Exercises involving the use of residue theory

Question 1 is a trig. integral and similar to what was asked in the exercises associated with chapter 5. The past exam questions in questions 5 and 6 just involve rational functions of z and be tackled as a result of what is taught in the first week of the material on chapter 8, i.e. from what is taught in week 23. Question 12 is a slight variation of something in the lecture notes with the difference here that a quarter of a circle is used instead of a half circle. Question 11 also just involves a rational function but has the additional difficulty in that the residue at a double pole must be obtained.

The past exam questions in questions 7, 8, and 9 all have an integrand which contains an exp(.) term and the material on this should be taught in week 24. Questions 2, 3, 4 and 10 all involve indented contours and the material on this should be taught in week 24. In the case of question 10 there is the additional difficulty of a double pole as well.

Questions 14 and 13 involve loops which are respectively a rectangle and a square. These can be considered at any time although they may be considered as among the more difficult questions.

1. Show the following by first using the substitution $z = \mathrm{e}^{i\theta}$.

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$$

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2. Suppose that $f(z)$ is analytic in an annulus $\{z : 0 < |z - x_0| < r\}$ and has a simple pole at $x_0 \in \mathbb{R}$. Let $0 < \epsilon < r$ and let $C_\epsilon^+ = \left\{x_0 + \epsilon \mathrm{e}^{i\theta} : 0 \le \theta \le \pi\right\}$ denote a half circle with centre at x_0 and radius ϵ. If the half circle is traversed once in the anti-clockwise direction then show that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon^+} f(z)\,\mathrm{d}z = \pi i \mathrm{Res}(f,\, x_0).$$

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3. Show the following.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(3x)}{x-1} \mathrm{d}x = -\pi\sin(3) \quad \text{and} \quad \text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(3x)}{x-1} \mathrm{d}x = \pi\cos(3).$$

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4. Verify that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

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5. *The following was part of question 4 in the May 2023 MA3614 exam paper. This part of the question was worth 9 marks of the 20 marks in the entire question.*

   Let a, b and c be real numbers with $a > 0$ and let

$$f(z) = \frac{1}{az^2 + bz + c}.$$

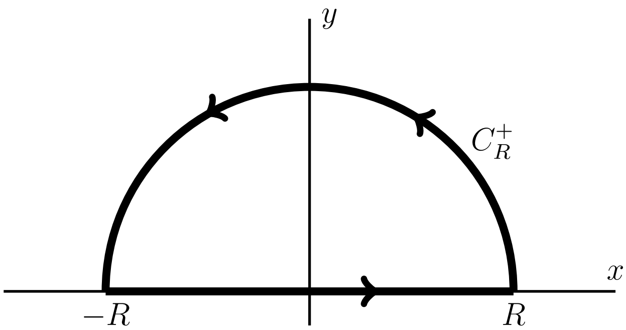
   (a) When $b^2 \neq 4ac$ indicate all the poles of $f(z)$ and determine the residue at each pole. Similarly, in the case $b^2 = 4ac$ indicate all the poles of $f(z)$ and determine the residue at each pole.

   (b) Let C_R denote the circle with centre 0 and radius $R > 0$ traversed once in the anti-clockwise sense. By any means explain why

$$\oint_{C_R} f(z)\,\mathrm{d}z = 0$$

   when R is sufficiently large.

   (c) Let C_R^+ denote the half circle with centre at 0 and radius $R > 0$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



   Use the ML inequality to explain why

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z = 0.$$

   Further, in the case $4ac > b^2$ use the loop Γ_R to determine an expression in terms of a, b and c of the value

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

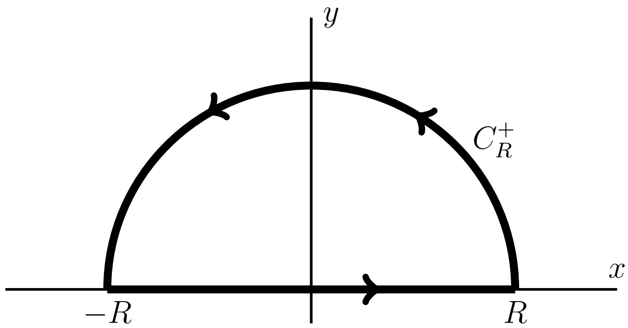
   You need to explain all your steps.

6. *The following was part of question 4 in the May 2022 MA3614 exam paper. This part of the question was worth 10 marks.*

Let

$$f(z) = \frac{1}{1 + z^2 + z^4}.$$

Let C_R^+ denote the half circle with centre at 0 and radius $R > 1$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



(a) The function $f(z)$ has simple poles at the points $\pm z_1$ and $\pm z_2$ where $z_1 = \mathrm{e}^{i\pi/3}$ and $z_2 = \mathrm{e}^{i2\pi/3}$. Indicate which two points are in the upper half plane, give the cartesian form of these points and give workings to confirm that $1 + z_1^2 + z_1^4 = 0$.

(b) Determine the residue at each of the two simple poles in the upper half plane and determine

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

(c) Determine, giving reasons, the value of

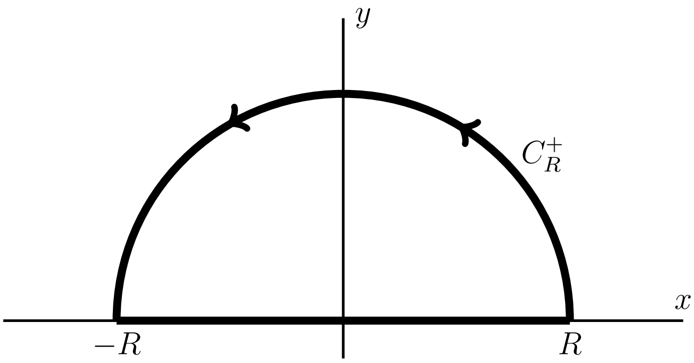
$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z.$$

(d) By using the loop Γ_R, determine

$$\int_0^\infty f(x)\,\mathrm{d}x.$$

7. *The following was part of question 4 in the May 2021 MA3614 exam paper. This part of the question was worth 10 marks.*

Let C_R^+ denote the half circle with centre at 0 and radius $R > 0$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



In the following which function you consider depends on the 4th digit of your 7-digit student id.. If your 4th digit is one of 0, 2, 4, 6, 8 then your function $f(z)$ is on the left hand side whilst if it is one of the digits 1, 3, 5, 7, 9 then your function $f(z)$ is on the right hand side.

$$f(z) = \frac{4 + \mathrm{e}^{3iz}}{1 + 2z^2} \quad \text{(even digit case)} \qquad \text{or} \qquad f(z) = \frac{2 - \mathrm{e}^{5iz}}{1 + 3z^2} \quad \text{(odd digit case).}$$

(a) Give all the poles of your version of the function $f(z)$ in the complex plane and determine the residue at each pole in the upper half plane.

(b) For your version of $f(z)$, determine, giving reasons, the value of

$$\lim_{R \to \infty} \int_{C_R^+} f(z)\,\mathrm{d}z.$$

(c) For your version of $f(z)$, determine, giving reasons, the value of the integrals

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{Re}(f(x))\,\mathrm{d}x.$$

Here $\mathrm{Re}(f(x))$ means the real part of $f(x)$.

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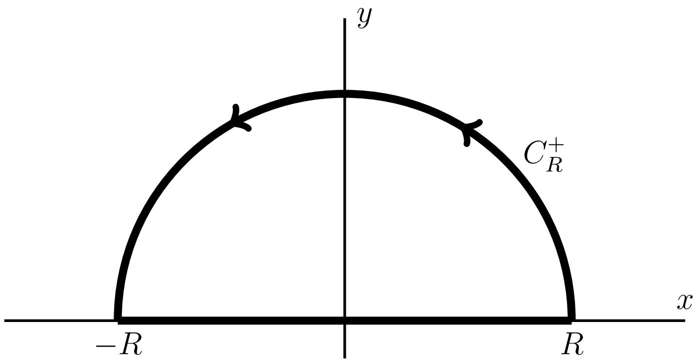
8. *The following was part of question 4 in the May 2020 MA3614 exam paper. This part of the question was worth 9 marks.*

Let C_R^+ denote the half circle with centre at 0 and radius $R > 1$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



Also let $a > 0$ and let

$$f(z) = \frac{\mathrm{e}^{iaz}}{4 + z^2}.$$

(a) Show that

$$\int_{C_R^+} f(z)\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

(b) When $R > 2$ determine, giving reasons,

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

(c) By giving appropriate reasoning, determine

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

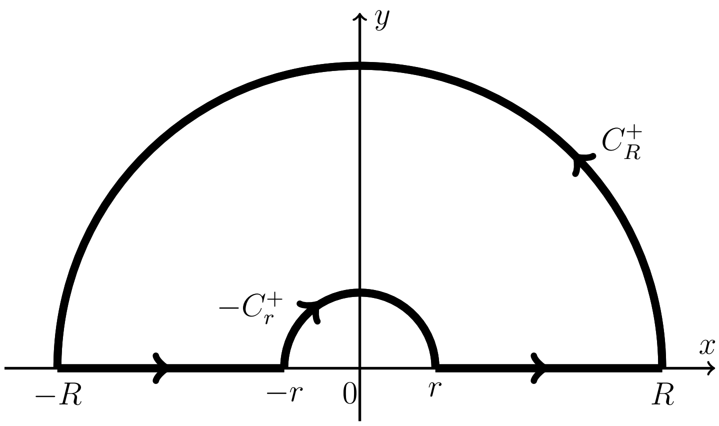
9. *The following was part of question 4 in the May 2019 MA3614 exam paper. This part of the question was worth 12 marks.*

   Let
   $$f(z) = \frac{1 - e^{iz}}{z^2(z^2 + 1)},$$
   and for any $\rho > 0$ let $C_\rho^+ = \left\{\rho e^{i\theta} : 0 \le \theta \le \pi\right\}$ denote an upper half circle. When contour integrals are considered on such half circles, the direction of integration corresponds to increasing θ. The notation $-C_\rho$ means the same path but in the opposite direction. For this function, it can be shown that
   $$\lim_{r \to 0} \int_{C_r^+} f(z)\, dz = \pi.$$

   (a) State all of the poles of $f(z)$ and determine the residue at each pole.

   (b) Explain why
   $$\lim_{R \to \infty} \int_{C_R^+} f(z)\, dz = 0.$$

   (c) For $0 < r < R$, let Γ_R^r denote the closed loop
   $$\Gamma_R^r = [r, R] \cup C_R^+ \cup [-R, -r] \cup (-C_r^+)$$
   illustrated below.



   When $r < 1 < R$ determine
   $$\oint_{\Gamma_R^r} f(z)\, dz.$$

   (d) By using the previous results, or otherwise, determine
   $$\int_0^\infty \frac{1 - \cos(x)}{x^2(x^2 + 1)}\, dx.$$

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10. By using the same contour Γ_R^r as in question 9 show that
$$\int_0^\infty \frac{\sin(2x)}{x(x^2 + 1)^2}\, dx = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right).$$

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11. Evaluate the following integral.

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, \quad a>0.$$

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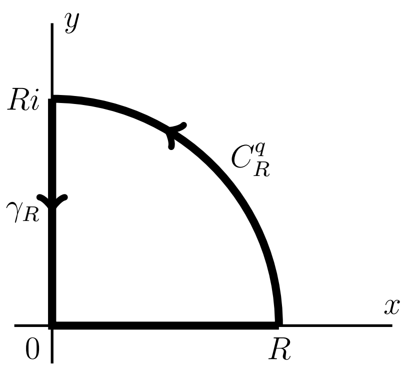
12. Let a function $f(z)$ and a quarter circle C_R^q of radius $R>2$ be given by

$$f(z)=\frac{1}{z^4+16}, \quad \text{and} \quad C_R^q=\left\{R\mathrm{e}^{it}:\ 0\le t\le \pi/2\right\}.$$

Also let Γ_R denote the closed loop composed of the real interval $[0,R]$ followed by the quarter circle C_R^q and followed by the segment γ_R of the imaginary axis from Ri to 0 as illustrated illustrated in the diagram.



(a) Explain why

$$\lim_{R\to\infty}\int_{C_R^q} f(z)\,\mathrm{d}z=0.$$

(b) Determine

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z.$$

(c) Explain why

$$\int_{\gamma_R} f(z)\,\mathrm{d}z=-i\int_0^R f(x)\,\mathrm{d}x.$$

(d) Using your previous results, or otherwise, to evaluate the real integral

$$\int_0^\infty \frac{1}{x^4+16}\,\mathrm{d}x.$$

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13. Let $f(z)$ be a function which is analytic except for a finite number of isolated singularities and let
$$g(z) = \pi\, \cot(\pi z)\, f(z).$$

(a) Show that if $f(z)$ does not have an isolated singularity at the integer n then
$$\mathrm{Res}(g,\, n) = f(n).$$

(b) In the case $f(z) = 1/z^2$ show that
$$\mathrm{Res}(g,\, 0) = -\frac{\pi^2}{3}.$$

(c) Let Γ_N be the square with vertices at $(N + 0.5)(\pm 1 \pm i)$. It can be shown that there is a constant $A > 0$ independent of N such that $|\pi\, \cot(\pi z)| \le A$ for all $z \in \Gamma_N$. In the case that $f(z) = 1/z^2$ show that
$$\int_{\Gamma_N} g(z)\, \mathrm{d}z \to 0 \quad \text{as } N \to \infty.$$

By using this result show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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14. (a) Let x and y be real. Determine the following limits.
$$\lim_{y\to\infty} \tan(x + iy) \quad \text{and} \quad \lim_{y\to\infty} \tan(x - iy).$$

(b) Let Γ_L denotes the straight line segment from $\pi + iL$ to iL where $L > 0$. Determine
$$\lim_{L\to\infty} \int_{\Gamma_L} \tan z\, \mathrm{d}z.$$

(c) By considering a closed loop in the anti-clockwise direction which is the rectangle with vertices 0, π, $\pi + iL$ and iL show that when $a \in \mathbb{R}$ and $a \neq 0$ we have
$$\int_0^{\pi} \tan(\theta + ia)\, \mathrm{d}\theta = \begin{cases} \pi i, & \text{when } a > 0, \\ -\pi i, & \text{when } a < 0. \end{cases}$$
