Exercises as part of the revision for the May exams

1. *Parts of this question are taken from the paper in May 2017 and May 2018.*

   (a) Let $z = x+iy$ with $x, y \in \mathbb{R}$. For each of the following functions determine whether or not it is analytic in the domain specified, giving reasons for your answers in each case.

   i.
   $$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(z) = x^2 + i2xy.$$

   ii.
   $$f_2 : \mathbb{C} \to \mathbb{C}, \quad f_2(z) = (2x^3 + 3x^2y - 6xy^2 - y^3) + i(-x^3 + 6x^2y + 3xy^2 - 2y^3).$$

   iii.
   $$f_3 : \mathbb{C} \to \mathbb{C}, \quad f_3(z) = \mathrm{e}^{-x}(\cos\, y + i \sin\, y).$$

   iv.
   $$f_4 : \mathbb{C} \to \mathbb{C}, \quad f_4(z) = \sinh\, x \cos\, y + i \cosh\, x \sin\, y.$$

   v.
   $$f_5 : \mathbb{C} \to \mathbb{C}, \quad f_5(z) = \frac{\partial^2 \phi}{\partial x^2} - i\frac{\partial^2 \phi}{\partial x \partial y}$$

   where ϕ is a harmonic function with continuous partial derivatives of all orders.

   (b) Show that the function
   $$u(x, y) = x^3y - xy^3$$
   is harmonic and determine the harmonic conjugate $v(x, y)$ satisfying $v(0, 0) = 2$. Express $u + iv$ in terms of z only.

   (c) Let $D = \{z : |z| < 1\}$ and let $f(z)$ be a function which is analytic in D. Also let $g_1(z)$ and $g_2(z)$ be functions defined in D by
   $$g_1(z) = f(\overline{z}), \quad g_2(z) = \overline{g_1(z)}.$$

   i. Let $z_0 \in D$. Explain why the following limit exists and give the limit in terms of f and/or its derivatives.
   $$\lim_{h\to 0} \frac{g_1(z_0 + h) - g_1(z_0)}{\overline{h}}.$$

   ii. Explain why $g_2(z)$ is analytic in D.

   iii. If the Maclaurin series representation of $f(z)$ is given by
   $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
   then give the Maclaurin series for $g_2(z)$.

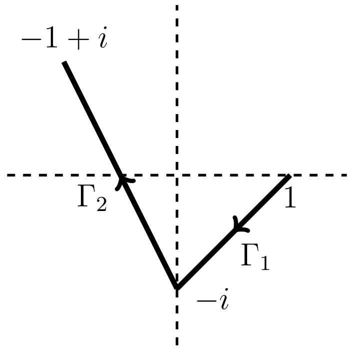
2. *Part of this question was question 2 of the Aug 2020 exam paper with some other parts from other years or are new exercises.*

   (a) Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.

   Suppose that $f(z)$ and the domain D are such that an anti-derivative F exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. We have the following result

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1)$$

   which you can use in the question below. Let Γ_1 and Γ_2 be the line segments illustrated below.



Γ_1 is from 1 to $-i$.
Γ_2 is from $-i$ to $-1+i$.

   Evaluate the following giving the value of each integral in cartesian form.

   i.
$$\int_{\Gamma_1} \mathrm{d}z.$$

   ii.
$$\int_{\Gamma_1 \cup \Gamma_2} 3z^2\,\mathrm{d}z.$$

   iii.
$$\int_{\Gamma_2} \frac{\mathrm{d}z}{z}.$$

$\cdots$ *question continues on the next page*

(b) Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

i.
$$\oint_\Gamma \frac{z\,\mathrm{e}^{3z}}{(z+2)^2} \, \mathrm{d}z.$$

ii.
$$\oint_\Gamma \frac{z^3}{(z+i)^4} \, \mathrm{d}z.$$

iii.
$$\oint_\Gamma \frac{\mathrm{Log}(z+4)}{(z+i)^2} \, \mathrm{d}z,$$

where Log denotes the principal valued logarithm.

(c) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. In the following let $0 < h < 1$ and let $\omega = \mathrm{e}^{\pi i/4}$. We have the following partial fraction representations which you can use.

$$\begin{aligned}
\frac{2h}{z^2-h^2} &= \frac{1}{z-h} - \frac{1}{z+h}, \\
\frac{2ih}{z^2+h^2} &= \frac{1}{z-ih} - \frac{1}{z+ih}, \\
\frac{4hz^2}{z^4-h^4} &= \frac{1}{z-h} - \frac{i}{z-ih} - \frac{1}{z+h} + \frac{i}{z+ih}, \\
\frac{4whz^2}{z^4+h^4} &= \frac{1}{z-wh} - \frac{i}{z-iwh} - \frac{1}{z+wh} + \frac{i}{z+iwh}.
\end{aligned}$$

By using the Cauchy integral formula (which is stated in the previous part) show that when we have the following.

$$\begin{aligned}
\frac{f(h)-f(-h)}{2h} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^2-h^2} \, \mathrm{d}z, \\
\frac{f(ih)-f(-ih)}{2ih} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^2+h^2} \, \mathrm{d}z, \\
\frac{f(h)-if(ih)-f(-h)+if(-ih)}{4h} &= \frac{1}{2\pi i} \oint_C \frac{z^2 f(z)}{z^4-h^4} \, \mathrm{d}z, \\
\frac{f(\omega h)-if(i\omega h)-f(-\omega h)+if(-i\omega h)}{4\omega h} &= \frac{1}{2\pi i} \oint_C \frac{z^2 f(z)}{z^4+h^4} \, \mathrm{d}z.
\end{aligned}$$

3. (a) Determine if the following power series define entire functions and if this is not the case then find the circle of convergence. In each case you must justify your answer.

   i.
$$\sum_{n=0}^{\infty} \frac{2n+1}{n!}(z+3)^n.$$

   ii.
$$\sum_{n=0}^{\infty} \frac{n}{2^n}(z-1)^n.$$

   (b) Determine the largest annulus of the form $0 \le r < |z| < R \le \infty$ for which the following Laurent series converges. You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^4}{4^n z^n} + \sum_{n=1}^{\infty} \frac{n^4 z^n}{4^n}.$$

   (c) Let

$$f(z) = \frac{-2z + z^2}{1 + 2\sin(z)}.$$

   Determine the radius of convergence of the Maclaurin series.

   Suppose that the Maclaurin series for your function is expressed in the form

$$a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots .$$

   Determine a_1, a_2, a_3 and a_4.

   (d) Let

$$\phi(z) = \frac{1}{(1+z)(2-z)}.$$

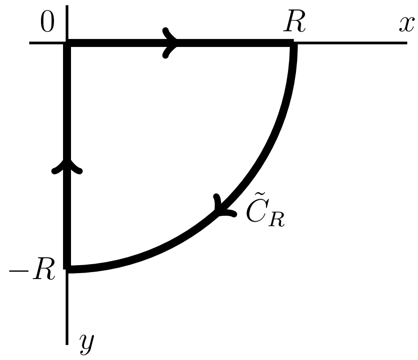
   Determine the partial fraction representation of $\phi(z)$ and determine the Laurent series valid for $|z| > 2$. In your answer you must give the coefficient of $1/z^n$ for $n \ge 1$.

4. *Part (a) was question 4a of the 2018 MA3614 paper and was worth 10 marks. It was also on the first exercise about integrals which was given out towards the end of term 1. Part (b) is a variation of what has been done in the lectures and exercises.*

   (a) By first using the substitution $z = \mathrm{e}^{i\theta}$, evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + 8\cos^2\theta}.$$

   (b) Let $R > 2$ and let $\tilde{C}_R$ denote the quarter circle with centre at 0 and radius $R > 0$ in the 4th quadrant traversed in the clockwise direction, and let Γ_R denote the closed loop composed of the real interval $[0, R]$ followed by the quarter circle $\tilde{C}^r_R$ followed by the pat of the imaginary axis from $-Ri$ to 0. The closed loop is illustrated in the diagram below.



Let $f(z)$ be the function

$$f(z) = \frac{1}{z^4 + 16}$$

Give all the points in the complex plane where this function has pole singularities and determine the residue at each pole which is inside the loop Γ_R.

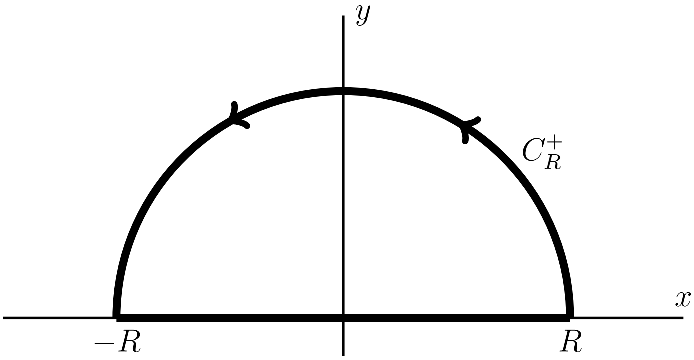
By considering an integral involving the loop Γ_R, evaluate

$$\int_0^{\infty} f(x)\,\mathrm{d}x.$$

For full marks you need to explain each of your steps.

5. *The following was part of question 4 of the May 2015 exam and was worth 10 marks.*

   Let C_R^+ denote the half circle with centre at 0 and radius $R > 2$ in the upper half plane traversed in the anti-clockwise direction and let Γ_R denote the closed loop composed of the real interval $[-R, R]$ followed by the half circle C_R^+, that is $\Gamma_R = [-R, R] \cup C_R^+$. The half circle C_R^+ and the closed loop are illustrated in the diagram below.



   By considering an integral involving the loop Γ_R evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)(4+x^2)}.$$

   For full marks you need to explain each of your steps.

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6. Let Γ denote the circle $\{z : \ |z-1| = 2\}$ traversed once in the anti-clockwise direction. Determine the following loop integral.

$$\oint_{\Gamma} \frac{\mathrm{e}^z}{z(4-z)} \, \mathrm{d}z.$$

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7. Let C be the unit circle $z(t) = \mathrm{e}^{it}$, $-\pi < t \le \pi$. By any means determine

$$\int_C z^{1/3} \, \mathrm{d}z,$$

   where $z^{1/3}$ denotes the principal value root function and where the direction of integration is the anti-clockwise direction.

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8. Consider the following series.

$$\sum_{n=0}^{\infty} \left(\frac{1}{3^n + 4^n} \right) (z-1)^n,$$

   (a) Give details to determine the circle of convergence using the ratio test.

   (b) Give details to determine the circle of convergence using the root test.

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9. Let a be real with $a > 1$. By using the substitution $z = \mathrm{e}^{i\theta}$ show that

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}.$$

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10. (a) Determine the Maclaurin series for $z \cos z$ and indicate where it converges.

(b) Determine the Laurent series of

$$\frac{\cos z}{z^2}$$

about the point $z = 0$ and indicate where it converges.

(c) Find the first 3 non-zero terms of the Laurent series of

$$\frac{z}{\sin z}$$

in a region of the form $0 < |z| < R$. State the largest value of R for which the Laurent series converges and give a reason to justify your answer.

11. Let $n \geq 1$ be an integer and let $f(z)$ denote a function which is analytic on the unit circle C and inside the unit circle. Also let $0 < h < 1$. The factorization

$$z^{n+1} - h^{n+1} = (z - h) \sum_{k=0}^{n} h^k z^{n-k}$$

rearranges to

$$z^{n+1} = (z - h) \sum_{k=0}^{n} h^k z^{n-k} + h^{n+1} \quad \text{and} \quad \frac{1}{z-h} - \frac{1}{z^{n+1}} \sum_{k=0}^{n} h^k z^{n-k} = \frac{h^{n+1}}{z^{n+1}(z-h)}.$$

Complete the steps to show that

$$f(h) - \sum_{k=0}^{n} \left(\frac{f^{(k)}(0)}{k!} \right) h^k = h^{n+1} \oint_C \frac{f(z)}{z^{n+1}(z-h)} \, \mathrm{d}z$$

where in the loop integral C is traversed once in the anti-clockwise sense.