Exercises involving the use of residue theory

Question 1 is a trig. integral and similar to what was asked in the exercises associated with chapter 5. The past exam questions in questions 5 and 6 just involve rational functions of z and be tackled as a result of what is taught in the first week of the material on chapter 8, i.e. from what is taught in week 23. Question 12 is a slight variation of something in the lecture notes with the difference here that a quarter of a circle is used instead of a half circle. Question 11 also just involves a rational function but has the additional difficulty in that the residue at a double pole must be obtained.

The past exam questions in questions 7, 8, and 9 all have an integrand which contains an exp(.) term and the material on this should be taught in week 24. Questions 2, 3, 4 and 10 all involve indented contours and the material on this should be taught in week 24. In the case of question 10 there is the additional difficulty of a double pole as well.

Questions 14 and 13 involve loops which are respectively a rectangle and a square. These can be considered at any time although they may be considered as among the more difficult questions.

1. Show the following by first using the substitution $z = \mathrm{e}^{i\theta}$.

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\sin\,\theta} = \frac{2\pi}{3}.$$

Solution

This is a trigonometric integral and the substitution $z = \mathrm{e}^{i\theta}$ gives a closed loop integral involving the unit circle traversed once in the anti-clockwise direction. We have

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad 4\sin(\theta) = \frac{2}{i}\left(z - \frac{1}{z}\right).$$

$$\left(\frac{1}{5 + 4\sin\,\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{i}F(z),$$

where

$$F(z) = \frac{1}{z}\left(\frac{1}{5 + \frac{2}{i}\left(z - \frac{1}{z}\right)}\right) = \frac{1}{5z - 2i(z^2 - 1)}.$$

Now

$$5z - 2i(z^2 - 1) = -i(2z^2 + 5iz - 2) = -i(2z + i)(z + 2i).$$

$F(z)$ has a simple pole inside the unit circle at $z = -i/2$ and for the residue the use of L'Hopital's rule gives

$$\begin{aligned}\mathrm{Res}(F,\, -i/2) &= \lim_{z\to -i/2}(z + i/2)F(z) \\ &= \left.\frac{1}{5 - 4iz}\right|_{z=-i/2} = \frac{1}{5 - 4i(-i/2)} = \frac{1}{5 - 2} = \frac{1}{3}.\end{aligned}$$

By the Residue theorem

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\sin\theta} = 2\pi \mathrm{Res}(F,\, -i/2) = \frac{2\pi}{3}.$$

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2. Suppose that $f(z)$ is analytic in an annulus $\{z : \ 0 < |z - x_0| < r\}$ and has a simple pole at $x_0 \in \mathbb{R}$. Let $0 < \epsilon < r$ and let $C_\epsilon^+ = \left\{x_0 + \epsilon \mathrm{e}^{i\theta} : \ 0 \le \theta \le \pi\right\}$ denote a half circle with centre at x_0 and radius ϵ. If the half circle is traversed once in the anti-clockwise direction then show that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon^+} f(z)\,\mathrm{d}z = \pi i \mathrm{Res}(f,\, x_0).$$

Solution

The function $f(z)$ has a Laurent series

$$f(z) = \frac{a_{-1}}{z - x_0} + \sum_{n=0}^{\infty} a_n (z - x_0)^n.$$

Let

$$g(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n.$$

This defines an analytic function.

$$\int_{C_\epsilon} f(z)\,\mathrm{d}z = a_{-1} \int_{C_\epsilon} \frac{\mathrm{d}z}{z - x_0} + \int_{C_\epsilon} g(z)\,\mathrm{d}z.$$

As $g(z)$ is analytic in the vicinity of x_0 it is bounded in magnitude by a constant M. By the ML inequality

$$\left| \int_{C_\epsilon} g(z)\,\mathrm{d}z \right| \le \pi \epsilon M \to 0 \quad \text{as } \epsilon \to 0.$$

With the parametrization $z(t) = x_0 + \epsilon \mathrm{e}^{it}$ we have

$$\int_{C_\epsilon} \frac{\mathrm{d}z}{z - x_0} = \int_0^{\pi} i \mathrm{d}t = \pi i.$$

Hence

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\,\mathrm{d}z = \pi i a_{-1}.$$

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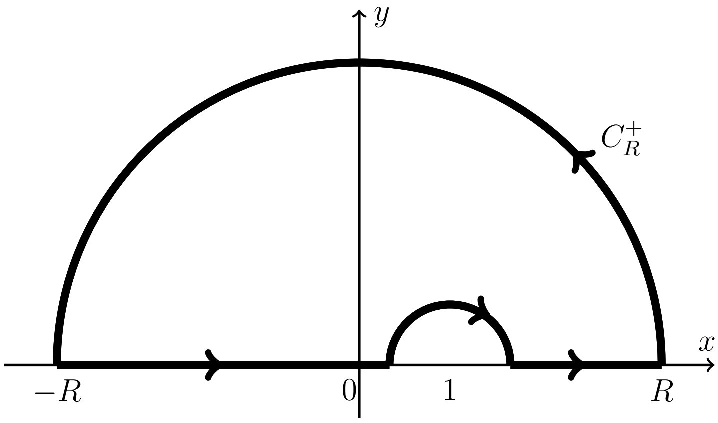
3. Show the following.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(3x)}{x - 1} \mathrm{d}x = -\pi \sin(3) \quad \text{and} \quad \text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(3x)}{x - 1} \mathrm{d}x = \pi \cos(3).$$

Solution

Let
$$f(z)=\frac{\mathrm{e}^{3iz}}{z-1}.$$
Consider the indented contour shown below with C_R^+ denoting the outer circle of radiius R considered in the anti-clockwise sense and also let C_ϵ^+ denote the half circle centered at 1 of radius ϵ considered in the anti-clockwise sence. The closed contour is
$$\Gamma_R=C_R^+\cup[-R,1-\epsilon]\cup(-C_\epsilon^+)\cup[1+\epsilon,R]$$
and hence as $f(z)$ is analytic inside the contour
$$\left(\int_{-R}^{1-\epsilon}+\int_{1+\epsilon}^{R}\right)f(x)\,\mathrm{d}x+\int_{C_R}f(z)\,hdz=\int_{C_\epsilon}f(z)\,hdz.$$



By Jordan's lemma
$$\int_{C_R}f(z)\,\mathrm{d}z\to 0\quad\text{as } R\to\infty.$$
By the result of the previous question
$$\int_{C_\epsilon}f(z)\,\mathrm{d}z\to\pi i\mathrm{Res}(f,\,1)\quad\text{as }\epsilon\to 0.$$
$$\lim_{R\to\infty,\epsilon\to0}\left(\int_{-R}^{1-\epsilon}+\int_{1+\epsilon}^{R}\right)f(x)\,\mathrm{d}x=\pi i\mathrm{Res}(f,\,1)=\pi i\mathrm{e}^{3i}=\pi(-\sin(3)+i\cos(3)).$$
By taking the real an imaginary parts gives the stated results.

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4. Verify that
$$\int_0^\infty\frac{\sin^2 x}{x^2}\,\mathrm{d}x=\frac{\pi}{2}.$$

Solution

First note that the integrand is even and that

$$\sin^2 x = \frac{1-\cos(2x)}{2}.$$

Thus

$$\begin{aligned} I &= \int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x \\ &= \frac{1}{4}\int_{-\infty}^\infty \frac{1-\cos(2x)}{x^2}\,\mathrm{d}x \\ &= \frac{1}{4}\,\mathrm{Re}\ \text{p.v.} \int_{-\infty}^\infty \frac{1-\mathrm{e}^{2ix}}{x^2}\,\mathrm{d}x. \end{aligned}$$

The principal value is needed here as the imaginary part has a pole at $x = 0$.

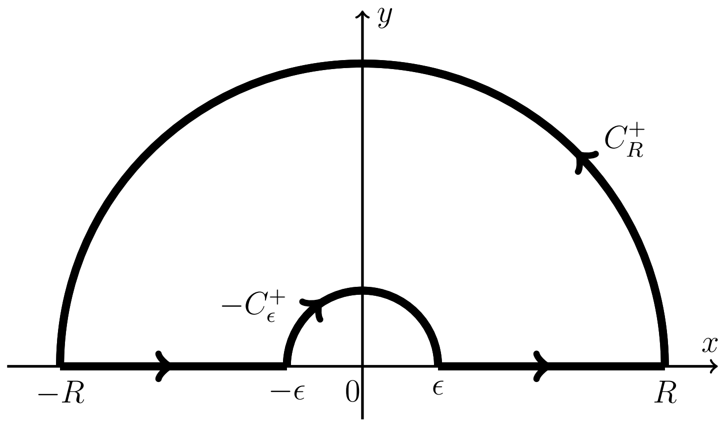
Let

$$f(z) = \frac{1-\mathrm{e}^{2iz}}{z^2}.$$

Let $0 < \epsilon < R$ and let

$$\Gamma_R = [-R,-\epsilon] \cup (-C_\epsilon^+) \cup [\epsilon, R] \cup C_R^+$$

where C_ϵ^+ and C_R^+ be half circles of radius ϵ and R respectively in the upper half plane as illustrated in the following diagram.



The function $f(z)$ is analytic inside Γ_R and hence

$$\int_{-R}^{-\epsilon} f(x)\,\mathrm{d}x + \int_{-C_\epsilon^+} f(z)\,\mathrm{d}z + \int_\epsilon^R f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = 0. \qquad (*)$$

On C_R^+ we have

$$\left|1-\mathrm{e}^{2iz}\right| \le 2 \quad \text{and} \quad |f(z)| \le \frac{2}{R^2}.$$

As the length of C_R^+ is πR we have

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \le \frac{2\pi R}{R^2} = \frac{2\pi}{R} \to 0 \quad \text{as } R\to\infty.$$

Thus letting $R\to\infty$ in $(*)$ gives

$$\int_{-\infty}^{-\epsilon} f(x)\,\mathrm{d}x + \int_{\epsilon}^{\infty} f(x)\,\mathrm{d}x = \int_{C_\epsilon^+} f(z)\,\mathrm{d}z.$$

Now $f(z)$ has a simple pole at $z=0$ and we can get the Laurent series as follows.

$$\begin{aligned}
\mathrm{e}^{2iz} &= 1 + (2iz) + \frac{(2iz)^2}{2} + \cdots \\
1-\mathrm{e}^{2iz} &= -(2iz) - \frac{(2iz)^2}{2} + \cdots \\
\frac{1-\mathrm{e}^{2iz}}{z^2} &= -\frac{2i}{z} - \text{analytic function.}
\end{aligned}$$

If we let $z=\epsilon\,\mathrm{e}^{i\theta}$ then

$$\int_{C_\epsilon^+} \frac{1}{z}\,\mathrm{d}z = i\int_0^{\pi}\mathrm{d}\theta = i\pi.$$

From this it follows that

$$\int_{C_\epsilon^+} f(z)\,\mathrm{d}z \to (\pi i)(-2i) = 2\pi \quad \text{as } \epsilon\to 0.$$

Hence

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{2ix}}{x^2}\,\mathrm{d}x = 2\pi$$

and as this is real we get

$$I = \left(\frac{1}{4}\right) 2\pi = \frac{\pi}{2}.$$

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5. *The following was part of question 4 in the May 2023 MA3614 exam paper. This part of the question was worth 9 marks of the 20 marks in the entire question.*

   Let a, b and c be real numbers with $a > 0$ and let

$$f(z) = \frac{1}{az^2 + bz + c}.$$

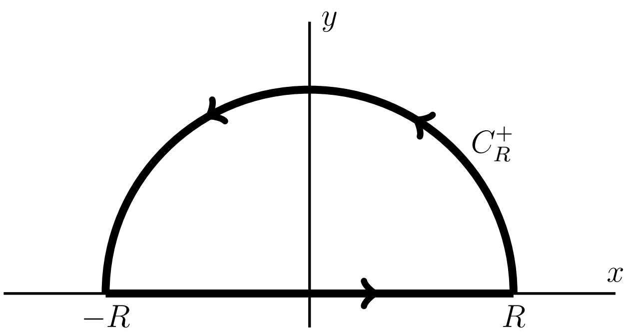
   (a) When $b^2 \neq 4ac$ indicate all the poles of $f(z)$ and determine the residue at each pole. Similarly, in the case $b^2 = 4ac$ indicate all the poles of $f(z)$ and determine the residue at each pole.

   (b) Let C_R denote the circle with centre 0 and radius $R > 0$ traversed once in the anti-clockwise sense. By any means explain why

$$\oint_{C_R} f(z)\,\mathrm{d}z = 0$$

   when R is sufficiently large.

   (c) Let C_R^+ denote the half circle with centre at 0 and radius $R > 0$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



   Use the ML inequality to explain why

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z = 0.$$

   Further, in the case $4ac > b^2$ use the loop Γ_R to determine an expression in terms of a, b and c of the value

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

   You need to explain all your steps.

Solution

(a) By the quadratic formula $f(z)$ has poles at

$$z_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad z_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 \neq 4ac$ these points are distinct and we have simple poles. By L'Hopital's rule the residues are

$$\begin{aligned} \mathrm{Res}(f,\, z_1) &= \lim_{z \to z_1} \frac{z - z_1}{az^2 + bz + c} = \frac{1}{2az_1 + b} = -\frac{1}{\sqrt{b^2 - 4ac}}, \\ \mathrm{Res}(f,\, z_2) &= \lim_{z \to z_2} \frac{z - z_2}{az^2 + bz + c} = \frac{1}{2az_2 + b} = \frac{1}{\sqrt{b^2 - 4ac}}. \end{aligned}$$

When $b^2 = 4ac$ we have $z_1 = z_2 = -b/(2a)$ and

$$f(z) = \frac{1}{a(z - z_1)^2}.$$

We just have one double pole and the residue is 0.

(b) When $b^2 = 4ac$ there is no residue and when $z_1 \neq z_2$ the sum of the residues is 0. The poles are inside C_R when R is sufficiently large. By the residue theorem the integral is 0.

(c) Let z be such that $|z| = R$.

$$|az^2| = aR^2, \quad |bz + c| \le |b|R + |c|, \quad |az^2 + bz + c| \ge aR^2 - (|b|R + |c|).$$

The right hand side is positive when R is sufficiently large and

$$|f(z)| \le \frac{1}{aR^2 - (|b|R + |c|)} =: M.$$

The length of C_R^+ is $L = \pi R$. By the ML inequality

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le ML = \frac{\pi R}{aR^2 - (|b|R + |c|)} \to 0 \quad \text{as } R \to \infty.$$

When $4ac > b^2$ the simple poles of $f(z)$ are a complex conjugate pair and the one in the upper half plane is

$$z_2 = \frac{-b + i\sqrt{4ac - b^2}}{2a}.$$

When R is sufficiently large this point is inside Γ_R and by the residue theorem

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f,\, z_2) = \frac{2\pi i}{i\sqrt{4ac - b^2}} = \frac{2\pi}{\sqrt{4ac - b^2}}.$$

As $\Gamma_R = [-R, R] \cup C_R^+$ we have for sufficiently large R that

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = \frac{2\pi}{\sqrt{4ac - b^2}}.$$

Letting $R \to \infty$ and using the result about the integral on C_R^+ tending to 0 we have

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{2\pi}{\sqrt{4ac - b^2}}.$$

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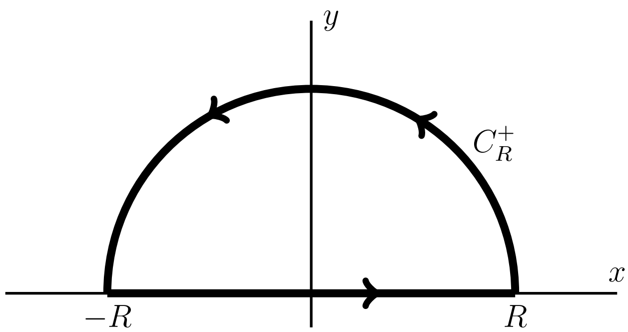
6. *The following was part of question 4 in the May 2022 MA3614 exam paper. This part of the question was worth 10 marks.*

Let
$$f(z)=\frac{1}{1+z^2+z^4}.$$
Let C_R^+ denote the half circle with centre at 0 and radius $R>1$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R,R]$ followed by the half circle C_R^+. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



(a) The function $f(z)$ has simple poles at the points $\pm z_1$ and $\pm z_2$ where $z_1=\mathrm{e}^{i\pi/3}$ and $z_2=\mathrm{e}^{i2\pi/3}$. Indicate which two points are in the upper half plane, give the cartesian form of these points and give workings to confirm that $1+z_1^2+z_1^4=0$.

(b) Determine the residue at each of the two simple poles in the upper half plane and determine
$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

(c) Determine, giving reasons, the value of
$$\lim_{R\to\infty}\int_{C_R^+} f(z)\,\mathrm{d}z.$$

(d) By using the loop Γ_R, determine
$$\int_0^\infty f(x)\,\mathrm{d}x.$$

Solution

(a) The points z_1 and $z_2 = z_1^2$ are in the upper half plane.

$$z_1 = \cos(\pi/3) + i\sin(\pi/3) = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$z_1^4 = \mathrm{e}^{4\pi/3} = -z_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Thus $z_1^2 + z_1^4 = z_2 + z_1^4 = -1$ and we have $1 + z_1^2 + z_1^4 = 0$.

(b) z_1 and z_2 are simple poles of $f(z)$.

$$\mathrm{Res}(f,\, z_1) = \lim_{z\to z_1} (z - z_1) f(z) = \lim_{z\to z_1} \frac{z - z_1}{1 + z^2 + z^4} = \frac{1}{2z_1 + 4z_1^3}.$$

As $z_1^3 = \mathrm{e}^{i\pi} = -1$ we have

$$\mathrm{Res}(f,\, z_1) = \frac{1}{2z_1 - 4} = \frac{1}{-3 + i\sqrt{3}} = \frac{-3 - i\sqrt{3}}{12}.$$

Similarly

$$\mathrm{Res}(f,\, z_2) = \lim_{z\to z_2} (z - z_2) f(z) = \frac{1}{2z_2 + 4z_2^3}.$$

As $z_2^3 = 1$ we have

$$\mathrm{Res}(f,\, z_2) = \frac{1}{2z_2 + 4} = \frac{1}{3 + i\sqrt{3}} = \frac{3 - i\sqrt{3}}{12}.$$

By the residue theorem

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i \left(\frac{-3 - i\sqrt{3}}{12} + \frac{3 - i\sqrt{3}}{12} \right) = 2\pi i \left(-2i\frac{\sqrt{3}}{12} \right) = \frac{\sqrt{3}\pi}{3}.$$

(c) For $|z| = R$ being large we have

$$|1 + z^2 + z^4| \ge R^4 - R^2 - 1 \quad \text{and} \quad |f(z)| \le \frac{1}{R^4 - R^2 - 1}.$$

The length of the half circle is πR. By the ML inequality

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \frac{\pi R}{R^4 - R^2 - 1} \to 0 \quad \text{as } R \to \infty.$$

(d) As the function $f(z)$ is even we have, using the previous parts,

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\int_0^R f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = \frac{\sqrt{3}\pi}{3}.$$

Letting $R \to \infty$ gives

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{\sqrt{3}\pi}{6}.$$

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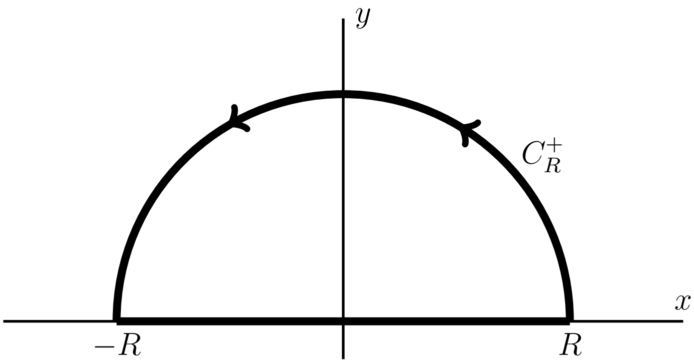
7. *The following was part of question 4 in the May 2021 MA3614 exam paper. This part of the question was worth 10 marks.*

Let C_R^+ denote the half circle with centre at 0 and radius $R > 0$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



In the following which function you consider depends on the 4th digit of your 7-digit student id.. If your 4th digit is one of 0, 2, 4, 6, 8 then your function $f(z)$ is on the left hand side whilst if it is one of the digits 1, 3, 5, 7, 9 then your function $f(z)$ is on the right hand side.

$$f(z) = \frac{4 + \mathrm{e}^{3iz}}{1 + 2z^2} \quad \text{(even digit case)} \qquad \text{or} \qquad f(z) = \frac{2 - \mathrm{e}^{5iz}}{1 + 3z^2} \quad \text{(odd digit case).}$$

(a) Give all the poles of your version of the function $f(z)$ in the complex plane and determine the residue at each pole in the upper half plane.

(b) For your version of $f(z)$, determine, giving reasons, the value of

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z.$$

(c) For your version of $f(z)$, determine, giving reasons, the value of the integrals

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{Re}(f(x))\,\mathrm{d}x.$$

Here $\mathrm{Re}(f(x))$ means the real part of $f(x)$.

Solution

This is the version for a 4th digit of 0, 2, 4, 6, 8.

(a) The only poles of the function are when $1 + 2z^2 = 0$ and the points are

$$z_1 = \frac{i}{\sqrt{2}}, \quad z_2 = -\frac{i}{\sqrt{2}}.$$

Only z_1 is in the upper half plane. z_1 is a simple pole and then by L'Hopital's rule and properties of limits

$$\mathrm{Res}(f,\, z_1) = \lim_{z\to z_1} (z - z_1) f(z) = \left(4 + \mathrm{e}^{3iz_1}\right) \lim_{z\to z_1} \frac{z - z_1}{1 + 2z^2} = \frac{4 + \mathrm{e}^{3iz_1}}{4z_1}.$$

$$3iz_1 = -\frac{3}{\sqrt{2}}, \quad 4z_1 = 2\sqrt{2}i, \quad \mathrm{Res}(f,\, z_1) = -i\left(\frac{4 + \mathrm{e}^{-3/\sqrt{2}}}{2\sqrt{2}}\right).$$

(b) When $z = x + iy \in C_R^+$, $y \ge 0$, $3iz = -3y + 3ix$. Thus

$$|\mathrm{e}^{3iz}| \le 1, \quad \text{and also } |1 + 2z^2| \ge 2R^2 - 1.$$

The length of C_R^+ is πR and on C_R^+ we have for sufficiently large R that

$$|f(z)| \le \frac{4+1}{2R^2 - 1} = \frac{5}{2R^2 - 1}.$$

By the ML inequality

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \frac{5\pi R}{2R^2 - 1} \to 0 \quad \text{as } R \to \infty.$$

(c) As Γ_R is the union of two parts the use of the residue theorem gives

$$\begin{aligned} \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\mathrm{d}z &= 2\pi i \mathrm{Res}(f,\, z_1) \\ &= 2\pi \left(\frac{4 + \mathrm{e}^{-3/\sqrt{2}}}{2\sqrt{2}}\right) = \pi \left(\frac{4 + \mathrm{e}^{-3/\sqrt{2}}}{\sqrt{2}}\right). \end{aligned}$$

Letting $R \to \infty$ and using the result of part (ii) gives

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi \left(\frac{4 + \mathrm{e}^{-3/\sqrt{2}}}{\sqrt{2}}\right).$$

As the value is real we also have

$$\int_{-\infty}^{\infty} \mathrm{Re}(f(x))\,\mathrm{d}x = \pi \left(\frac{4 + \mathrm{e}^{-3/\sqrt{2}}}{\sqrt{2}}\right).$$

This is the version for a 4th digit of 1, 3, 5, 7, 9.

(a) The only poles of the function are when $1+3z^2=0$ and the points are

$$z_1 = \frac{i}{\sqrt{3}}, \quad z_2 = -\frac{i}{\sqrt{3}}.$$

Only z_1 is in the upper half plane. z_1 is a simple pole and then by L'Hopital's rule and properties of limits

$$\mathrm{Res}(f,\, z_1) = \lim_{z\to z_1} (z-z_1) f(z) = \left(2 - \mathrm{e}^{5iz_1}\right) \lim_{z\to z_1} \frac{z-z_1}{1+3z^2} = \frac{2-\mathrm{e}^{5iz_1}}{6z_1}.$$

$$5iz_1 = -\frac{5}{\sqrt{3}}, \quad 6z_1 = \frac{6}{\sqrt{3}} i = 2\sqrt{3} i, \quad \mathrm{Res}(f,\, z_1) = -i\left(\frac{2-\mathrm{e}^{-5/\sqrt{3}}}{2\sqrt{3}}\right).$$

(b) When $z = x+iy \in C_R^+$, $y \ge 0$, $5iz = -5y + 5ix$. Thus

$$|\mathrm{e}^{5iz}| \le 1, \quad \text{and also } |1+3z^2| \ge 3R^2 - 1.$$

The length of C_R^+ is πR and on C_R^+ we have for sufficiently large R that

$$|f(z)| \le \frac{2+1}{3R^2-1} = \frac{3}{3R^2-1}.$$

By the ML inequality

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \frac{3\pi R}{3R^2-1} \to 0 \quad \text{as } R \to \infty.$$

(c) As Γ_R is the union of two parts the use of the residue theorem gives

$$\begin{aligned} \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\mathrm{d}z &= 2\pi i \mathrm{Res}(f,\, z_1) \\ &= 2\pi\left(\frac{2-\mathrm{e}^{-5/\sqrt{3}}}{2\sqrt{3}}\right) = \pi\left(\frac{2-\mathrm{e}^{-5/\sqrt{3}}}{\sqrt{3}}\right). \end{aligned}$$

Letting $R\to\infty$ and using the result of part (ii) gives

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi\left(\frac{2-\mathrm{e}^{-5/\sqrt{3}}}{\sqrt{3}}\right).$$

As the value is real we also have

$$\int_{-\infty}^{\infty} \mathrm{Re}(f(x))\,\mathrm{d}x = \pi\left(\frac{2-\mathrm{e}^{-5/\sqrt{3}}}{\sqrt{3}}\right).$$

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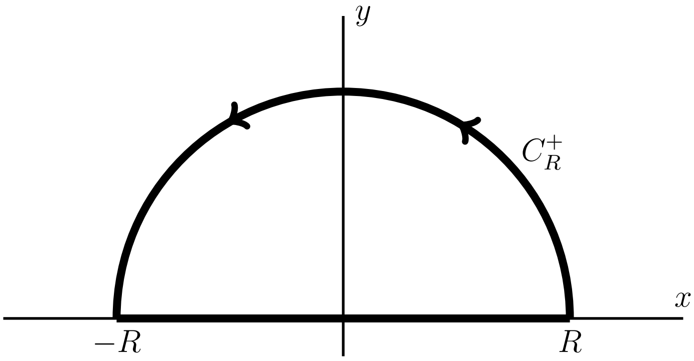
8. *The following was part of question 4 in the May 2020 MA3614 exam paper. This part of the question was worth 9 marks.*

   Let C_R^+ denote the half circle with centre at 0 and radius $R > 1$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



   Also let $a > 0$ and let

$$f(z) = \frac{\mathrm{e}^{iaz}}{4 + z^2}.$$

   (a) Show that

$$\int_{C_R^+} f(z)\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

   (b) When $R > 2$ determine, giving reasons,

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

   (c) By giving appropriate reasoning, determine

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Solution

(a) Now if $x+iy \in C_R^+$ then $y \geq 0$ and

$$\mathrm{e}^{iaz} = \mathrm{e}^{iax}\mathrm{e}^{-ay} \quad \text{and} \quad \left|\mathrm{e}^{iaz}\right| = \mathrm{e}^{-ay} \leq 1.$$

When $R > 2$ the denominator in the expression for $f(z)$ is bounded below by

$$|4+z^2| \geq R^2 - 4.$$

Hence on C_R^+ we have

$$|f(z)| \leq \frac{1}{R^2-4}$$

and as the length of C_R^+ is πR the use of the ML inequality gives

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-4} \to 0 \quad \text{as } R \to \infty.$$

(b) The function $f(z)$ has simple poles at $\pm 2i$ but only $z_1 = 2i$ is in the upper half plane. z_1 is inside Γ_R when $R > 2$. By the residue theorem the value of the loop integral is

$$2\pi i \mathrm{Res}(f,\, z_1).$$

$$\mathrm{Res}(f,\, z_1) = \lim_{z\to z_1} \frac{(z-z_1)\mathrm{e}^{iaz}}{4+z^2} = \lim_{z\to z_1} \mathrm{e}^{iaz} \lim_{z\to z_1} \frac{(z-z_1)}{4+z^2} = \frac{\mathrm{e}^{-2a}}{2z_1} = \frac{\mathrm{e}^{-2a}}{4i}.$$

Thus the value is

$$\frac{\pi \mathrm{e}^{-2a}}{2}.$$

(c) As the loop is the union of 2 parts we have, when $R > 2$,

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = \frac{\pi \mathrm{e}^{-2a}}{2}.$$

Letting $R \to \infty$ and using the previous part gives

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi \mathrm{e}^{-2a}}{2}.$$

9. *The following was part of question 4 in the May 2019 MA3614 exam paper. This part of the question was worth 12 marks.*

Let

$$f(z) = \frac{1-\mathrm{e}^{iz}}{z^2(z^2+1)},$$

and for any $\rho > 0$ let $C_\rho^+ = \left\{\rho \mathrm{e}^{i\theta} : 0 \leq \theta \leq \pi\right\}$ denote an upper half circle. When contour integrals are considered on such half circles, the direction of integration corresponds to increasing θ. The notation $-C_\rho$ means the same path but in the opposite direction. For this function, it can be shown that

$$\lim_{r\to 0} \int_{C_r^+} f(z)\,\mathrm{d}z = \pi.$$

(a) State all of the poles of $f(z)$ and determine the residue at each pole.

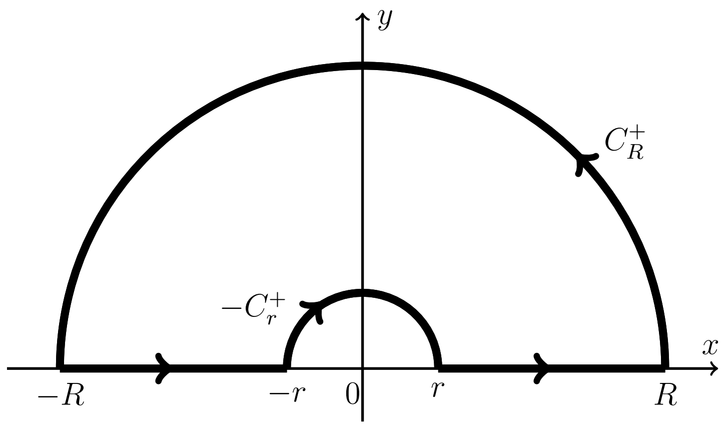
(b) Explain why

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z = 0.$$

(c) For $0 < r < R$, let Γ_R^r denote the closed loop

$$\Gamma_R^r = [r, R] \cup C_R^+ \cup [-R, -r] \cup (-C_r^+)$$

illustrated below.



When $r < 1 < R$ determine

$$\oint_{\Gamma_R^r} f(z)\,\mathrm{d}z.$$

(d) By using the previous results, or otherwise, determine

$$\int_0^\infty \frac{1-\cos(x)}{x^2(x^2+1)}\,\mathrm{d}x.$$

Solution

(a) $f(z)$ has simple poles at the points 0 and $\pm i$.

$$\begin{aligned}
\mathrm{Res}(f,\,0) &= \lim_{z\to 0} z f(z) = \left(\left.\frac{1}{z^2+1}\right|_{z=0} \right) \lim_{z\to 0} \frac{1-\mathrm{e}^{iz}}{z} = -i,\\
\mathrm{Res}(f,\,i) &= \lim_{z\to i} (z-i) f(z) = \left(\left.\left(\frac{1-\mathrm{e}^{iz}}{z^2} \right)\right|_{z=i} \right) \left(\lim_{z\to i} \frac{z-i}{z^2+1} \right)\\
&= (\mathrm{e}^{-1}-1)\frac{1}{2i} = \left(\frac{1-\mathrm{e}^{-1}}{2} \right) i,\\
\mathrm{Res}(f,\,-i) &= \lim_{z\to -i} (z+i) f(z) = \left(\left.\left(\frac{1-\mathrm{e}^{iz}}{z^2} \right)\right|_{z=-i} \right) \left(\lim_{z\to -i} \frac{z+i}{z^2+1} \right)\\
&= (\mathrm{e}-1)\frac{1}{-2i} = \left(\frac{\mathrm{e}-1}{2} \right) i
\end{aligned}$$

(b) The length of C_R^+ is πR. When $z = x + iy$ with $y \ge 0$, $iz = -y + ix$ and $|e^{iz}| = e^{-y} \le 1$.
With $|z| = R > 1$

$$|1 - e^{iz}| \le 2 \quad \text{and} \quad |z^2(z^2+1)| \ge R^2(R^2-1).$$

Thus on the half circle

$$|f(z)| \le \frac{2}{R^2(R^2-1)} =: M$$

and by the ML inequality

$$\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| \le \frac{2\pi R}{R^2(R^2-1)} = \frac{2\pi}{R(R^2-1)} \to 0 \quad \text{as } R \to \infty.$$

(c) The only pole inside the loop is at $z = i$ and hence by the residue theorem

$$\oint_{\Gamma_R^r} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}(f,\, i) = -\pi \left(1 - e^{-1}\right).$$

(d) When $x \in \mathbb{R}$ the real part of $f(x)$ is

$$\frac{1-\cos(x)}{x^2(x^2+1)}.$$

As the loop is the union of 4 parts we have

$$-\pi\left(1 - e^{-1}\right) = \left(\int_{-R}^{-r} + \int_r^R\right) f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z - \int_{C_r^+} f(z)\,\mathrm{d}z.$$

Letting $R \to \infty$ and $r \to 0$ and using previous results we have

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi e^{-1}.$$

As $f(x)$ is even it follows that

$$\int_0^{\infty} f(x)\,\mathrm{d}x = \frac{\pi e^{-1}}{2}.$$

---

10. By using the same contour Γ_R^r as in question 9 show that

$$\int_0^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,\mathrm{d}x = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right).$$

Solution

If we let

$$f(z)=\frac{\mathrm{e}^{2iz}}{z(z^2+1)^2}$$

then our integrand is given by

$$\frac{\sin(2x)}{x(x^2+1)^2}=\mathrm{Im}\frac{\mathrm{e}^{2ix}}{x(x^2+1)^2}=\mathrm{Im}f(x).$$

$\mathrm{Im}f(x)$ is even in x and thus

$$\int_0^\infty \frac{\sin(2x)}{x(x^2+1)^2}\,\mathrm{d}x=\frac{1}{2}\int_{-\infty}^\infty \frac{\sin(2x)}{x(x^2+1)^2}\,\mathrm{d}x.$$

$f(z)$ has a simple pole at $z=z_0=0$ and double poles at $\pm i$ but we only need to consider the pole at $z_1=i$ which is in the upper half plane.

Let Γ_R^r denote the indented loop. When we take $0<r<1<R$ the function $f(z)$ only has one pole inside the this loop and thus by the residue theorem

$$\oint_{\Gamma_R^r} f(z)\,\mathrm{d}z=2\pi i\mathrm{Res}(f,\, z_1).$$

As Γ_R^r is the union of two half circles and part of the real line we also have

$$\oint_{\Gamma_R^r} f(z)\,\mathrm{d}z=\left(\int_{-R}^{-r}+\int_r^R\right)f(x)\,\mathrm{d}x-\int_{C_r^+}f(z)\,\mathrm{d}z+\int_{C_R^+}f(z)\,\mathrm{d}z.$$

From the result in question 2 we have

$$\lim_{r\to 0}\int_{C_r^+}f(z)\,\mathrm{d}z=\pi i\mathrm{Res}(f,\, z_0).$$

Hence if we can show that

$$\lim_{R\to\infty}\int_{C_R^+}f(z)\,\mathrm{d}z=0$$

then

$$2\pi i\mathrm{Res}(f,\, z_1)=\int_{-\infty}^\infty f(x)\,\mathrm{d}x-\pi i\mathrm{Res}(f,\, z_0)$$

so that our result is

$$\mathrm{Im}\int_0^\infty f(x)\,\mathrm{d}x=\frac{1}{2}\mathrm{Im}\left(2\pi i\mathrm{Res}(f,\, z_1)+\pi i\mathrm{Res}(f,\, z_0)\right).$$

We first explain why the integral on C_R^+ tends to 0 as $R\to\infty$. Let $z=x+iy\in C_R^+$ and thus $|z|=R$ and $y\ge 0$. This implies that $2iz=-2y+2ix$ and $|\mathrm{e}^{2iz}|=\mathrm{e}^{-2y}\le 1$. Thus

$$|f(z)|\le\frac{1}{R(R^2-1)^2}$$

and as the length of C_R^+ is πR the ML inequality gives

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \le \frac{\pi R}{R(R^2-1)^2} = \frac{\pi}{(R^2-1)^2} \to 0 \quad \text{as } R\to\infty.$$

For the residue at $z_0 = 0$ we have

$$\mathrm{Res}(f,\,0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{\mathrm{e}^{2iz}}{(z^2+1)^2} = 1.$$

For the double pole at $z_i = i$ we first note that $z^2+1 = (z+i)(z-i)$ so that

$$(z-i)^2 f(z) = \frac{\mathrm{e}^{2iz}}{z}\frac{1}{(z+i)^2} = c_{-2} + c_{-1}(z-i) + \cdots$$

and

$$\mathrm{Res}(f,\,i) = c_{-1} = \left((z-i)^2 f(z)\right)'\Big|_{z=i}.$$

Now for the derivative

$$\left((z-i)^2 f(z)\right)' = \left(\frac{\mathrm{e}^{2iz}}{z}\right)\left(\frac{-2}{(z+i)^3}\right) + \left(\frac{\mathrm{e}^{2iz}}{z}\right)'\left(\frac{1}{(z+i)^2}\right).$$

with

$$\left(\frac{\mathrm{e}^{2iz}}{z}\right)' = \frac{z(2i\mathrm{e}^{2iz}) - \mathrm{e}^{2iz}}{z^2}.$$

We just need the value at i and this is given by

$$\mathrm{Res}(f,\,i) = \left(\frac{\mathrm{e}^{-2}}{i}\right)\left(\frac{-2}{(2i)^3}\right) + \left(\frac{-2\mathrm{e}^{-2}-\mathrm{e}^{-2}}{-1}\right)\left(\frac{1}{(2i)^2}\right) = \frac{\mathrm{e}^{-2}}{4}(-1-3) = -\mathrm{e}^{-2}.$$

Both residues are real and thus

$$\mathrm{Im}\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\left(2\pi\mathrm{Res}(f,\,z_1) + \pi\mathrm{Res}(f,\,z_0)\right) = \pi\left(-\mathrm{e}^{-2} + \frac{1}{2}\right)$$

as required.

---

11. Evaluate the following integral.

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a > 0.$$

Solution

Let

$$f(z) = \frac{1}{(z^2+a^2)^2}.$$

This function is even and

$$I=\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}.$$

Let $\Gamma_R=[-R,R]\cup C_R^+$ denote the closed contour with C_R^+ denoting the upper half circle with centre at 0 and radius R. The length of C_R^+ in πR and for $z\in C_R^+$ we have

$$|f(z)|\le \frac{1}{(R^2-a^2)^2}$$

which gives

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right|\le \frac{\pi R}{(R^2-a^2)^2}\to 0 \quad \text{as } R\to\infty.$$

As $z^2+a^2=(z+ai)(z-ai)$ we have

$$\frac{1}{(z^2+a^2)^2}=\frac{1}{(z+ai)^2(z-ai)^2}.$$

This has double poles at $\pm ai$ with $z_1=ai$ being in the upper half plane.

By considering the closed loop and the residue theorem we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z=\int_{-R}^R f(x)\,\mathrm{d}x+\int_{C_R^+} f(z)\,\mathrm{d}z=2\pi i\mathrm{Res}(f,\,z_1).$$

To get the residue note that

$$(z-z_1)^2 f(z)=\frac{1}{(z+ai)^2}=a_{-2}+a_{-1}(z-z_1)+\cdots$$

which gives

$$a_{-1}=\left(\frac{1}{(z+ai)^2}\right)'\Bigg|_{z=ai}=\frac{-2}{(z+ai)^3}\Bigg|_{z=ai}=\frac{-2}{(2ai)^3}=\frac{1}{4a^3 i}.$$

Thus

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x=2\pi i\left(\frac{1}{4a^3 i}\right)=\frac{\pi}{2a^3}$$

and

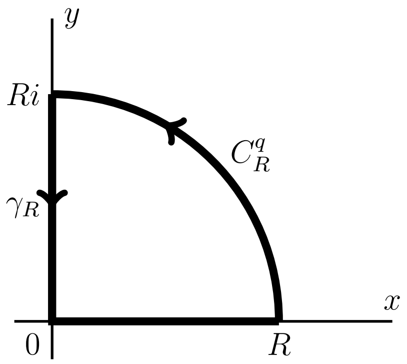
$$\int_0^\infty f(x)\,\mathrm{d}x=\frac{\pi}{4a^3}.$$

---

12. Let a function $f(z)$ and a quarter circle C_R^q of radius $R>2$ be given by

$$f(z)=\frac{1}{z^4+16},\quad \text{and} \quad C_R^q=\left\{R\mathrm{e}^{it}:\ 0\le t\le \pi/2\right\}.$$

Also let Γ_R denote the closed loop composed of the real interval $[0,R]$ followed by the quarter circle C_R^q and followed by the segment γ_R of the imaginary axis from Ri to 0 as illustrated illustrated in the diagram.



(a) Explain why

$$\lim_{R\to\infty} \int_{C_R^q} f(z)\,\mathrm{d}z = 0.$$

(b) Determine

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

(c) Explain why

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = -i\int_0^R f(x)\,\mathrm{d}x.$$

(d) Using your previous results, or otherwise, to evaluate the real integral

$$\int_0^\infty \frac{1}{x^4+16}\,\mathrm{d}x.$$

Solution

(a)

$$f(z) = \frac{1}{z^4+16}.$$

When $|z| = R$ and R is large the magnitude of the denominator is bounded below by

$$R^4 - 16$$

and hence

$$|f(z)| \le \frac{1}{R^4-16}.$$

The length of the quarter circle is $\pi R/2$. By the ML inequality we have

$$\left|\int_{C_R^q} f(z)\,\mathrm{d}z\right| \le \frac{\pi R/2}{R^4-16} \to 0 \quad \text{as } R\to\infty.$$

(b) $f(z)$ has simple poles when $z^4 = -16 = -2^4$ and thus $f(z)$ has 4 simple poles on the circle $|z| = 2$. There is one simple pole inside the quarter circle at

$$z_1 = 2\mathrm{e}^{i\pi/4} = \sqrt{2}(1+i).$$

Let I denote the loop integral. By the residue theorem

$$I = 2\pi i \mathrm{Res}(f,\, z_1).$$

$$\begin{aligned}\mathrm{Res}(f,\, z_1) &= \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}\frac{z-z_1}{z^4+4} \\ &= \frac{1}{4z_1^3} = \frac{z_1}{4z_1^4} = -\frac{z_1}{64}.\end{aligned}$$

Hence

$$I = -2\pi i\frac{z_1}{64} = \frac{\sqrt{2}\pi}{64}(-i)(1+i) = \frac{\sqrt{2}\pi}{32}(1-i).$$

(c) We consider first the integral on $-\gamma_R = \{z(t) = it :\; 0 \le t \le R\}$.

$$z(t) = it, \quad z'(t) = i, \quad f(z(t)) = \frac{1}{1+(it)^4} = \frac{1}{1+t^4}.$$

Thus by the definition of the integral on $-\gamma_R$ we have

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = -\int_{-\gamma_R} f(z)\,\mathrm{d}z = -\int_{-\gamma_R} f(z(t))z'(t)\,\mathrm{d}t = -\int_0^R \frac{1}{1+t^4}\,\mathrm{d}t.$$

(d) As Γ_R is the union of 3 parts we have

$$\begin{aligned}I = \oint_{\Gamma_R} f(z)\,\mathrm{d}z &= \int_0^R f(x)\,\mathrm{d}x + \int_{C_R^q} f(z)\,\mathrm{d}z + \int_{\gamma_R} f(z)\,\mathrm{d}z \\ &= (1-i)\int_0^R f(x) + \int_{C_R^q} f(z)\,\mathrm{d}z.\end{aligned}$$

Letting $R \to \infty$ and using the result of part (i) we have

$$\lim_{R\to\infty}\oint_{\Gamma_R} f(z)\,\mathrm{d}z = (1-i)\int_0^\infty f(x)\,\mathrm{d}x.$$

Thus

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{\sqrt{2}\pi}{32}.$$

---

13. Let $f(z)$ be a function which is analytic except for a finite number of isolated singularities and let
$$g(z) = \pi \cot(\pi z)\, f(z).$$

(a) Show that if $f(z)$ does not have an isolated singularity at the integer n then
$$\text{Res}(g,\, n) = f(n).$$

(b) In the case $f(z) = 1/z^2$ show that
$$\text{Res}(g,\, 0) = -\frac{\pi^2}{3}.$$

(c) Let Γ_N be the square with vertices at $(N + 0.5)(\pm 1 \pm i)$. It can be shown that there is a constant $A > 0$ independent of N such that $|\pi \cot(\pi z)| \le A$ for all $z \in \Gamma_N$. In the case that $f(z) = 1/z^2$ show that
$$\int_{\Gamma_N} g(z)\,\mathrm{d}z \to 0 \quad \text{as } N \to \infty.$$
By using this result show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution

(a) $\cot(\pi z)$ has simple zeros at the integers and if $f(z)$ is analytic and non-zero at an an integer n then $g(z) = \pi \cot(\pi z)\, f(z)$ has a simple pole at $z = n$.
$$\begin{aligned}\text{Res}(g,\, n) &= \lim_{z\to n}(z-n)g(z) = \lim_{z\to n}(z-n)\pi\frac{\cos(\pi z)}{\sin(\pi z)}f(z)\\ &= \pi\cos(n\pi)f(n)\lim_{z\to n}\frac{z-n}{\sin(\pi z)} = \pi\cos(n\pi)f(n)\frac{1}{\cos(n\pi)} = f(n).\end{aligned}$$

(b) As $f(z) = 1/z^2$ has a double pole at $z = 0$ and $\cos(\pi z)$ has a simple pole at $z = 0$ the $g(z)$ has a pole of order 3 at $z = 0$, We can get the residue at $z = 0$ by considering the series. Now as $\sin w$ only involves odd powers and $\cos w$ only involves even powers the Laurent series for $\cot w$ only involves odd powers and with
$$\cot w = \frac{a_{-1}}{w} + a_1 w + \cdots$$
the relation $\cot w \sin w = \cos w$ gives
$$\left(\frac{a_{-1}}{w} + a_1 w + \cdots\right)\left(w - \frac{w^3}{6} + \cdots\right) = 1 - \frac{w^2}{2} + \cdots$$

Equating the constant terms gives

$$a_{-1} = 1.$$

Equating the w^2 terms gives

$$a_1 - \frac{a_{-1}}{6} = -\frac{1}{2}, \quad a_1 = -\frac{1}{2} + \frac{a_{-1}}{6} = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}.$$

Thus

$$\cot(\pi z) = \frac{1}{\pi z} - \frac{\pi z}{3} + \cdots$$

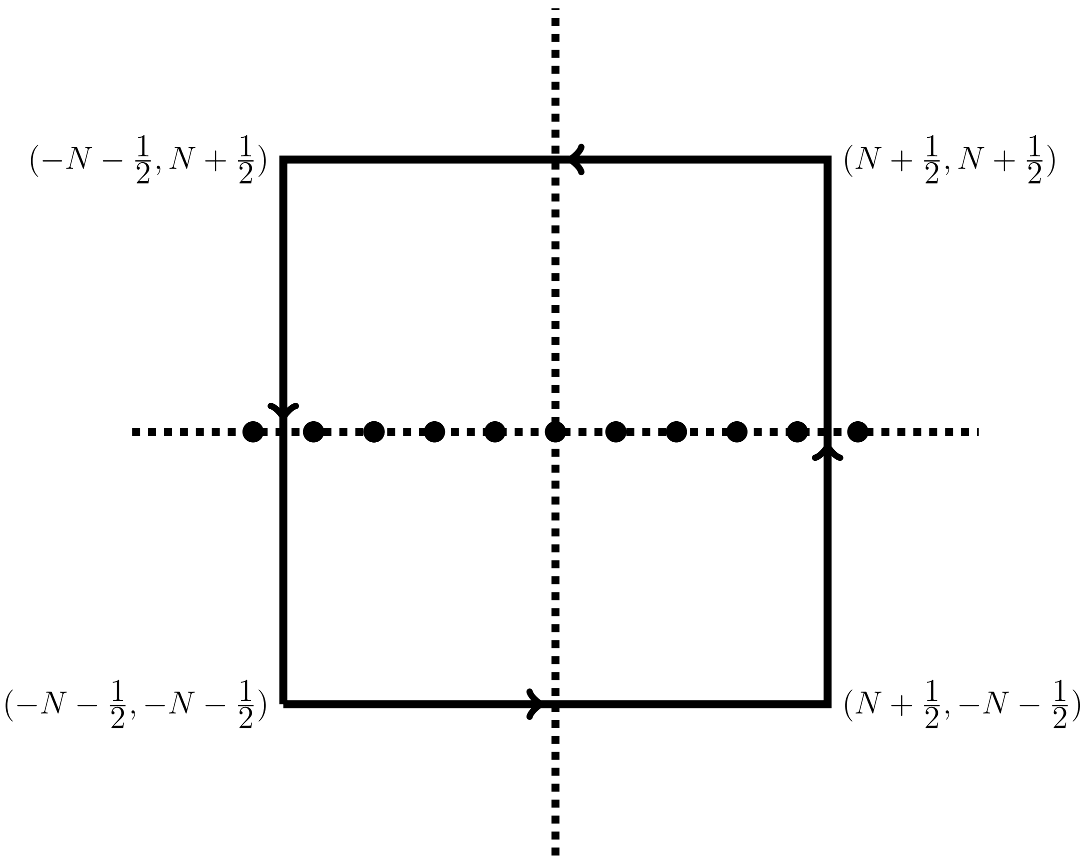
and

$$g(z) = \pi \left(\frac{1}{\pi z^3} - \frac{\pi}{3z} + \cdots \right)$$

and hence

$$\mathrm{Res}(g, 0) = -\frac{\pi^2}{3}.$$

(c) The closed contour Γ_N is shown below and is such that it crosses the real line at points where $g(z)$ is zero.



We are given that $\pi \cot(\pi z)$ is bounded on Γ_N and thus

$$|g(z)| \le A|f(z)| \le \frac{A}{N^2}.$$

Each of the 4 sides on Γ_N has length $2N+1$ and thus

$$\left| \oint_{\Gamma_N} g(z)\,\mathrm{d}z \right| \le 4(2N+1)\frac{A}{N^2} \to 0 \quad \text{as } N \to \infty.$$

Inside the closed loop Γ_N there are singularities at $z = 0$ and $\pm 1, \pm 2, \ldots, \pm N$. By the residue theorem

$$\oint_{\Gamma_N} g(z)\,\mathrm{d}z = 2\pi i \left(\mathrm{Res}(g,\, 0) + \sum_{k=1}^{N} (\mathrm{Res}(g,\, -k) + \mathrm{Res}(g,\, k)) \right).$$

As $f(z)$ is even

$$\mathrm{Res}(g,\, -k) = \mathrm{Res}(g,\, k) = f(k) = \frac{1}{k^2}, \quad k \ge 1.$$

Letting $N \to \infty$ and using the result that the integral around Γ_N tends to 0 gives

$$2\sum_{k=1}^{\infty} \frac{1}{k^2} - \mathrm{Res}(g,\, 0) = 2\sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{\pi^2}{3} = 0$$

and the result follows.

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14. (a) Let x and y be real. Determine the following limits.

$$\lim_{y\to\infty} \tan(x+iy) \quad \text{and} \quad \lim_{y\to\infty} \tan(x-iy).$$

(b) Let Γ_L denotes the straight line segment from $\pi + iL$ to iL where $L > 0$. Determine

$$\lim_{L\to\infty} \int_{\Gamma_L} \tan z\,\mathrm{d}z.$$

(c) By considering a closed loop in the anti-clockwise direction which is the rectangle with vertices 0, π, $\pi + iL$ and iL show that when $a \in \mathbb{R}$ and $a \ne 0$ we have

$$\int_0^{\pi} \tan(\theta + ia)\,\mathrm{d}\theta = \begin{cases} \pi i, & \text{when } a > 0, \\ -\pi i, & \text{when } a < 0. \end{cases}$$

Solution

(a)

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$$
$$= \frac{1}{i}\left(\frac{e^{2iz} - 1}{e^{2iz} + 1}\right) = \frac{1}{i}\left(\frac{1 - e^{-2iz}}{1 + e^{-2iz}}\right).$$

With $z = x + iy$, $iz = -y + ix$ and $|e^{iz}| = e^{-y}$, $|e^{-iz}| = e^{y}$.
As $y \to \infty$ we have $|e^{iz}| \to 0$ and hence $\tan(x + iy) \to i$.
As $y \to -\infty$ we have $|e^{-iz}| \to 0$. and hence $\tan(x + iy) \to -i$.

(b) Let

$$J_L = \int_{\Gamma_L} \tan\, z \mathrm{d}z = \int_{\pi}^{0} \tan(x + iL)\,\mathrm{d}x = -\int_{0}^{\pi} \tan(x + iL)\,\mathrm{d}x.$$

$$J_L + i\pi = \int_0^{\pi} (i - \tan(x + iL))\,\mathrm{d}x \to 0 \quad \text{as } L \to \infty.$$

Thus $J_L \to -i\pi$ as $L \to \infty$.

(c) The function $f(z) = \tan(z + ia)$ has simple poles at points such that

$$z + ia = \frac{\pi}{2} + m\pi, \quad m \in \mathbb{Z}.$$

Let R_L denote the rectangular loop. When $a > 0$ these points are in the lower half plane and hence are not inside the loop R_L and in this case

$$\int_{R_L} \tan(z + ia)\,\mathrm{d}z = 0.$$

When $a < 0$ there is one simple pole at $z_1 = \pi/2 - ia$ inside the loop and

$$\int_{R_L} \tan(z + ia)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f,\, z_1).$$

For the residue

$$\lim_{z\to z_1} (z - z_1)\frac{\sin(z + ia)}{\cos(z + ia)} = \sin(z + ia) \lim_{z\to z_1} \frac{z - z_1}{\cos(z + ia)} = -1.$$

Thus when $a < 0$

$$\int_{R_L} \tan(z + ia)\,\mathrm{d}z = -2\pi i.$$

The rectangle has 4 sides and the sides parallel to the imaginary axis the periodic property of $\tan(z)$ implies that

$$f(iy) = f(\pi + iy).$$

The integral on the part from π to $\pi + iL$ is in the opposite direction to the integral from iL to 0 and thus the contribution to the loop integral from these two sides is 0. Thus by considering the other two sides gives

$$\int_0^\pi \tan(\theta + ia)\,\mathrm{d}\theta + J_L = \begin{cases} 0 & \text{if } a > 0, \\ -2\pi i & \text{if } a < 0. \end{cases}$$

By letting $L \to \infty$ and using the result of part (b) that $J_L \to -\pi i$ gives the result.
