to\pi/2} \sin\, z\right)\left(\lim_{z\to\pi/2} \frac{(z - \pi/2)}{\cos\, z}\right) \\ &= \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1. \end{aligned}$$

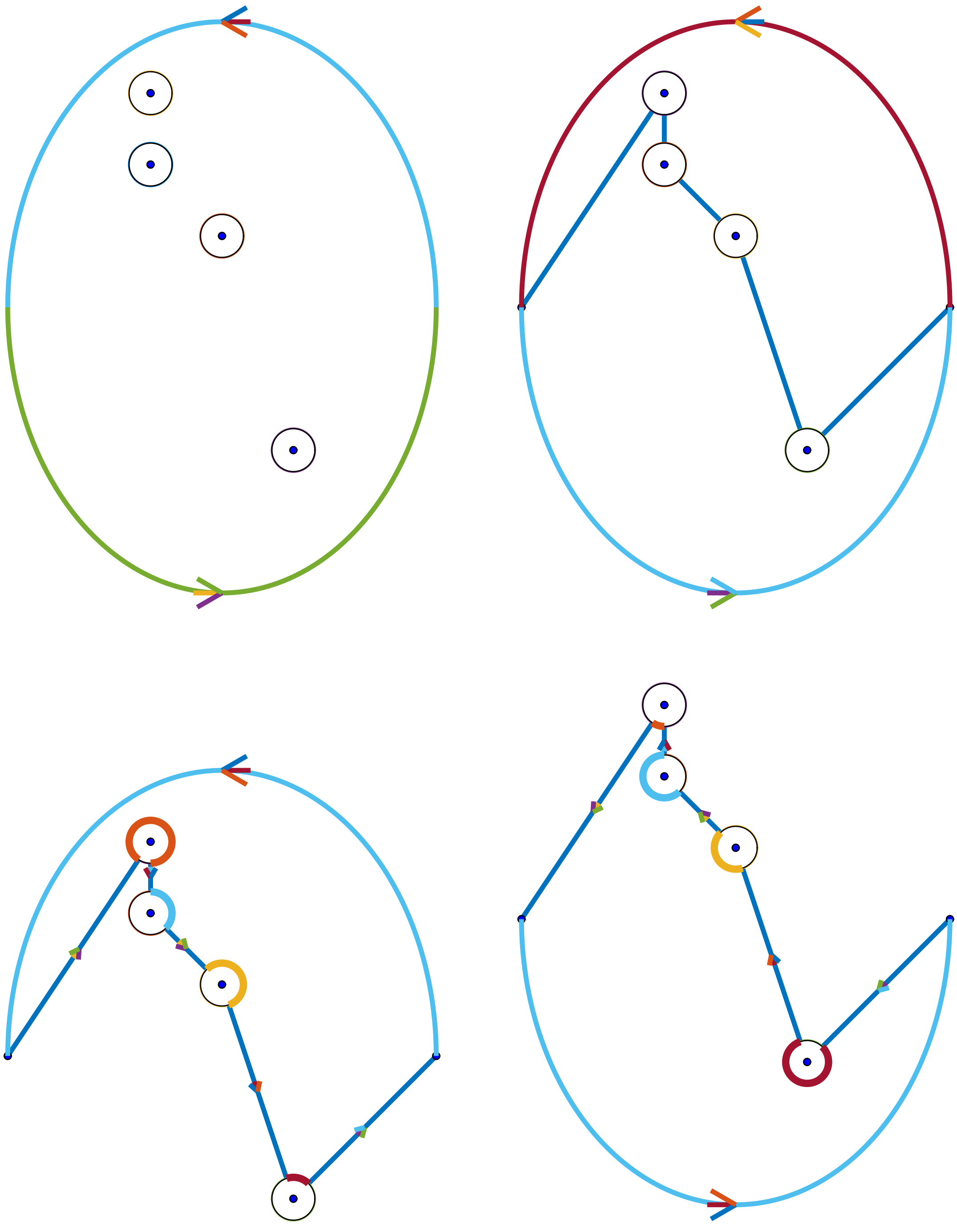


Figure 8.2: A contour containing 4 isolated singularities (top left). Lines joining the singular points and the boundary are added to divide the region into two part (top right). The contour for the top part (bottom left) and the contour for the bottom part (bottom right).

The same working is involved with all the residues and we have

$$\mathrm{Res}(f,\,-3\pi/2)=\mathrm{Res}(f,\,-\pi/2)=\mathrm{Res}(f,\,\pi/2)=\mathrm{Res}(f,\,3\pi/2)=-1$$

so that

$$\oint_C \tan\, z\,\mathrm{d}z = 2\pi i\,(-1-1-1-1) = -8\pi i.$$

A technique for determining the residue when we have a pole of order m

To use the Cauchy Residue theorem we need to be able to obtain the residue at each point inside Γ at which $f(z)$ has an isolated singularity and as the previous example illustrated it is not usually necessary to first obtain the entire Laurent series expansion. All we need is knowledge of the existence of the Laurent expansion and a technique for determining a_{-1} and in the case of a pole at z_0 of order m this can be done as follows.

As we know that $f(z)$ has a pole of order m at z_0 we know that the Laurent series close to z_0 is of the form

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+\frac{a_{-m+1}}{(z-z_0)^{m-1}}+\cdots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$$

To get a function which is finite as $z\to z_0$ we multiply by $(z-z_0)^m$ giving

$$(z-z_0)^m f(z)=a_{-m}+a_{-m+1}(z-z_0)+\cdots+a_{-1}(z-z_0)^{m-1}+a_0(z-z_0)^m+\cdots.$$

If we differentiate $m-1$ times then the part before a_{-1} disappears and we have

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left((z-z_0)^m f(z)\right)=(m-1)!a_{-1}+m!a_0(z-z_0)+\cdots$$

and if we take the limit as $z\to z_0$ this leads to an expression for a_{-1}. Thus if $m=1$ no differentiation is needed and we have

$$a_{-1}=\lim_{z\to z_0}(z-z_0)f(z)$$

and when $m\ge 2$ we have

$$a_{-1}=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left((z-z_0)^m f(z)\right).$$

8.4 Trigonometric integrals over $[0,2\pi]$ — using the the unit circle

This section is short as we have already considered integrals of this type in section 5.6.1. We just restrict the details here to the integrals involved and the technique used. The integrands contain $\cos\,\theta$ and $\sin\,\theta$ over a 2π-range and the integrals are of the form

$$\int_0^{2\pi} R(\cos\,\theta,\sin\,\theta)\,\mathrm{d}\theta$$

where $R(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

The technique to determine each of these is by using the substitution

$$z = \mathrm{e}^{i\theta}, \quad \text{and note that } \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz},$$

and as θ varies from 0 to 2π this corresponds to z traversing the unit circle once in the anti-clockwise direction. In terms of z we have

$$\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Let C denote the unit circle. In the general case this gives

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_C R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{\mathrm{d}z}{iz}.$$

Let

$$F(z) = \frac{1}{z}R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right).$$

As $R(.,.)$ is a rational function of its arguments it follows that $F(z)$ is a rational function of z and hence $F(z)$ has isolated singularities which are pole singularities. If the poles of $F(z)$ which are inside the unit circle are at $z_1, z_2, \ldots, z_m$ then by the Residue theorem

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_C \frac{1}{i}F(z)\,\mathrm{d}z = 2\pi\sum_{k=1}^{m}\mathrm{Res}(F,\, z_k).$$

8.5 $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ integrals — using a half circle

We now consider using the Residue theorem to help in evaluating certain real integrals of the type

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x. \tag{8.5.1}$$

Before the technique is described we need to first indicate what an integral with infinite limits means. In the case that the upper limit only is infinite and $a \in \mathbb{R}$ we define

$$\int_a^{\infty} f(x)\,\mathrm{d}x := \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x \tag{8.5.2}$$

and in the case that the lower limit only is infinite (i.e. $-\infty$) we define

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x := \lim_{c\to-\infty}\int_c^a f(x)\,\mathrm{d}x \tag{8.5.3}$$

provided the limit exists in each case. When both limits exist we define

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_a^{\infty} f(x)\,\mathrm{d}x. \tag{8.5.4}$$

Thus when the usual integral notation is used the integral in (8.5.1) exists when two different limits exist.

Sometimes the meaning of (8.5.1) is extended to deal with situations in which the limit in (8.5.2) and the limit in (8.5.3) do not exist but that

$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,\mathrm{d}x$$

does exist. When this limit exists we write

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,\mathrm{d}x$$

and the value is called the **Cauchy principal value** of the integral of f over $(-\infty,\infty)$. When (8.5.2) and (8.5.3) both exist the Cauchy principal value is the same as the value (8.5.4). In the examples considered in this section the function $f(x)$ being considered is always such that the value (8.5.4) exists and we do not need the Cauchy principal value notation here although you do sometimes see this in books even when it is not strictly needed. The Cauchy principal value notation is needed later in section 8.7 when we deal with integrands which are infinite at certain finite points on the contour being considered.

We illustrate the technique of using the Residue theorem to evaluate integrals of this type with some examples and then summarize the procedure.

1. Determine

$$I = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+2x+2}.$$

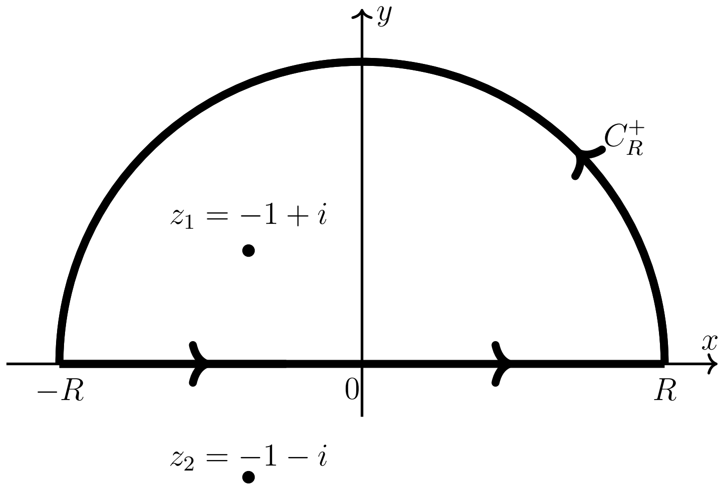
To start let

$$f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z+1)^2+1}$$

and observe that this has simple poles at the points

$$z_1 = -1+i \quad\text{and}\quad z_2 = \overline{z_1} = -1-i.$$

To use the Residue theorem we need a closed loop and for this to help in determining the value I we need part of the loop to be part of the real axis. A convenient choice for such a loop is to consider a half circle in the upper half plane and specifically we choose to take a half circle C_R^+ in the upper half plane in the anti-clockwise sense joined to the interval $[-R,R]$ as illustrated in the diagram and we let Γ_R denote the closed loop. In the diagram $R>\sqrt{2}$ so that z_1 is inside the loop with z_2 always being outside of the loop.



By the Residue theorem we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f,\, z_1).$$

We only have a simple pole and the residue is given by

$$\mathrm{Res}(f,\, z_1) = \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}\frac{z-z_1}{z^2+2z+2} = \frac{1}{2z_1+2} = \frac{1}{2i}.$$

Hence for $R > \sqrt{2}$ we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \pi.$$

To use this to determine I note that as $\Gamma_R = [-R, R] \cup C_R^+$ we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z. \tag{8.5.5}$$

Now on C_R^+ the integrand is "small" in magnitude when R is large but the length of the half circle is πR and the length increases with R and for the integral to tend to 0 as $R \to \infty$ we need the integrand to tend to 0 sufficiently fast. This can be established quite easily in this case as the denominator is dominated by the z^2 term when R is large so that for $z \in C_R^+$ the triangle inequality gives

$$|2z+2| \le 2R+2 \quad \text{and} \quad |z^2+2z+2| \ge |z^2| - |2z+2| \ge R^2 - (2R+2)$$

and hence

$$|f(z)| \le \frac{1}{R^2-2R-2}.$$

Thus

$$\begin{aligned}\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| &\le (\text{length of } C_R^+)(\text{bound on } |f(z)|)\\ &= \frac{\pi R}{R^2-2R-2} \to 0 \quad \text{as } R\to\infty.\end{aligned}$$

Thus by taking the limit as $R \to \infty$ in (8.5.5) we get

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi.$$

As some comments on this example, note that in the justification that the contribution from the part involving C_R^+ tends to 0 as $R \to \infty$ we can take R to be as large as we want when using the triangle inequality and we do not need to get the sharpest possible inequalities as we only interested in showing that the term tends to 0 as $R \to \infty$. This particular example can of course be evaluated using techniques that you have met before year 3 as

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

and if we make the substitution

$$x = -1 + \tan\theta$$

then the limits on θ are $-\pi/2$ to $\pi/2$ and

$$(x+1)^2 + 1 = 1 + \tan^2\theta = \sec^2\theta \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$$

so that

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+2x+2} = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta = \pi.$$

2. Determine

$$I = \int_0^{\infty} \frac{\mathrm{d}x}{x^4+16}.$$

Let

$$f(z) = \frac{1}{z^4+16}.$$

As this function $f(z)$ is even we have

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+16}.$$

and thus we can evaluate the integral by considering the integral on $(-\infty, \infty)$.

There are 4 solutions to

$$z^4 = -16 = 2^4 \mathrm{e}^{i\pi}$$

and two of the solutions are in the upper half and are given by

$$z_1 = 2\mathrm{e}^{i\pi/4} = \sqrt{2}(1+i), \quad z_2 = 2\mathrm{e}^{3i\pi/4} = \sqrt{2}(-1+i).$$

The function $f(z)$ has simple poles at these points. Let $R > 0$. With C_R^+ denoting the upper half circle and with $\Gamma_R = [-R, R] \cup C_R^+$ being a loop we need $R > 2$ for the 2 points to be inside the loop. By the residue theorem

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\left(\mathrm{Res}(f,\, z_1) + \mathrm{Res}(f,\, z_2)\right).$$

For the residues we have

$$\mathrm{Res}(f,\ z_1) = \lim_{z\to z_1} (z-z_1)f(z) = \lim_{z\to z_1} \frac{z-z_1}{z^4+16} = \frac{1}{4z_1^3} = \frac{z_1}{4z_1^4} = -\frac{z_1}{64}.$$

Similarly

$$\mathrm{Res}(f,\ z_2) = \lim_{z\to z_2} (z-z_2)f(z) = \lim_{z\to z_2} \frac{z-z_2}{z^4+16} = \frac{1}{4z_2^3} = \frac{z_2}{4z_2^4} = -\frac{z_2}{64}.$$

Hence the loop integral value is

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\left(-\frac{(z_1+z_2)}{64}\right) = 2\pi\left(\frac{2\sqrt{2}}{64}\right) = \frac{\pi\sqrt{2}}{16}.$$

To use this to determine I note that as $\Gamma_R = [-R, R] \cup C_R^+$ we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z.$$

When $z \in C_R^+$ we have $|z| = R$ and $|z^4+16| \ge R^4 - 16$ so that

$$|f(z)| \le \frac{1}{R^4-16}.$$

Thus

$$\begin{aligned}\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| &\le (\text{length of } C_R^+)(\text{bound on } |f(z)|) \\ &= \frac{\pi R}{R^4-1} \to 0 \quad \text{as } R\to\infty.\end{aligned}$$

Thus by taking the limit as $R \to \infty$ in the relation relating the loop integral to the contributions from $[-R, R]$ and C_R^+ we have

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi\sqrt{2}}{16}$$

and

$$I = \int_0^{\infty} f(x)\,\mathrm{d}x = \frac{\pi\sqrt{2}}{32}.$$

The general technique for this type of integral

As the examples have illustrated, we take a half circle C_R^+ and a closed loop

$$\Gamma_R = [-R, R] \cup C_R^+$$

and we have a function $f(x)$ with no singularities on the real line. If $f(z)$ has isolated singularities at points $z_1, \ldots, z_m$ in the upper half plane then these are inside the loop Γ_R when R is sufficiently large and by the Residue theorem we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f,\ z_k).$$

Provided $f(z)$ is such that

$$\int_{C_R^+} f(z)\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty \tag{8.5.6}$$

we get

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f,\ z_k).$$

We consider now a sufficient condition on $f(z)$ for (8.5.6) to be true and suppose that $M(R)$ is such that $|f(z)| \le M(R)$ for $|z| = R$. Then as πR is the length of C_R^+ we have

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \pi R M(R)$$

and we have the required outcome if $RM(R) \to 0$ as $R \to \infty$. Suppose that $f(z)$ is a rational function of the form

$$f(z) = \frac{P(z)}{Q(z)}, \quad \text{with } P(z) = a_p z^p + \cdots + a_1 z + a_0, \quad Q(z) = b_q z^q + \cdots + b_1 z + b_0$$

and with $a_p \ne 0$, $b_q \ne 0$. If we divide the numerator and denominator by the highest power of z in each case then we have

$$f(z) = \frac{z^p}{z^q} \left(\frac{a_p + a_{p-1}/z + \cdots a_0/(z^p)}{b_q + b_{q-1}/z + \cdots + b_0/(z^q)} \right) = z^{p-q} \left(\frac{a_p + a_{p-1}/z + \cdots a_0/(z^p)}{b_q + b_{q-1}/z + \cdots + b_0/(z^q)} \right)$$

so that

$$z^{q-p} f(z) \to \frac{a_p}{b_q} \quad \text{as } |z| \to \infty.$$

This tells us the behaviour of $|f(z)|$ as $|z| \to \infty$. When $|z| = R$ is sufficiently large there exists $M(R)$ such that

$$R^{q-p} M(R) \le 2 \left| \frac{a_p}{b_q} \right| \quad \text{and hence} \quad RM(R) \le 2 \left| \frac{a_p}{b_q} \right| \frac{1}{R^{q-p-1}}. \tag{8.5.7}$$

As the degrees p and q are integers the bound on the right hand side of (8.5.7) tends to 0 as $R \to \infty$ provided

$$q - p - 1 \ge 1, \quad \text{giving } q \ge p + 2.$$

In the examples this condition was satisfied with $p = 0$ and $q = 2$ when

$$f(z) = \frac{1}{z^2 + 2z + 2}$$

and with $p = 0$ and $q = 4$ when

$$f(z) = \frac{1}{z^4 + 16}.$$

8.6 $\int_{-\infty}^{\infty} f(x)\cos mx \,\mathrm{d}x$ and $\int_{-\infty}^{\infty} f(x)\sin mx \,\mathrm{d}x$, integrals — using half circles in the upper and lower half planes

We now consider evaluating integrals of the form

$$\int_{-\infty}^{\infty} f(x)\cos(mx)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin(mx)\,\mathrm{d}x,$$

where m is real and $f(x)$ is a real valued rational function, i.e. $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials with real coefficients and we assume that $Q(x)$ does not have a zero on the real line. With such integrals the integrand is now of the form

$$g(x) = f(x)\cos(mx) \quad \text{or} \quad g(x) = f(x)\sin(mx)$$

and as we will see the oscillation of the integrand leads to the integrals existing when $\deg(Q) \ge \deg(P) + 1$. We can consider similar techniques to section 8.5 but adjustments are needed as to what we take for $g(z)$ and we will now use half circles in the upper half plane and the lower half plane, in the most general case, depending on the integrand being considered which we next discuss.

The previous section suggests that we consider

$$g(z) = f(z)\cos(mz) \quad \text{or} \quad g(z) = f(z)\sin(mz) \tag{8.6.1}$$

but this does not help in this case as

$$\cos(mz) = \frac{1}{2}\left(\mathrm{e}^{imz} + \mathrm{e}^{-imz}\right) \quad \text{and} \quad \sin(mz) = \frac{1}{2i}\left(\mathrm{e}^{imz} - \mathrm{e}^{-imz}\right)$$

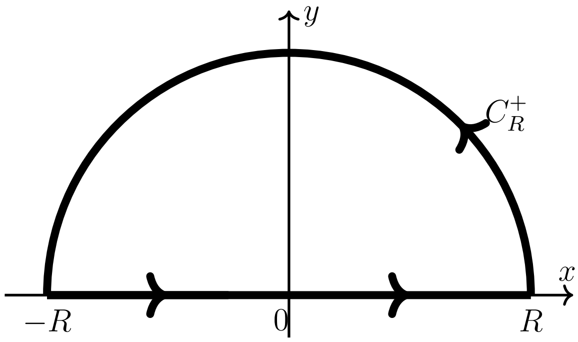
and each of the terms e^{imz} and e^{-imz} is bounded in one of the half planes and unbounded in the other half plane. Thus if we take $g(z)$ as just given then it is unbounded on a half circle in the upper half plane and also on a half circle in the lower half plane. Having determined why (8.6.1) does not help we now instead consider separately the cases

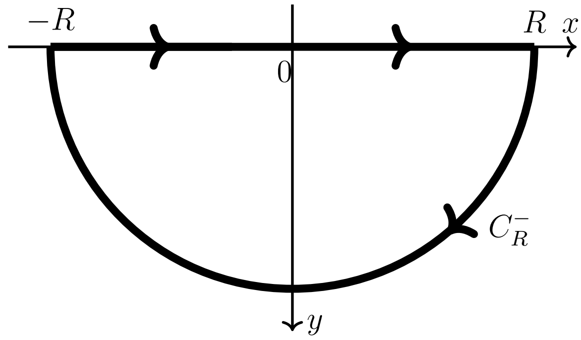
$$g(z) = f(z)\mathrm{e}^{imz} \quad \text{or} \quad g(z) = f(z)\mathrm{e}^{-imz}$$

and we later combine the results to get the integral we want to evaluate. With $z = x + iy$ and with $m > 0$ we have

$$\mathrm{e}^{imz} = \mathrm{e}^{-my}\mathrm{e}^{imx} \quad \text{and} \quad \mathrm{e}^{-imz} = \mathrm{e}^{my}\mathrm{e}^{-imx}.$$

When we consider half circles we need to take the upper half circle C_R^+ in the case of e^{imz} and we need to take the lower half circle C_R^- in the case of e^{-imz} so that the exponential term has magnitude $\mathrm{e}^{-|my|} \le 1$ on the half circle being used. We show below the two half circles to consider and we let now Γ_R^+ denote the closed half circle in the upper half plane, which is in the anti-clockwise direction, and let Γ_R^- denote the closed half circle in the lower half plane, which is in the clockwise direction.





As a note, in all the examples that will be considered in this module the need to use the lower half circle can be avoided.

Example

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\,\mathrm{d}x.$$

This is an integral of the type being considered corresponding to $m = 1$ (as $\sin(mx)$ is just $\sin x$) and as indicated above we first consider the integral involving

$$g(z) = \left(\frac{z}{1+z^2}\right) \mathrm{e}^{iz}.$$

This has simple poles at $\pm i$. With $R > 1$ and with $\Gamma_R^+ = [-R, R] \cup C_R^+$ the Residue theorem gives

$$\oint_{\Gamma_R^+} g(z)\,\mathrm{d}z = 2\pi i \,\mathrm{Res}(g,\, i).$$

Now for the residue at $z = i$ we have

$$\mathrm{Res}(g,\, i) = i\mathrm{e}^{-1} \lim_{z\to i} \frac{z-i}{1+z^2} = i\mathrm{e}^{-1}\left(\frac{1}{2i}\right) = \frac{\mathrm{e}^{-1}}{2}$$

and

$$\oint_{\Gamma_R^+} g(z)\,\mathrm{d}z = \pi i\,\mathrm{e}^{-1}.$$

In this particular example if we use a result that the integral over C_R^+ tends to 0 as $R \to \infty$ then we get

$$\int_{-\infty}^{\infty} \frac{x\,\mathrm{e}^{ix}}{1+x^2}\,\mathrm{d}x = \pi i \mathrm{e}^{-1}$$

and if we take the imaginary part then we get

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\,\mathrm{d}x = \pi \mathrm{e}^{-1}.$$

If the above had not been spotted then we would also need we consider the integral involving

$$g(z) = \left(\frac{z}{1+z^2}\right) \mathrm{e}^{-iz}$$

along the closed curve Γ_R^- which is in the clockwise direction and we do this now to illustrate how this is handled. As the direction is the clockwise direction the Residue theorem gives

$$\oint_{\Gamma_R^-} g(z)\,\mathrm{d}z = -2\pi i \,\mathrm{Res}(g,\,-i).$$

Now for the residue at $z = -i$ we have

$$\mathrm{Res}(g,\,-i) = -i\mathrm{e}^{-1} \lim_{z\to -i} \frac{z+i}{1+z^2} = -i\mathrm{e}^{-1}\left(\frac{1}{-2i}\right) = \frac{\mathrm{e}^{-1}}{2}$$

and

$$\oint_{\Gamma_R^-} g(z)\,\mathrm{d}z = -\pi i\,\mathrm{e}^{-1}.$$

As $R\to\infty$ the integrals over the half circles C_R^+ and C_R^- tend to 0, as we show in a moment, and as $\Gamma_R^+ = [-R,R]\cup C_R^+$ and $\Gamma_R^- = [-R,R]\cup C_R^-$ we get

$$\begin{aligned} I = \int_{-\infty}^{\infty} \frac{x\,\sin\,x}{1+x^2}\,\mathrm{d}x &= \frac{1}{2i}\left(\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{ix}}{1+x^2}\,\mathrm{d}x - \int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{-ix}}{1+x^2}\,\mathrm{d}x\right) \\ &= \frac{1}{2i}\left(\pi i\mathrm{e}^{-1} - (-\pi i\mathrm{e}^{-1})\right) = \pi\mathrm{e}^{-1}. \end{aligned}$$

We have obtained the answer in the example but we have not justified yet that

$$\int_{C_R^+}\left(\frac{z}{1+z^2}\right)\mathrm{e}^{iz}\,\mathrm{d}z \to 0 \quad \text{and} \quad \int_{C_R^-}\left(\frac{z}{1+z^2}\right)\mathrm{e}^{-iz}\,\mathrm{d}z \to 0 \quad \text{as } R\to\infty.$$

This is a bit harder to do and the result which covers this type of situation is known as Jordan's lemma which we consider next.

Theorem 8.6.1 *If $m > 0$ and P/Q is the quotient of two polynomials such that*

$$\deg(Q) \ge 1 + \deg(P)$$

then

$$\lim_{R\to\infty}\int_{C_R^+} e^{imz}\frac{P(z)}{Q(z)}\,dz = 0$$

where C_R^+ is the upper half circle of radius R and if $m < 0$ then

$$\lim_{R\to\infty}\int_{C_R^-} e^{-imz}\frac{P(z)}{Q(z)}\,dz = 0$$

where C_R^- is the lower half circle of radius R.

Proof: On the upper half circle

$$z = R\,\mathrm{e}^{i\theta} = R(\cos\,\theta + i\sin\,\theta), \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = iR\,\mathrm{e}^{i\theta}, \quad 0 \le \theta \le \pi$$

and

$$\int_{C_R^+} \mathrm{e}^{imz}\frac{P(z)}{Q(z)}\,\mathrm{d}z = i\int_0^\pi \exp(imR(\cos\,\theta + i\sin\,\theta))\frac{P(R\mathrm{e}^{i\theta})}{Q(R\mathrm{e}^{i\theta})}R\mathrm{e}^{i\theta}\,\mathrm{d}\theta.$$

The condition that $\deg(Q) \ge 1 + \deg(P)$ implies that for sufficiently large R there exists a constant A such that

$$\left| i\frac{P(R\mathrm{e}^{i\theta})}{Q(R\mathrm{e}^{i\theta})}R\mathrm{e}^{i\theta}\right| \le A.$$

Now

$$|\exp(imR(\cos\,\theta + i\sin\,\theta))| = \exp(-imR\sin\,\theta)$$

and thus we get the bound

$$\left|\int_{C_R^+} \mathrm{e}^{imz}\frac{P(z)}{Q(z)}\,\mathrm{d}z\right| \le A\int_0^\pi \exp(-imR\sin\,\theta)\,\mathrm{d}\theta.$$

Now as $\sin\,\theta$ has symmetry about $\theta = \pi/2$ the integrand above has symmetry about $\theta = \pi/2$, i.e.

$$\int_0^{\pi/2} \exp(-imR\sin\,\theta)\,\mathrm{d}\theta = \int_{\pi/2}^{\pi} \exp(-imR\sin\,\theta)\,\mathrm{d}\theta,$$

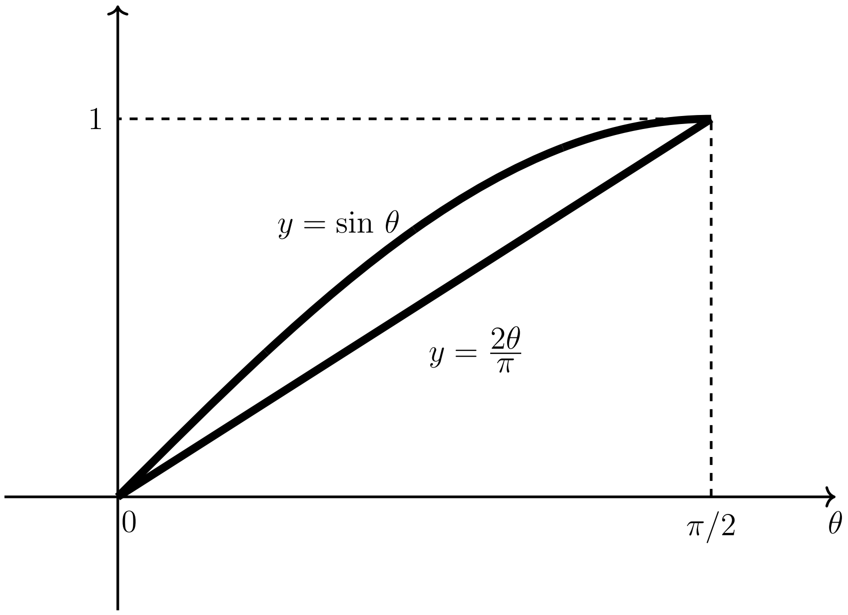
and we can further simplify slightly to

$$\left|\int_{C_R^+} \mathrm{e}^{imz}\frac{P(z)}{Q(z)}\,\mathrm{d}z\right| \le 2A\int_0^{\pi/2} \exp(-imR\sin\,\theta)\,\mathrm{d}\theta.$$

To complete the proof when $m > 0$ we need to show that the right hand side above tends to 0 as $R \to \infty$ which we do as follows.

When R is large $\exp(-imR\sin\,\theta)$ is very small at $\theta = \pi/2$ and over most of the range $[0, \pi/2]$ but it is 1 at $\theta = 0$. The part of $[0, \pi/2]$ where this is close to 1 diminishes as R increases and we need to make use of this observation to show that the integral gets smaller in magnitude as R increases. This can be done by considering the graph of $\sin\theta$ to observe that in $(0, \pi/2)$ the curve is above the straight line joining $(0, 0)$ to $(\pi/2, 1)$ to give the inequality

$$\sin\theta > \frac{2\theta}{\pi}.$$



To prove this inequality without need for the diagram we can define

$$h(\theta) = \sin\,\theta - \frac{2\theta}{\pi}, \quad h'(\theta) = \cos\,\theta - \frac{2}{\pi}.$$

We have $h(0) = h(\pi/2) = 0$ with $h'(\theta) > 0$ for $0 \le \theta < \alpha$ and $h'(\theta) < 0$ for $\alpha < \theta \le \pi/2$ where α is such that $h'(\alpha) = 0$, i.e. $\cos\,\alpha = 2/\pi$. $h(\theta)$ increases in $(0, \alpha)$ and decreases in $(\alpha, \pi/2)$ and and the property that $h(0) = h(\pi/2) = 0$ is sufficient to establish that it is positive in $(0, \pi/2)$.

With this lower bound on $\sin\,\theta$ we have for $0 < \theta < \pi/2$ that

$$\exp(-mR\sin\,\theta) \le \exp(-2mR\theta/\pi).$$

The right hand side here, i.e. the bound, is something that we can give expression for its integral. Thus we can bound the integral by

$$\begin{aligned}
\int_0^{\pi/2} \exp(-mR\sin\,\theta)\,\mathrm{d}\theta &\le \int_0^{\pi/2} \exp(-2mR\theta/\pi)\,\mathrm{d}\theta \\
&= \frac{\pi}{2mR}\left[\exp(-2mR\theta/\pi)\right]_{\pi/2}^{0} \\
&= \frac{\pi}{2mR}\left(1 - \exp(-mR)\right) \\
&\le \frac{\pi}{2mR} \to 0 \quad \text{as } R \to \infty.
\end{aligned}$$

We have hence shown that

$$\int_{C_R^+} \mathrm{e}^{imz}\frac{P(z)}{Q(z)}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

The proof for the lower half circle when $m < 0$ that

$$\int_{C_R^-} \mathrm{e}^{-imz}\frac{P(z)}{Q(z)}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty$$

is similar.

□

8.7 Indented contours

In all the previous integrals considered the integrand has been bounded on the contour in the integral. We now consider cases in which the integrand can be unbounded at certain points with the integral still being finite. A standard example of this type is with $\epsilon > 0$ the integral

$$\int_\epsilon^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2\left[\sqrt{x}\right]_\epsilon^1 = 2(1-\sqrt{\epsilon}) \to 2 \quad \text{as } \epsilon \to 0.$$

We thus say that

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2.$$

The integrand becomes unbounded as $x \to 0$ but the integral is finite.

As another example, suppose $a < 0 < b$, and again assume that $\epsilon > 0$, and consider

$$\int_\epsilon^b \frac{\mathrm{d}x}{x} = \ln(b) - \ln(\epsilon)$$

and similarly

$$\int_a^{-\epsilon} \frac{\mathrm{d}x}{x} = -\int_\epsilon^{|a|} \frac{\mathrm{d}x}{x} = \ln(\epsilon) - \ln(|a|).$$

In both these cases the limit does not exist as $\epsilon \to 0$ but the sum of the two terms remains constant. This leads to the Cauchy principal value which in this case is defined as

$$\text{p.v.}\int_a^b \frac{\mathrm{d}x}{x} := \lim_{\epsilon\to 0}\left(\int_a^{-\epsilon}\frac{\mathrm{d}x}{x} + \int_\epsilon^b \frac{\mathrm{d}x}{x}\right) = \ln(b) - \ln(|a|).$$

This is an example in which the integral only exists in a Cauchy principal value sense and as we see in a moment when we use contour integration to evaluate certain integrals we obtain the Cauchy principal value.

We consider next the example of evaluating

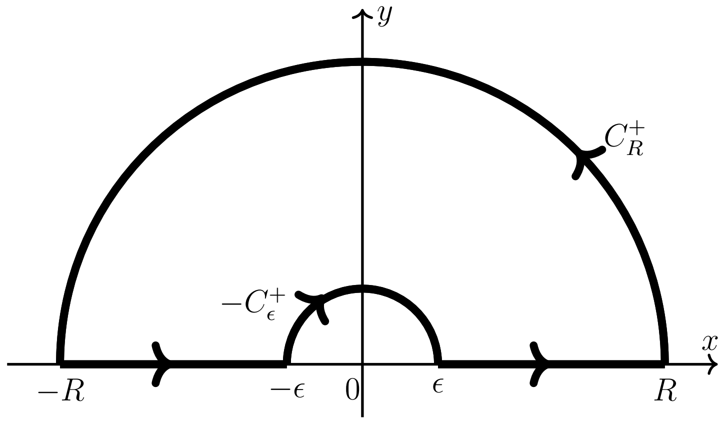
$$I = \text{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x$$

and once determined we obtain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \mathrm{Im}(I)$$

with this integral existing in the "usual sense" as $\sin(x)/x$ has a removable singularity at $x = 0$.

To deal with this integral we take a closed contour Γ_R in the anti-clockwise sense which involves a half circle C_R^+ of radius R, a half circle $-C_\epsilon^+$ of radius ϵ (traversed in the clockwise sense) and line segments $[-R, -\epsilon]$ and $[\epsilon, R]$ as shown in the diagram. The contour Γ_R is described as an indented contour.



Let

$$f(z) = \frac{e^{iz}}{z}.$$

The function $f(z)$ is analytic inside Γ_R and thus

$$\oint_{\Gamma_R} f(z)\,dz = 0.$$

As the closed loop Γ_R is the union of 4 parts this implies that

$$\oint_{\Gamma_R} f(z)\,d = \int_{-R}^{\epsilon} f(x)\,dx + \int_{\epsilon}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz - \int_{C_\epsilon^+} f(z)\,dz = 0. \qquad (8.7.1)$$

By Jordan's lemma on page 8-16 we have

$$\int_{C_R^+} f(z)\,dz \to 0 \quad \text{as } R \to \infty.$$

To deal with the half circle of radius ϵ we make use of the Laurent series representation of $f(z)$ in the vicinity of $z = 0$. The Laurent series is

$$f(z) = \frac{1}{z}\left(1 + (iz) + \frac{(iz)^2}{2} + \cdots\right) = \frac{1}{z} + (\text{analytic function}).$$

On C_ϵ^+ we have $z = \epsilon e^{i\theta}$, $0 \le \theta \le \pi$ giving

$$\int_{C_\epsilon^+} \frac{dz}{z} = i\int_0^\pi d\theta = i\pi$$

and as the length of C_ϵ^+ tends to 0 as $\epsilon \to 0$ we have

$$\int_{C_\epsilon^+} (\text{analytic function})\,dz \to 0 \quad \text{as } \epsilon \to 0.$$

From (8.7.1) if we let $R \to \infty$ and let $\epsilon \to 0$ then we have

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi.$$

If we take the real and imaginary parts then we obtain

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{x} \, \mathrm{d}x = 0$$

and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \pi.$$

As already indicated, there is no need for the principal value in the last case as $(\sin z)/z$ is bounded with the limit as $z \to 0$ being 1. As the integrand here is even we have

$$\int_{-\infty}^{0} \frac{\sin x}{x} \, \mathrm{d}x = \int_{0}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x$$

and hence

$$\int_{0}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \frac{\pi}{2}.$$

8.8 The argument principle and Rouche's theorem

The Residue theorem applies to functions which are analytic except for a finite number of isolated singularities and if we restrict to functions whose isolated singularities are poles, i.e. no essential singularities, then we say that the function is **meromorphic**.

Let f be a meromorphic function and let

$$G(z) = \frac{f'(z)}{f(z)}.$$

The function $G(z)$ is also meromorphic and has poles at the poles of $f(z)$ and also at the zeros of $f(z)$ and we investigate the isolated singularities of $G(z)$ next.

If $f(z)$ has a zero at z_0 of multiplicity m then the derivative has a zero at z_0 of multiplicity $m-1$ and we have the representations

$$\begin{aligned} f(z) &= (z-z_0)^m h(z), \\ f'(z) &= m(z-z_0)^{m-1} h(z) + (z-z_0)^m h'(z), \end{aligned}$$

where $h(z)$ is analytic in a vicinity of z_0 and $h(z_0) \neq 0$. These relations imply that in a vicinity of z_0 we have

$$G(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{h'(z)}{h(z)}.$$

Hence $G(z)$ has a simple pole at z_0 and the residue is the integer m which is the multiplicity of the zero.

If $f(z)$ has a pole at z_p of order n then it has a representation in the vicinity of z_p of the form

$$f(z) = \frac{\phi(z)}{(z-z_p)^n},$$

where $\phi(z)$ is analytic in a vicinity of z_p and $\phi(z_p) \neq 0$. Differentiating gives

$$f'(z) = \frac{-n}{(z-z_p)^{n+1}} \phi(z) + \frac{\phi'(z)}{(z-z_p)^n}$$

and

$$G(z) = \frac{f'(z)}{f(z)} = \frac{-n}{(z-z_p)} + \frac{\phi'(z)}{\phi(z)}.$$

Hence $G(z)$ has a simple pole at z_p and the residue is the negative integer $-n$.

All the singularities of $G(z)$ are thus simple poles and the residues are integers indicating the multiplicity of a zero or the order of a pole. If we apply the Residue theorem to $G(z)$ then we get what is known as the **argument principle** which we state next.

Theorem 8.8.1 *If f is analytic and non-zero at each point of a simple closed positively orientated contour C and it is meromorphic inside C then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N_0(f) - N_p(f),$$

where

$$\begin{aligned} N_0(f) &= \text{number of zeros inside } C \text{ (counting multiplicities)},\\ N_p(f) &= \text{number of poles inside } C \text{ (counting orders appropriately)}. \end{aligned}$$

When we restrict to analytic functions then we have the following corollary.

Corollary 8.8.1 *If f is analytic and non-zero at each point of a simple closed positively orientated contour C and it is analytic inside C then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \text{number of zeros inside } C \text{ (counting multiplicities)}. \qquad (8.8.1)$$

This corollary gives information about the location of zeros of an analytic function. Firstly, the value in (8.8.1) must be an integer and if can estimate it sufficiently accurately to determine the integer then we know exactly how many zeros lie inside C. Related to this last comment, if $g(z)$ is another analytic function which is "small" relative to $f(z)$ on C then we would expect

$$N_0(f) = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z \quad \text{and} \quad N_0(f+g) = \frac{1}{2\pi i}\oint_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,\mathrm{d}z$$

to be the same integer, i.e. $f+g$ has the same number of zeros inside C as does f. This is essentially Rouche's theorem which we state and prove next.

Theorem 8.8.2 *If f and g are analytic inside and on a simple closed contour C and if the strict inequality*

$$|g(z)| < |f(z)|$$

holds at each point of C, then f and $f+g$ must have the same total number of zeros (counting multiplicities) inside C.

Proof: Let

$$F(z) = \frac{g(z)}{f(z)}$$

so that for $z \in C$

$$|F(z)| < 1$$

and note that

$$g = fF, \quad \text{and } g' = fF' + f'F.$$

Now

$$\frac{f' + g'}{f + g} = \frac{f' + fF' + f'F}{f + fF} = \frac{f'(1 + F) + fF'}{f(1 + F)} = \frac{f'}{f} + \frac{F'}{1 + F}.$$

Hence

$$N_0(f + g) - N_0(f) = \frac{1}{2\pi i} \oint_C \frac{F'}{1 + F} \, \mathrm{d}z.$$

On the contour C we have $|F(z)| < 1$ and we have the geometric series expansion

$$\frac{F'}{1 + F} = F'(1 - F + F^2 - F^3 + \cdots)$$

and as the series converges uniformly term-by-term integration is valid and gives

$$\oint_C \frac{F'}{1 + F} \, \mathrm{d}z = \sum_{k=0}^{\infty} (-1)^k \oint_C F^k F' \, \mathrm{d}z.$$

As $F^k F'$ has the anti-derivative $F^{k+1}/(k + 1)$ which is analytic on C we have

$$\oint_C F^k F' \, \mathrm{d}z = 0$$

giving

$$N_0(f + g) - N_0(f) = \frac{1}{2\pi i} \oint_C \frac{F'}{1 + F} \, \mathrm{d}z = 0.$$

□

As a remark, the last part of the proof could have been done differently by observing that

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Log}(1 + F(z)) = \frac{F'(z)}{1 + F(z)}.$$

The property that $|F(z)| < 1$ for all $z \in C$ implies that $1 + F(z)$ is in the right half plane, i.e.

$$\mathrm{Re}(1 + F(z)) \ge 1 - |F(z)| > 0 \quad \forall z \in C,$$

and as a consequence $\mathrm{Log}(1 + F(z))$ is continuous on C. Thus our integrand has a continuous anti-derivative on the loop and we hence have

$$\oint_C \frac{F'}{1 + F} \, \mathrm{d}z = 0.$$

Examples

1. The following was taken from Question 1(b) of the May 2011 exam paper and was concerned with showing that all the zeros of a given polynomial lie in the annulus $1/2 < |z| < 2$.

   Let
   $$h(z) = z^5 + 3z^3 - 1.$$
   To investigate the zeros of this polynomial we take
   $$f(z) = z^5 \quad \text{and} \quad g(z) = 3z^3 - 1.$$
   $f(z)$ has a zero of multiplicity 5 at $z = 0$. Now on $|z| = 2$ we have $|f(z)| = 32$ and
   $$|g(z)| \le 2|z|^3 + 1 = 3 \times 8 + 1 = 25 < 32.$$
   Thus on the circle $|z| = 2$ we have $|g(z)| < |f(z)|$ and $h = f + g$ also has 5 zeros in $|z| < 2$ by Rouche's theorem.

   If we consider what is known as the reverse polynomial then we have
   $$z^5 h(1/z) = z^5(1/z^5 + 3/z^3 - 1) = 1 + 3z^2 - z^5$$
   and if we now define f and g by
   $$f(z) = -z^5, \quad g(z) = 1 + 3z^2$$
   then we have on $|z| = 2$ that $|f(z)| = 32$ and $|g(z)| \le 1 + 3|z|^2 \le 13 < 32$. Thus the reverse polynomial also has 5 zeros inside $|z| = 2$ by Rouche's theorem. Now if $z_1, \ldots, z_5$ are the roots of $h(z)$ then $1/z_1, \ldots, 1/z_5$ are the roots of the reverse polynomial and we have shown that
   $$|z_k| < 2 \quad \text{and} \quad \left|\frac{1}{z_k}\right| < 2$$
   so that
   $$\frac{1}{2} < |z_k| < 2, \quad k = 1, 2, 3, 4, 5.$$
   Rouche's theorem applied twice hence shows that all the zeros of $h(z)$ lie in the annulus $1/2 < |z| < 2$.

   This was not part of the question but to check this you can use the following statements in matlab.

```
a=[1 0 3 0 0 -1]';
r=roots(a);
[real(r), imag(r), abs(r)]
```

   which gives

```
-0.054470434414810   1.738049470684216   1.738902812342030
-0.054470434414810  -1.738049470684216   1.738902812342030
 0.385725592900579   0.591941938660390   0.706526356028934
 0.385725592900579  -0.591941938660390   0.706526356028934
-0.662510316971538   0.000000000000000   0.662510316971538
```

We have one real root and two complex conjugate pairs of roots and all have magnitude between 1/2 and 2 as the application of Rouche's theorem above has shown.

2. We can use Rouche's theorem to prove the fundamental theorem of algebra which was previously done on page 6-13. If

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0$$

and we take $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ then for sufficiently large R we have $|f(z)| > |g(z)|$ for all z satisfying $|z| = R$. We know that $f(z)$ has a zero at 0 of multiplicity n and thus $p(z) = f(z) + g(z)$ also has n zeros (counting multiplicities) inside the circle $|z| = R$.