Exercises as part of the revision for the May exams

1. *Parts of this question are taken from the paper in May 2017 and May 2018.*

   (a) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

   i.
   $$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 + i2xy.$$

   ii.
   $$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = (2x^3 + 3x^2y - 6xy^2 - y^3) + i(-x^3 + 6x^2y + 3xy^2 - 2y^3).$$

   iii.
   $$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \mathrm{e}^{-x}(\cos y + i \sin y).$$

   iv.
   $$f_4 : \mathbb{C} \to \mathbb{C}, \quad f_4(z) = \sinh x \cos y + i \cosh x \sin y.$$

   v.
   $$f_5 : \mathbb{C} \to \mathbb{C}, \quad f_5(z) = \frac{\partial^2 \phi}{\partial x^2} - i \frac{\partial^2 \phi}{\partial x \partial y}$$

   where ϕ is a harmonic function with continuous partial derivatives of all orders.

   (b) Show that the function
   $$u(x, y) = x^3y - xy^3$$
   is harmonic and determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 2$. Express $u + iv$ in terms of z only.

   (c) Let $D = \{z : |z| < 1\}$ and let $f(z)$ be a function which is analytic in D. Also let $g_1(z)$ and $g_2(z)$ be functions defined in D by
   $$g_1(z) = f(\overline{z}), \quad g_2(z) = \overline{g_1(z)}.$$

   i. Let $z_0 \in D$. Explain why the following limit exists and give the limit in terms of f and/or its derivatives.
   $$\lim_{h\to 0} \frac{g_1(z_0 + h) - g_1(z_0)}{\overline{h}}.$$

   ii. Explain why $g_2(z)$ is analytic in D.

   iii. If the Maclaurin series representation of $f(z)$ is given by
   $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
   then give the Maclaurin series for $g_2(z)$.

Solution

(a) i. Let $u = x^2$ and $v = 2xy$.

$$\frac{\partial u}{\partial y} = 0 \quad \text{but} \quad -\frac{\partial v}{\partial x} = -2y.$$

These are the same at $y = 0$ but not in a neighbourhood of $y = 0$ and thus f_1 is not analytic.

ii. Let $u = 2x^3 + 3x^2y - 6xy^2 - y^3$ and $v = -x^3 + 6x^2y + 3xy^2 - 2y^3$.

$$\frac{\partial u}{\partial x} = 6x^2 + 6xy - 6y^2 = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = 3x^2 - 12xy - 3y^2, \quad \frac{\partial v}{\partial x} = -3x^2 + 12xy + 3y^2.$$

Both Cauchy Riemann equations are satisfied and hence f_2 is analytic.

iii. Let $u = \mathrm{e}^{-x}\cos y$ and $v = \mathrm{e}^{-x}\sin y$.

$$\frac{\partial u}{\partial x} = -u \quad \text{and} \quad \frac{\partial v}{\partial y} = u.$$

These are only the same when $\cos y = 0$ but not in a neighbourhood of any of these values of y and thus f_3 is not analytic.

iv. Let $u = \sinh x \cos y$ and $v = \cosh x \sin y$.

$$\frac{\partial u}{\partial x} = \cosh x \cos y = \frac{\partial y}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -\sinh x \sin y, \quad \frac{\partial v}{\partial x} = \sinh x \sin y.$$

Both Cauchy Riemann equations are satisfied and thus f_4 is analytic.

v. $f_5(z) = \dfrac{\partial^2 \phi}{\partial x^2} - i\dfrac{\partial^2 \phi}{\partial x \partial y}$ gives $u = \dfrac{\partial^2 \phi}{\partial x^2}$ and $v = -\dfrac{\partial^2 \phi}{\partial x \partial y}$.

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} = \frac{\partial}{\partial x}\nabla^2 \phi = 0$$

as ϕ is harmonic.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^3 \phi}{\partial y \partial x^2} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$$

as mixed partial derivatives do not depend on the order. The Cauchy Riemann equations hold at all points and thus the function f_5 is analytic everywhere.

(b)

$$\frac{\partial u}{\partial x} = 3x^2y - y^3, \quad \frac{\partial^2 u}{\partial x^2} = 6xy$$

and

$$\frac{\partial u}{\partial y} = x^3 - 3xy^2, \quad \frac{\partial^2 u}{\partial y^2} = -6xy.$$

Hence $\nabla^2 u = 0$. The harmonic conjugate v is such that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3xy^2 - x^3.$$

Integrating partially with respect to x gives

$$v = 3\left(\frac{x^2}{2}\right)y^2 - \frac{x^4}{4} + g(y)$$

for any function $g(y)$. Using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = 3x^2 y + g'(y) = \frac{\partial u}{\partial x} = 3x^2 y - y^3.$$

Thus $g'(y) = -y^3$ and

$$g(y) = -\frac{y^4}{4} + C$$

where C is a constant. As we require $v(0,0) = 2$ this gives $C = 2$.

$$f = u + iv = (x^3 y - xy^3) + i\left(\left(\frac{6x^2y^2 - x^4 - y^4}{4}\right) + 2\right).$$

As by construction this is analytic it is a polynomial in z of degree 4. We use the Maclaurin series to get the representation and get the derivatives by differentiating in the x-direction.

$$\begin{aligned} f'(z) &= 3x^2y - y^3 + i(3xy^2 - x^3), \\ f''(z) &= 6xy + i(3y^2 - 3x^2), \\ f'''(z) &= 6y - i(6x), \\ f''''(z) &= -6i. \end{aligned}$$

Thus

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \frac{f'''(0)}{6}z^3 + \frac{f''''(0)}{24}z^4 = \left(\frac{-i}{4}\right)z^4 + 2i.$$

(c) i.

$$\frac{g_1(z_0+h) - g_1(z_0)}{\overline{h}} = \frac{f(\overline{z_0+h}) - (\overline{z_0})}{\overline{h}} = \frac{f(\overline{z_0} + \overline{h}) - f(\overline{z_0})}{\overline{h}}.$$

$z_0 \in D$ implies that $\overline{z_0} \in D$ and as f is analytic at $\overline{z_0}$ we have from the definition of complex differentiability

$$\lim_{h\to 0} \frac{g_1(z_0+h) - g_1(z_0)}{\overline{h}} = f'(\overline{z_0}).$$

ii.

$$\frac{g_2(z_0+h) - g_2(z_0)}{h} = \frac{\overline{g_1(z_0+h)} - \overline{g_1(z_0)}}{h}.$$

This is the complex conjugate of the expression in the previous part and thus

$$\lim_{h\to 0} \frac{g_2(z_0+h) - g_2(z_0)}{h} = \overline{f'(\overline{z_0})}.$$

As the limit exists at all points in D the function $g_2(z)$ is analytic in D.

iii. The Maclaurin series representation of $g_2(z)$ is given by

$$g_2(z) = \sum_{n=0}^{\infty} \overline{a_n} z^n.$$

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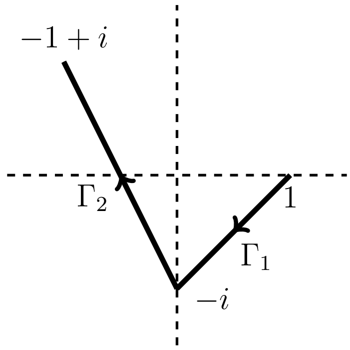
2. *Part of this question was question 2 of the Aug 2020 exam paper with some other parts from other years or are new exercises.*

   (a) Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.

   Suppose that $f(z)$ and the domain D are such that an anti-derivative F exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. We have the following result

$$\int_{\Gamma} f(z)\,\mathrm{d}z = F(z_2) - F(z_1)$$

   which you can use in the question below. Let Γ_1 and Γ_2 be the line segments illustrated below.



Γ_1 is from 1 to $-i$.
Γ_2 is from $-i$ to $-1+i$.

   Evaluate the following giving the value of each integral in cartesian form.

   i.
$$\int_{\Gamma_1} \mathrm{d}z.$$

   ii.
$$\int_{\Gamma_1\cup\Gamma_2} 3z^2\,\mathrm{d}z.$$

   iii.
$$\int_{\Gamma_2} \frac{\mathrm{d}z}{z}.$$

··· *question continues on the next page*

(b) Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

i.
$$\oint_\Gamma \frac{z\,\mathrm{e}^{3z}}{(z+2)^2} \, \mathrm{d}z.$$

ii.
$$\oint_\Gamma \frac{z^3}{(z+i)^4} \, \mathrm{d}z.$$

iii.
$$\oint_\Gamma \frac{\mathrm{Log}(z+4)}{(z+i)^2} \, \mathrm{d}z,$$

where Log denotes the principal valued logarithm.

(c) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. In the following let $0 < h < 1$ and let $\omega = \mathrm{e}^{\pi i/4}$. We have the following partial fraction representations which you can use.

$$\begin{aligned}
\frac{2h}{z^2-h^2} &= \frac{1}{z-h} - \frac{1}{z+h}, \\
\frac{2ih}{z^2+h^2} &= \frac{1}{z-ih} - \frac{1}{z+ih}, \\
\frac{4hz^2}{z^4-h^4} &= \frac{1}{z-h} - \frac{i}{z-ih} - \frac{1}{z+h} + \frac{i}{z+ih}, \\
\frac{4whz^2}{z^4+h^4} &= \frac{1}{z-wh} - \frac{i}{z-iwh} - \frac{1}{z+wh} + \frac{i}{z+iwh}.
\end{aligned}$$

By using the Cauchy integral formula (which is stated in the previous part) show that when we have the following.

$$\begin{aligned}
\frac{f(h)-f(-h)}{2h} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^2-h^2} \, \mathrm{d}z, \\
\frac{f(ih)-f(-ih)}{2ih} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^2+h^2} \, \mathrm{d}z, \\
\frac{f(h)-if(ih)-f(-h)+if(-ih)}{4h} &= \frac{1}{2\pi i} \oint_C \frac{z^2 f(z)}{z^4-h^4} \, \mathrm{d}z, \\
\frac{f(\omega h)-if(i\omega h)-f(-\omega h)+if(-i\omega h)}{4\omega h} &= \frac{1}{2\pi i} \oint_C \frac{z^2 f(z)}{z^4+h^4} \, \mathrm{d}z.
\end{aligned}$$

Solution

(a) i. With $f(z) = 1$ and $F(z) = z$ the value is $-i - 1$.

ii. With $f(z) = 3z^2$ we have the anti-derivative

$$F(z) = z^3.$$

The path of $\Gamma_1 \cup \Gamma_2$ starts at 1 and ends at $-1 + i$.

$$F(1) = 1 \quad \text{and} \quad F(-1+i) = (-1+i)^3 = -1 + 3i - 3i^2 + i^3 = 2 + 2i.$$

The value of the integral is

$$(2 + 2i) - 1 = 1 + 2i.$$

iii. With $f(z) = 1/z$ possible anti-derivatives are $\mathrm{Log}(z)$ and $\mathrm{Log}(-z)$. As the segment Γ_2 crosses the negative real axis the one to take is $F(z) = \mathrm{Log}(-z)$ as this is continuous on Γ_2.

$$F(-i) = \mathrm{Log}(i) = i\frac{\pi}{2}, \quad F(-1+i) = \mathrm{Log}(1-i) = \ln(\sqrt{2}) - i\frac{\pi}{4}.$$

The value of the integral is

$$\frac{1}{2}\ln(2) - i\frac{3\pi}{4}.$$

(b) i.

$$\frac{z\,\mathrm{e}^{3z}}{(z+2)^2} = \frac{f(z)}{(z-z_0)^{n+1}}$$

with $z_0 = -2$, $n + 1 = 2$ and $f(z) = z\mathrm{e}^{3z}$. The value of the integral is

$$2\pi i f'(-2).$$

$$f'(z) = \mathrm{e}^{3z}(1 + 3z).$$

The value is hence

$$2(-5)\pi i\mathrm{e}^{-6} = -10\pi\mathrm{e}^{-6}i.$$

ii.

$$\frac{z^3}{(z+i)^4} = \frac{f(z)}{(z-z_0)^{n+1}}$$

with $f(z) = z^3$, $n + 1 = 4$ and $z_0 = -i$.

$$f'''(z) = 6.$$

The value of the integral is

$$\frac{2\pi i}{3!}f'''(i) = 2\pi i.$$

iii.
$$\frac{\mathrm{Log}(z+4)}{(z+i)^2} = \frac{f(z)}{(z-z_0)^{n+1}}$$

with $f(z) = \mathrm{Log}(z+4)$, $n+1 = 2$ and $z_0 = -i$. The function $f(z)$ has a branch point at $z = -4$ and is analytic on and inside the circle being considered.

$$f'(z) = \frac{1}{z+4}, \quad f'(-i) = \frac{1}{-i+4} = \frac{4+i}{17}.$$

The value of the integral is

$$\frac{2\pi i}{1!} f'(-i) = (2\pi i)\left(\frac{4+i}{17}\right) = \frac{2\pi}{17}(-1+4i).$$

(c) The Cauchy integral formula for $f(h)$ and $f(-h)$ gives

$$f(h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-h}\,\mathrm{d}z, \quad f(-h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z+h}\,\mathrm{d}z,$$

Hence

$$\begin{aligned} f(h) - f(-h) &= \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-h} - \frac{1}{z+h}\right)\mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_C f(z)\frac{2h}{z^2-h^2}\,\mathrm{d}z \end{aligned}$$

by using one of the given partial fraction representations. Dividing by $2h$ gives the identity.

The second identity follows by replacing h by ih.

The Cauchy integral formula for $f(ih)$ and $f(-ih)$ gives

$$f(ih) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-ih}\,\mathrm{d}z, \quad f(-ih) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z+ih}\,\mathrm{d}z,$$

Hence

$$\begin{aligned} & f(h) - if(ih) - f(-h) + if(-ih) \\ = {} & \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-h} - \frac{i}{z-ih} - \frac{1}{z+h} + \frac{i}{z+ih}\right)\mathrm{d}z \\ = {} & \frac{1}{2\pi i}\oint_C \frac{4hz^2 f(z)}{z^4-h^4}\,\mathrm{d}z \end{aligned}$$

by using one of the given partial fraction representations.. Dividing by $4h$ gives the third identity.

The fourth identity follows by replacing h by ωh as $\omega^4 = \mathrm{e}^{i\pi} = -1$.

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3. (a) Determine if the following power series define entire functions and if this is not the case then find the circle of convergence. In each case you must justify your answer.

   i.
$$\sum_{n=0}^{\infty} \frac{2n+1}{n!}(z+3)^n.$$

   ii.
$$\sum_{n=0}^{\infty} \frac{n}{2^n}(z-1)^n.$$

(b) Determine the largest annulus of the form $0 \le r < |z| < R \le \infty$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^4}{4^n z^n} + \sum_{n=1}^{\infty} \frac{n^4 z^n}{4^n}.$$

(c) Let

$$f(z) = \frac{-2z+z^2}{1+2\sin(z)}.$$

Determine the radius of convergence of the Maclaurin series.

Suppose that the Maclaurin series for your function is expressed in the form

$$a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

Determine a_1, a_2, a_3 and a_4.

(d) Let

$$\phi(z) = \frac{1}{(1+z)(2-z)}.$$

Determine the partial fraction representation of $\phi(z)$ and determine the Laurent series valid for $|z| > 2$. In your answer you must give the coefficient of $1/z^n$ for $n \ge 1$.

Solution

(a) i. Let $a_n = (2n+1)/n!$ and let $b_n = a_n(z+3)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{2n+1}\left(\frac{1}{n+1}\right) = \frac{2+3/n}{2+1/n}\left(\frac{1}{n+1}\right) \to 0 \quad \text{as } n \to \infty.$$

Hence for all z

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(z+3) \to 0 \quad \text{as } n \to \infty.$$

By the ratio test the series converges for all z and the function is an entire function.

ii. Let now $a_n = n/2^n$ and let $b_n = a_n(z-1)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\left(\frac{1}{2}\right) \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Hence

$$\frac{|b_{n+1}|}{|b_n|} \to \frac{|z-1|}{2} \quad \text{as } n \to \infty.$$

By the ratio test the series converges when $|z-1| < 2$ and diverges when $|z-1| > 2$ and thus the circle $|z-1| = 2$ is the circle of convergence.

(b) We try the ratio test on the term with positive powers of z. Let

$$a_n = \frac{n^4}{4^n}, \quad b_n = a_n z^n.$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+1)^4}{n^4}\right)\left(\frac{4^n}{4^{n+1}}\right) = \left(1+\frac{1}{n}\right)^4 \frac{1}{4} \to \frac{1}{4}$$

as $n \to \infty$.

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n} z \to \frac{z}{4}$$

as $n \to \infty$. By the ratio test the series of this part converges when $|z| < 4$ and diverges when $|z| > 4$ and thus $R = 4$.

We try the ratio test on the term with negative powers of z. Let

$$c_n = \frac{a_n}{z^n}.$$

From the earlier part

$$\frac{c_{n+1}}{c_n} \to \frac{1}{4z}$$

as $n \to \infty$. By the ratio test the series of this part converges when $|4z| > 1$ and diverges when $|4z| < 4$ and thus $r = 1/4$.

(c) $1 + 2\sin(z) = 0$ when $\sin(z) = -1/2$ and the nearest points to 0 is at $-\pi/6$. $f(z)$ is analytic in $|z| < \pi/6$ and has a simple pole on $|z| = \pi/6$. The radius of convergence of the Maclaurin series is $\pi/6$.

We consider

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)(1 + 2\sin(z)) = -2z + z^2$$

$$1 + 2\sin(z) = 1 + 2\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right) = 1 + 2z - \frac{z^3}{3} + \frac{z^5}{60} + \cdots$$

Hence

$$(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)\left(1 + 2z - \frac{z^3}{3} + \frac{z^5}{60} + \cdots\right) = -2z + z^2.$$

Equating the coefficients of z gives

$$a_1 = -2.$$

Equating the coefficients of z^2 gives

$$a_2 + 2a_1 = a_2 - 4 = 1, \quad a_2 = 5.$$

Equating the coefficients of z^3 gives

$$a_3 + 2a_2 = a_3 + 10 = 0, \quad a_3 = -10.$$

Equating the coefficients of z^4 gives

$$a_4 + 2a_3 - \frac{a_1}{3} = a_4 - 20 + \frac{2}{3} = 0, \quad a_4 = 20 - \frac{2}{3} = \frac{58}{3}.$$

(d)

$$\phi(z) = \frac{1}{(1+z)(2-z)} = \frac{A}{1+z} + \frac{B}{2-z}.$$

$$A = \lim_{z\to -1}(1+z)\phi(z) = \frac{1}{3}.$$

$$B = \lim_{z\to 2}(2-z)\phi(z) = \frac{1}{3}.$$

$$1 + z = z(1/z + 1) = z(1 - (-1/z)) \quad \text{and} \quad 2 - z = z(2/z - 1) = -z(1 - 2/z).$$

By the geometric series

$$\frac{1}{1+z} = \left(\frac{1}{z}\right)\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

which is valid for $|z| > 1$.

$$\frac{1}{2-z} = -\left(\frac{1}{z}\right)\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right)$$

which is valid for $|z| > 2$.

We get the Laurent series for $\phi(z)$ valid for $|z| > 2$ by appropriately combining these two series . The coefficient of $1/z^{n+1}$, $n \ge 0$, in the Laurent series for $\phi(z)$ valid for $|z| > 2$ is

$$(-1)^n A - 2^n B = \frac{(-1)^n - 2^n}{3}.$$

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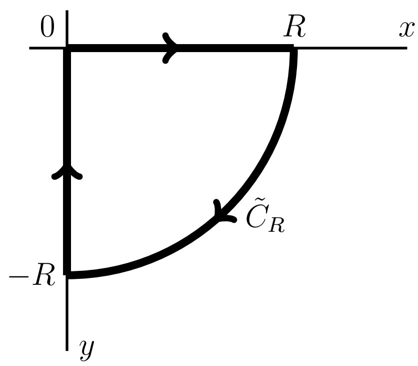
4. *Part (a) was question 4a of the 2018 MA3614 paper and was worth 10 marks. It was also on the first exercise about integrals which was given out towards the end of term 1. Part (b) is a variation of what has been done in the lectures and exercises.*

   (a) By first using the substitution $z = \mathrm{e}^{i\theta}$, evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + 8\cos^2\theta}.$$

   (b) Let $R > 2$ and let $\tilde{C}_R$ denote the quarter circle with centre at 0 and radius $R > 0$ in the 4th quadrant traversed in the clockwise direction, and let Γ_R denote the closed loop composed of the real interval $[0, R]$ followed by the quarter circle $\tilde{C}^r_R$ followed by the pat of the imaginary axis from $-Ri$ to 0. The closed loop is illustrated in the diagram below.



Let $f(z)$ be the function

$$f(z) = \frac{1}{z^4 + 16}$$

Give all the points in the complex plane where this function has pole singularities and determine the residue at each pole which is inside the loop Γ_R.

By considering an integral involving the loop Γ_R, evaluate

$$\int_0^\infty f(x)\,\mathrm{d}x.$$

For full marks you need to explain each of your steps.

Solution

(a)
$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}).$$

The interval $0 \le \theta \le 2\pi$ maps to the unit circle C traversed once in the anti-clockwise direction and we have that the integral is

$$I = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z,$$

where

$$F(z) = \frac{1}{z}\left(\frac{1}{1 + 2(z + 1/z)^2}\right) = \frac{z}{z^2 + 2(z^2+1)^2} = \frac{z}{2z^4 + 5z^2 + 2}.$$

The denominator vanishes when

$$2z^4 + 5z^2 + 2 = 0.$$

This is a quadratic in z^2 and the smaller in magnitude root z_1 satisfies

$$\begin{aligned} z_1^2 &= \frac{-5 + \sqrt{25 - 16}}{4} \\ &= \frac{-5+3}{4} = -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \mathrm{Res}(F,\, z_1) &= \lim_{z \to z_1} (z - z_1) F(z) = z_1 \lim_{z \to z_1} \frac{z - z_1}{2z^4 + 5z^2 + 2} \\ &= \frac{z_1}{8z_1^3 + 10 z_1} \\ &= \frac{1}{8z_1^2 + 10} = \frac{1}{-4 + 10} = \frac{1}{6} \end{aligned}$$

by using L'Hopital's rule and the expression for z_1. By similar workings the residue at $-z_1$ is the same value. As both z_1 and $-z_1$ are inside C the residue theorem gives

$$I = 2\pi(\mathrm{Res}(F,\, z_1) + \mathrm{Res}(F,\, -z_1)) = 2\pi\left(\frac{1}{6} + \frac{1}{6}\right) = \frac{2\pi}{3}.$$

(b) The simple poles of $f(z)$ are when

$$z^4 = -16 = 16\mathrm{e}^{i\pi}.$$

In polar form the positions are

$$z_k = 2\exp\left(\frac{i\pi}{4} + \frac{k\pi}{2}\right), \quad k = 0, 1, 2, 3.$$

The pole which is inside the loop is

$$z_3 = 2\mathrm{e}^{-\pi/4} = \sqrt{2}(1 - i).$$

The residue at this point is

$$\mathrm{Res}(f,\, z_3) = \lim_{z\to z_3} \frac{z - z_3}{z^4+16} = \frac{1}{4z_3^3} = \frac{z_3}{4z_3^4} = -\frac{z_3}{64}.$$

As the loop is clockwise the residue theorem gives

$$\oint_{\Gamma_R} \frac{\mathrm{d}z}{z^4+16} = -2\pi i \mathrm{Res}(f,\, z_3) = \frac{\pi i z_3}{32} = \frac{(1+i)\pi\sqrt{2}}{32}.$$

For $z \in \tilde{C}_R$ we have $|z| = R$ and

$$|f(z)| \le \frac{1}{R^4 - 16}.$$

The length of $\tilde{C}_R$ is $\pi R/2$ and thus by the ML inequality

$$\left| \int_{\tilde{C}_R} f(z)\,\mathrm{d}z \right| \le \frac{\pi R/2}{R^4-16} \to 0 \quad \text{as } R \to \infty.$$

Let γ_R denote the line segment joininy $-Ri$ and 0. It follows that a parametrization of $-\gamma_R$ is

$$z(t) = -it, \quad 0 \le t \le R, \quad z'(t) = -i.$$

$$\int_{-\gamma_R} f(z)\,\mathrm{d}z = \int_0^R -if(-it)\,\mathrm{d}t = \int_0^R -if(t)\,\mathrm{d}t.$$

Hence

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = \int_0^R if(t)\,\mathrm{d}t.$$

As the Γ_R is the union of 3 parts and two of them combine we have

$$(1+i)\int_0^R f(x)\,\mathrm{d}x + \int_{\tilde{C}_R} f(z)\,\mathrm{d}z = \frac{(1+i)\sqrt{2}\pi}{32}.$$

Letting $R \to \infty$ gives

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{\pi\sqrt{2}}{32}.$$

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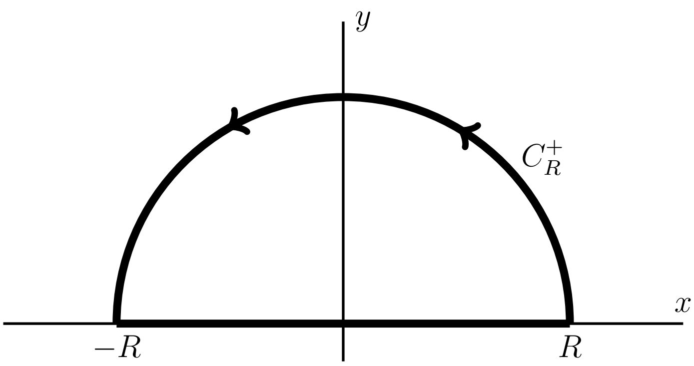
5. *The following was part of question 4 of the May 2015 exam and was worth 10 marks.*

   Let C_R^+ denote the half circle with centre at 0 and radius $R > 2$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



   By considering an integral involving the loop Γ_R evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)(4+x^2)}.$$

   For full marks you need to explain each of your steps.

Solution

Let

$$f(z) = \frac{1}{(1+z^2)(4+z^2)}.$$

This function has simple poles at $\pm i$ and $\pm 2i$. Label the poles in the upper half plane as $z_1 = i$ and $z_2 = 2i$.

When $R > 2$ the points z_1 and z_2 are inside Γ_R and by the residue theorem

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i(\mathrm{Res}(f,\, z_1) + \mathrm{Res}(f,\, z_2)).$$

For the residues

$$\begin{aligned}
\mathrm{Res}(f,\, z_1) &= \lim_{z\to z_1}(z-z_1)f(z) = \left.\frac{1}{4+z^2}\right|_{z=i} \lim_{z\to i}\frac{z-i}{1+z^2} \\
&= \left(\frac{1}{3}\right)\left(\frac{1}{2i}\right) = \frac{1}{6i}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}(f,\, z_2) &= \lim_{z\to z_2}(z-z_2)f(z) = \left.\frac{1}{1+z^2}\right|_{z=2i} \lim_{z\to 2i}\frac{z-2i}{4+z^2} \\
&= \left(\frac{1}{-3}\right)\left(\frac{1}{4i}\right) = -\frac{1}{12i}.
\end{aligned}$$

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\left(\frac{1}{6i} - \frac{1}{12i}\right) = \frac{\pi}{6}.$$

As Γ_R is the union of two parts we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{C_R} f(z)\,\mathrm{d}z + \int_{-R}^{R} f(x)\,\mathrm{d}x.$$

By letting $R \to \infty$ we obtain the result

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi}{6}$$

and to justify we need to show that the integral around the semi-circle tends to 0 as $R \to \infty$.

On C_R we have $|1+z^2| \ge |R^2-1|$ and $|4+z^2| \ge |R^2-4|$ and thus

$$|f(z)| \le \frac{1}{(R^2-1)(R^2-4)}.$$

As the length of C_R is πR the ML inequality gives the bound

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \le \frac{\pi R}{(R^2-1)(R^2-4)} \to 0 \quad \text{as } R \to \infty$$

as the denominator is a higher degree polynomial in R than the numerator in the last expression.

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6. Let Γ denote the circle $\{z : \; |z-1| = 2\}$ traversed once in the anti-clockwise direction. Determine the following loop integral.

$$\oint_{\Gamma} \frac{\mathrm{e}^z}{z(4-z)}\,\mathrm{d}z.$$

Solution

The integrand only has a singularity inside the contour at $z_0 = 0$ and the conditions of the Cauchy integral formula hold if we take

$$f(z) = \frac{e^z}{4-z}.$$

The value of the integral is

$$2\pi i f(0) = \frac{2\pi i}{4} = \frac{\pi i}{2}.$$

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7. Let C be the unit circle $z(t) = e^{it}$, $-\pi < t \le \pi$. By any means determine

$$\int_C z^{1/3}\,dz,$$

where $z^{1/3}$ denotes the principal value root function and where the direction of integration is the anti-clockwise direction.

Solution

By using the given parametrization we have

$$\frac{dz}{dt} = i\,e^{it}$$

the integral is

$$\int_{-\pi}^{\pi} ie^{i(t/3+t)}\,dt = i\left[\frac{e^{i(4t/3)}}{(4i/3)}\right]_{-\pi}^{\pi} = \frac{3}{4}\left(e^{4i\pi/3} - e^{-4i\pi/3}\right) = \frac{3i}{2}\sin\left(\frac{4\pi}{3}\right).$$

An alternative solution is to note that we have an anti-derivative

$$F(z) = \frac{z^{4/3}}{(4/3)} = \frac{3}{4}z^{4/3}.$$

Let $F(-1+i0)$ denote the value on the branch cut of the function and let $F(-1-i0)$ denote the limit as you approach the point from below the negative real axis. These values are as follows.

$$F(-1+i0) = \frac{3}{4}e^{i4\pi/3} \quad \text{and} \quad F(-1-i0) = \frac{3}{4}e^{-i4\pi/3}.$$

Then

$$\begin{aligned} F(-1+i0) - F(-1-i0) &= \frac{3}{4}\left(e^{i4\pi/3} - e^{-i4\pi/3}\right) \\ &= \frac{3}{4}2i\sin\left(\frac{4\pi}{3}\right) = \frac{3i}{2}\sin\left(\frac{4\pi}{3}\right). \end{aligned}$$

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8. Consider the following series.

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n+4^n}\right)(z-1)^n,$$

(a) Give details to determine the circle of convergence using the ratio test.

(b) Give details to determine the circle of convergence using the root test.

Solution

(a)

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n+4^n}\right)(z-1)^n, \quad \text{let } a_n=\frac{1}{3^n+4^n}, \quad b_n=a_n(z-1)^n.$$

$$\frac{a_{n+1}}{a_n}=\frac{3^n+4^n}{3^{n+1}+4^{n+1}}=\frac{(3/4)^n+1}{3(3/4)^n+4}\to\frac{1}{4} \quad \text{as } n\to\infty.$$

Thus

$$\left|\frac{b_{n+1}}{b_n}\right|\to\frac{|z-1|}{4} \quad \text{as } n\to\infty.$$

By the ratio test we have convergence when $|z-1|<4$ and divergence when $|z-1|>4$. Thus by the ratio test the circle of convergence is $\{z:\ |z-1|=4\}$.

(b) With a_n and b_n as in the previous part we have

$$3^n+4^n=4^n\left(1+\left(\frac{3}{4}\right)^n\right)$$

and

$$a_n^{1/n}=\frac{1}{4}\left(1+\left(\frac{3}{4}\right)^n\right)^{-1/n}\to\frac{1}{4} \quad \text{as } n\to\infty.$$

$$|b_n|^{1/n}=|a_n|^{1/n}\,|z-1|\to\frac{|z-1|}{4} \quad \text{as } n\to\infty.$$

By the root test we have convergence when $|z-1|<4$ and divergence when $|z-1|>4$. Thus by the root test the circle of convergence is $\{z:\ |z-1|=4\}$.

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9. Let a be real with $a>1$. By using the substitution $z=\mathrm{e}^{i\theta}$ show that

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{a+\cos\theta}=\frac{2\pi}{\sqrt{a^2-1}}.$$

Solution

$$\frac{\mathrm{d}z}{\mathrm{d}\theta}=iz \quad \text{gives } \frac{\mathrm{d}\theta}{\mathrm{d}z}=\frac{1}{iz} \quad \text{and } \cos\theta=\frac{1}{2}(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta})=\frac{1}{2}\left(z+\frac{1}{z}\right).$$

For the integrand we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{1}{a+\cos\theta}\right)=\frac{1}{i}F(z)$$

where

$$F(z)=\frac{1}{z}\left(\frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)}\right)=\frac{2}{2az+z^2+1}.$$

Now

$$z^2 + 2az + 1 = 0 \quad \text{when } z = -a \pm \sqrt{a^2 - 1}.$$

The negative sign case gives a value which is less than -1 and hence outside of the unit disk and hence we only need to consider

$$z_1 = -a + \sqrt{a^2 - 1}.$$

With I denoting the integral we have by the residue theorem that

$$I = 2\pi \mathrm{Res}(F,\ z_1)$$

and

$$\mathrm{Res}(F,\ z_1) = 2 \lim_{z \to z_1} \frac{(z - z_1)}{z^2 + 2az + 1} = 2\left(\frac{1}{2z_1 + 2a}\right) = \frac{1}{z_1 + a} = \frac{1}{\sqrt{a^2 - 1}}.$$

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10. (a) Determine the Maclaurin series for $z\cos z$ and indicate where it converges.

(b) Determine the Laurent series of

$$\frac{\cos z}{z^2}$$

about the point $z = 0$ and indicate where it converges.

(c) Find the first 3 non-zero terms of the Laurent series of

$$\frac{z}{\sin z}$$

in a region of the form $0 < |z| < R$. State the largest value of R for which the Laurent series converges and give a reason to justify your answer.

Solution

(a) The Maclaurin series for $\cos z$ is a standard series and is given by

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$

Thus

$$z\cos z = z - \frac{z^3}{2} + \frac{z^5}{24} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n)!} + \cdots$$

and this converges for all z.

(b) Using the series for $\cos z$ the Laurent series is

$$\frac{\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{2} + \frac{z^2}{24} + \cdots + (-1)^n \frac{z^{2n-2}}{(2n)!} + \cdots$$

and this converges in $\{z :\ 0 < |z| < \infty\}$.

(c) As the zeros of $\sin z$ are on the real axis and these are at $k\pi$ where k is an integer it follows that the given function is analytic in $0 < |z| < R$ with $R = \pi$.

The function is bounded as $z \to 0$ and thus it has a removable singularity at $z = 0$. It is an even function and hence only even powers are involved and hence for $0 < |z| < \pi$

$$\frac{z}{\sin z} = a_0 + a_2 z^2 + a_4 z^4 + \cdots .$$

Rearranging and using the known series for $\sin z$ gives

$$z = (a_0 + a_2 z^2 + a_4 z^4 + \cdots)\left(z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right).$$

Equating the coefficient of z gives $a_0 = 1$.

Equating the coefficient of z^3 gives

$$0 = a_2 - \frac{a_0}{6}, \qquad a_2 = \frac{1}{6}.$$

Equating the coefficient of z^5 gives

$$0 = a_4 - \frac{a_2}{6} + \frac{a_0}{120}, \qquad a_4 = \frac{a_2}{6} - \frac{a_0}{120} = \frac{1}{36} - \frac{1}{120} = \frac{7}{360}.$$

11. Let $n \geq 1$ be an integer and let $f(z)$ denote a function which is analytic on the unit circle C and inside the unit circle. Also let $0 < h < 1$. The factorization

$$z^{n+1} - h^{n+1} = (z-h)\sum_{k=0}^{n} h^k z^{n-k}$$

rearranges to

$$z^{n+1} = (z-h)\sum_{k=0}^{n} h^k z^{n-k} + h^{n+1} \quad \text{and} \quad \frac{1}{z-h} - \frac{1}{z^{n+1}}\sum_{k=0}^{n} h^k z^{n-k} = \frac{h^{n+1}}{z^{n+1}(z-h)}.$$

Complete the steps to show that

$$f(h) - \sum_{k=0}^{n} \left(\frac{f^{(k)}(0)}{k!}\right) h^k = h^{n+1} \oint_C \frac{f(z)}{z^{n+1}(z-h)}\,\mathrm{d}z$$

where in the loop integral C is traversed once in the anti-clockwise sense.

Solution

The last identity can be written as

$$\frac{1}{z-h} - \sum_{k=0}^{n} h^k z^{-(k+1)} = \frac{h^{n+1}}{z^{n+1}(z-h)}.$$

Now if we multiply by $f(z)/(2\pi i)$ and integrate using the loop C then we have

$$\frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-h} - \sum_{k=0}^{n} h^k z^{-(k+1)}\right)\mathrm{d}z = \frac{h^{n+1}}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}(z-h)}\,\mathrm{d}z.$$

The right hand side expression matches what we want to show. By linearity of the integral the left hand side can be written as

$$\frac{1}{2\pi i}\left(\oint_C \frac{f(z)}{z-h}\,\mathrm{d}z - h^k \sum_{k=0}^{n} \oint_C \frac{f(z)}{z^{k+1}}\,\mathrm{d}z\right).$$

The Cauchy integral formula and the generalised Caucy integral formula give

$$f(h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-h}\,\mathrm{d}z, \quad \text{and} \quad \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{k+1}}\,\mathrm{d}z$$

and the result follows.