Exercises related to the introduction chapter

The following are a collection of questions obtained from various sources which include some past MA3614 class tests that I have set, the book of Saff and Snider as well as questions that I have created myself. One of the hours each week will usually just be exercises and we can decide which is the most popular hour to use when we meet. All parts of this exercise sheet can be attempted from the start of the module although some parts may seem hard until you become a bit more familiar with with manipulations with complex numbers in cartesian form, polar form, properties of $e^{i\theta}$ and simplifications when you only want the magnitude just to mention a few things. You will note that some questions are from past class tests which were either in week 11 or were in the winter exam weeks.

1. Let $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $w = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ be complex numbers. Compute zw and z/w. Express your answer in the form $x+iy$ $(x, y \in \mathbb{R})$ and in polar form.

Solution

$$zw = 16\exp\left(i\frac{5\pi}{6}\right), \quad z/w = \frac{1}{4}\exp\left(-i\frac{\pi}{2}\right) = -\frac{i}{4}.$$

To write zw in $x+iy$ form note that

$$\exp\left(i\frac{5\pi}{6}\right) = -\exp\left(-i\frac{\pi}{6}\right) = -\cos(\pi/6) + i\sin(\pi/6) = -\frac{\sqrt{3}}{2} + i\frac{1}{2}.$$

Thus

$$zw = 8(-\sqrt{3} + i).$$

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2. Determine the real and imaginary parts of the following

$$\text{(a)}\ \frac{1+i}{1-i}, \qquad \text{(b)}\ \frac{1-i}{1+i}, \qquad \text{(c)}\ (1+i)^8, \qquad \text{(d)}\ \left(\frac{1}{1+i} - \frac{1}{1-i}\right)^2.$$

Solution

(a)

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i^2+2i}{2} = i.$$

The real part is 0 and the imaginary part is 1.

(b)

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{2} = -i.$$

The real part is 0 and the imaginary part is -1.

(c)

$$(1+i)^2 = 1 + i^2 + 2i = 2i, \quad (1+i)^4 = (2i)^2 = -4, \quad (1+i)^8 = (-4)^2 = 16.$$

Alternatively you may have noted that the magnitude is $\sqrt{2}$ and the argument is $\pi/4$ and

$$(1+i)^8 = \left(\sqrt{2}\exp(i\pi/4)\right)^8 = 2^4\exp(2\pi i) = 16.$$

The real part is 16 and the imaginary part is 0.

(d)

$$\frac{1}{1+i} = \frac{1-i}{2}, \quad \frac{1}{1-i} = \frac{1+i}{2}, \quad \frac{1}{1+i} - \frac{1}{1-i} = -i.$$

Thus the result is $(-i)^2 = -1$.

The real part is -1 and the imaginary part is 0.

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3. *This was Q1 of the December 2022 class test. It was work 15 of the 100 marks on the class test.*

   Let

   $$f(z) = \frac{4}{4-z}.$$

   (a) Determine $f(1)$, $f(i)$ and $f(-i)$ expressing the value in cartesian form.

   (b) Let $\theta \in \mathbb{R}$. Explain why

   $$1 + \left(\frac{e^{i\theta}}{4}\right) + \left(\frac{e^{i\theta}}{4}\right)^2 + \cdots + \left(\frac{e^{i\theta}}{4}\right)^n + \cdots = f\left(e^{i\theta}\right).$$

   Further explain why the real part of $f\left(e^{i\theta}\right)$ is

   $$\frac{16 - 4\cos(\theta)}{17 - 8\cos(\theta)}.$$

Solution

(a)

$$f(1) = \frac{4}{3}.$$

$$f(i) = \frac{4}{4-i} = \frac{4(4+i)}{17}.$$

$$f(-i) = \frac{4}{4+i} = \frac{4(4-i)}{17}.$$

(b) Let $w = \left(e^{i\theta}\right)/4$. Now $|w| = 1/4 < 1$ and we have a convergent geometric series

$$\begin{aligned} 1 + w + w^2 + \cdots + w^n + \cdots &= \frac{1}{1-w} \\ &= \frac{1}{1 - e^{i\theta}/4} \\ &= \frac{4}{4 - e^{i\theta}} = f\left(e^{i\theta}\right). \end{aligned}$$

Let $c = \cos(\theta)$ and $s = \sin(\theta)$.

$$\begin{aligned} f\left(\mathrm{e}^{i\theta}\right) &= \frac{4}{(4-c)-is} \\ &= \frac{4((4-c)+is)}{(4-c)^2+s^2}. \end{aligned}$$

Now for the denominator

$$(4-c)^2 + s^2 = 16 - 8c + c^2 + s^2 = 17 - 8c, \quad \text{using } c^2 + s^2 = 1.$$

The real part of the numerator is $4(4-c) = 16 - 4c$ and we have shown the result.

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4. *This was Q1 of the December 2021 class test. The test was on-campus but close to the time of the test there was a possibility that we may have had to switch an online at-home exam.*

   In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 1, 2, 3, 4 then

$$f(z) = \frac{2z-1}{z-1}$$

   whilst if the last digit is one of the digits 5, 6, 7, 8, 9 then

$$f(z) = \frac{3z+1}{z+1}.$$

   (i) For your version of $f(z)$ indicate which of the following is not defined and give in cartesian form the complex number in the other three cases.

$$f(1), \quad f(-1), \quad f(i) \quad \text{and} \quad f(-i).$$

   (ii) For your version of $f(z)$ show that the real part is a constant for values of the form $z = \mathrm{e}^{it}$, $0 \le t < 2\pi$ for which you are able to evaluate $f(z)$.

Solution

This is the version if the last digit is one of the digits of 0, 1, 2, 3, 4.

(i) $f(z)$ is not defined when $z = 1$.

$$f(-1) = \frac{-3}{-2} = \frac{3}{2}.$$

$$f(i) = \frac{-1+2i}{-1+i} = \frac{(-1+2i)(-1-i)}{2} = \frac{3-i}{2}.$$

$$f(-i) = \frac{-1-2i}{-1-i} = \frac{(-1-2i)(-1+i)}{2} = \frac{3+i}{2}.$$

(ii) Let $c = \cos(t)$ and $s = \sin(t)$.

$$f(\mathrm{e}^{it}) = \frac{2(c+is)-1}{c+is-1}.$$

Now

$$|c+is-1|^2 = (c-1)^2 + s^2 = c^2 - 2c + 1 + s^2 = 2 - 2c$$

as $c^2 + s^2 = 1$. Thus

$$f(\mathrm{e}^{it}) = \frac{(2c-1) + 2is)((c-1) - is)}{2-2c}.$$

The real part is

$$\begin{aligned} \frac{(2c-1)(c-1) + 2s^2}{2-2c} &= \frac{2c^2 - 3c + 1 + 2s^2}{2-2c} \\ &= \frac{3-3c}{2-2c} = \frac{3}{2}, \end{aligned}$$

using again $c^2 + s^2 = 1$.

This is the version if the last digit is one of the digits of 5, 6, 7, 8, 9.

(i) $f(z)$ is not defined when $z = -1$.

$$f(1) = \frac{4}{2} = 2.$$

$$f(i) = \frac{1+3i}{1+i} = \frac{(1+3i)(1-i)}{2} = \frac{4+2i}{2} = 2+i.$$

$$f(-i) = \frac{1-3i}{1-i} = \frac{(1-3i)(1+i)}{2} = \frac{4-2i}{2} = 2-i.$$

(ii) Let $c = \cos(t)$ and $s = \sin(t)$.

$$f(\mathrm{e}^{it}) = \frac{3(c+is)+1}{c+is+1}.$$

Now

$$|c+is+1|^2 = (c+1)^2 + s^2 = c^2 + 2c + 1 + s^2 = 2 + 2c$$

as $c^2 + s^2 = 1$. Thus

$$f(\mathrm{e}^{it}) = \frac{(3c+1) + 3is)((c+1) - is)}{2+2c}.$$

The real part is

$$\begin{aligned} \frac{(3c+1)(c+1) + 3s^2}{2+2c} &= \frac{3c^2 + 4c + 1 + 3s^2}{2+2c} \\ &= \frac{4+4c}{2+2c} = 2, \end{aligned}$$

using again $c^2 + s^2 = 1$.

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5. *This was Q1 of the January 2021 class test. The exam was an online at-home exam due to covid.*

In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits $0, 1, 2, 3, 4$ then

$$f(z) = \frac{2z-1}{z-2}$$

whilst if the last digit is one of the digits $5, 6, 7, 8, 9$ then

$$f(z) = \frac{1+3z}{3+z}.$$

(a) For your version of $f(z)$ give in cartesian form the following complex numbers. $f(1)$, $f(-1)$, $f(i)$ and $f(-i)$.

(b) For your version of $f(z)$ determine z such that

$$f(z) = i.$$

You need to express z in cartesian form.

(c) For your version of $f(z)$ show that $|f(\mathrm{e}^{i\theta})| = 1$ when $\theta \in \mathbb{R}$.

Solution

This is the version for a last digit of $0, 1, 2, 3, 4$.

(a)

$$f(1) = 1/(-1) = -1, \quad f(-1) = -3/(-3) = 1.$$

$$f(i) = \frac{-1+2i}{-2+i} = \frac{(-1+2i)(-2-i)}{5} = \frac{4-3i}{5}.$$

$$f(-i) = \frac{-1-2i}{-2-i} = \frac{(-1-2i)(-2+i)}{5} = \frac{4+3i}{5} = \overline{f(i)}.$$

(b)

$$\frac{2z-1}{z-2} = i, \quad 2z-1 = iz-2i, \quad (2-i)z = 1-2i$$

$$z = \frac{1-2i}{2-i} = \frac{(1-2i)(2+i)}{5} = \frac{4-3i}{5}.$$

(c) Let $c = \cos(\theta)$ and $s = \sin(\theta)$. For the numerator in $f(z)$ we have

$$2z-1 = 2c-1+i(2s), \quad |2z-1|^2 = (2c-1)^2+4s^2 = 4c^2-4c+1+4s^2 = 5-4c.$$

For the denominator in $f(z)$ we have

$$z-2 = c-2+is, \quad |z-2|^2 = (c-2)^2+s^2 = c^2-4c+4+s^2 = 5-4c.$$

The numerator and denominator have the same magnitude when $z = \mathrm{e}^{i\theta}$ and hence $|f(z)| = 1$.

This is the version for a last digit of $5, 6, 7, 8, 9$.

(a)

$$f(1) = 4/4 = 1, \quad f(-1) = -2/2 = -1.$$

$$f(i) = \frac{1+3i}{3+i} = \frac{(1+3i)(3-i)}{10} = \frac{6+8i}{10} = \frac{3+4i}{5}.$$

$$f(-i) = \frac{1-3i}{3-i} = \frac{(1-3i)(3+i)}{10} = \frac{6-8i}{10} = \frac{3-4i}{5} = \overline{f(i)}.$$

(b)

$$\frac{1+3z}{3+z} = i, \quad 1+3z = i(3+z), \quad (3-i)z = -1+3i.$$

$$z = \frac{-1+3i}{3-i} = \frac{(-1+3i)(3+i)}{10} = \frac{-6+8i}{10} = \frac{-3+4i}{5}.$$

(c) Let $c = \cos(\theta)$ and $s = \sin(\theta)$. For the numerator in $f(z)$ we have

$$1+3z = 1+3c+i(3s), \quad |1+3z|^2 = (1+3c)^2+(3s)^2 = 1+6c+9c^2+9s^2 = 10+6c.$$

For the denominator in $f(z)$ we have

$$3+z = 3+c+is, \quad |3+z|^2 = (3+c)^2+s^2 = 9+6c+c^2+s^2 = 10+6c.$$

The numerator and denominator have the same magnitude when $z = \mathrm{e}^{i\theta}$ and hence $|f(z)| = 1$.

6. *This was Q1 of the December 2019 class test.*

For $z \in \mathbb{C}$ with $z \neq i$ let $f(z)$ be defined by

$$f(z) = \frac{z-1}{i-z}.$$

(a) Give in cartesian form the following complex numbers: $f(1)$, $f(-1)$ and $f(-i)$.

(b) Determine z in the form $z = x + iy$, $x, y \in \mathbb{R}$, such that $f(z) = 1 - i$.

(c) Let $t \in \mathbb{R}$. If $f(z) = t(1-i)$ then show that $|z| = 1$.

Solution

(a) $f(1) = 0$.

$$f(-1) = \frac{-2}{i+1} = \frac{-2(1-i)}{2} = -1 + i.$$

$$f(-i) = \frac{-i-1}{2i} = \frac{-1+i}{2}.$$

(b) $f(z) = 1 - i$ gives

$$z - 1 = (1-i)(i-z) = (i+1) - (1-i)z \quad \text{and thus} \quad (2-i)z = i + 2.$$

$$z = \frac{2+i}{2-i} = \frac{(2+i)(2+i)}{5} = \frac{3+4i}{5}.$$

(c) $f(z) = t(1-i)$ gives

$$z-1 = t(1-i)(i-z) = t((i+1)-(1-i)z) \quad \text{and thus} \quad z(1+t(1-i)) = 1+t(i+1).$$

Hence

$$z = \frac{1+t+it}{1+t-it}.$$

As the numerator is the complex conjugate of the denominator they have the same magnitude and thus $|z| = 1$.

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7. *This was Q1 of the December 2018 class test.*

For $z \in \mathbb{C}$ and $z \neq i$ let the function f be defined by

$$f(z) = \frac{i}{i-z}.$$

Determine the real and imaginary parts of the following.

(a) $f(1)$.

(b) $f(-1)$.

(c) $f(-i)$.

For $0 < \theta < 2\pi$ show that

$$f(\mathrm{e}^{i\theta+i\pi/2}) = \frac{1}{2}\left(1 + i\cot\left(\frac{\theta}{2}\right)\right).$$

Solution

(a)
$$f(1) = \frac{i}{i-1} = \frac{i(-i-1)}{2} = \frac{1}{2} - i\frac{1}{2}.$$
The real part is $1/2$ and the imaginary part is $-1/2$.

(b)
$$f(-1) = \frac{i}{i+1} = \frac{i(1-i)}{2} = \frac{1}{2} + i\frac{1}{2}.$$
The real part is $1/2$ and the imaginary part is $1/2$.

(c)
$$f(-i) = \frac{i}{2i} = \frac{1}{2}.$$
The real part is $1/2$ and the imaginary part is 0.

$i = \mathrm{e}^{i\pi/2}$ and thus

$$\mathrm{e}^{i\theta+i\pi/2} = i\mathrm{e}^{i\theta}, \quad f(\mathrm{e}^{i\theta+i\pi/2}) = \frac{i}{i - i\mathrm{e}^{i\theta}} = \frac{1}{1-\mathrm{e}^{i\theta}}$$

By multiplying numerator and denominator by $\mathrm{e}^{-i\theta/2}$ we get

$$\frac{\mathrm{e}^{-i\theta/2}}{\mathrm{e}^{-i\theta/2} - \mathrm{e}^{i\theta/2}}.$$

In terms of sines and cosines we have

$$\frac{\cos(\theta/2) - i\sin(\theta/2)}{-2i\sin(\theta/2)}$$

and this can be re-written as

$$\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)} = \frac{1}{2}\left(1 + i\cot\left(\frac{\theta}{2}\right)\right).$$

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8. *This was Q1 of the December 2017 class test.*

Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = 2 + \frac{3}{z-2}.$$

(a) Express in cartesian form each of the following.

i. $f(1)$.
ii. $f(i)$.
iii. $f(-i)$.
iv. $f(\mathrm{e}^{i\theta})$, where $\theta \in \mathbb{R}$.

(b) Show that

$$(5 - 4\cos\,\theta)^2 = (4 - 5\cos\,\theta)^2 + 9\sin^2\theta.$$

Hence, or otherwise, determine $|f(\mathrm{e}^{i\theta})|$.

Solution

(a)

$$f(1) = 2 + \frac{3}{1-2} = 2 - 3 = -1.$$

(b)

$$f(i) = 2 + \frac{3}{i-2} = 2 + \frac{3(-2-i)}{5} = \frac{4}{5} - \frac{3}{5}i.$$

(c)

$$f(i) = 2 + \frac{3}{-i-2} = 2 + \frac{3(-2+i)}{5} = \frac{4}{5} + \frac{3}{5}i.$$

(d)

$$f(\mathrm{e}^{i\theta}) = 2 + \frac{3}{\mathrm{e}^{i\theta} - 2} = 2 + \frac{3(-2 + \mathrm{e}^{-i\theta})}{|-2 + \mathrm{e}^{i\theta}|^2}.$$

Now for the term in the denominator and with $c = \cos\,\theta$ and $s = \sin\,\theta$ we have

$$|-2 + \mathrm{e}^{i\theta}|^2 = (-2 + c)^2 + s^2 = 4 - 4c + c^2 + s^2 = 5 - 4c.$$

Hence

$$f(\mathrm{e}^{i\theta}) = \frac{2(5-4c) + 3(-2 + c - is)}{5 - 4c} = \frac{(4 - 5c) - 3si}{5 - 4c}.$$

For the magnitude squared of the numerator we have

$$\begin{aligned} (4-5c)^2 + 9s^2 &= 16 - 40c + 25c^2 + 9s^2 \\ &= (16 + 9) - 40c + 16c^2 = 25 - 40c + 16c^2 = (5 - 4c)^2. \end{aligned}$$

Thus the numerator and denominator have the same magnitude which implies that

$$|f(\mathrm{e}^{i\theta})| = 1.$$

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9. *This was Q2 of the December 2022 class test. It was work 6 of the 100 marks on the class test.*

Give in polar form all solutions of

$$z^6 = 8i.$$

Indicate all the solutions which are also in the following set of 4 complex numbers.

$$\{-1-i,\, 1-i,\, 1+i,\, -1+i\}\,.$$

Solution

In polars the right hand side is

$$8i = 8\mathrm{e}^{i\pi/2} = 2^3\mathrm{e}^{i\pi/2}.$$

The principal solution is

$$z_0 = 2^{1/2}\mathrm{e}^{i\pi/12}.$$

All 6 solutions are equally spaced on the circle with centre 0 and radius $\sqrt{2}$ and are given by

$$z_k = \sqrt{2}\exp\left(i\frac{\pi}{12} + k\frac{\pi}{3}\right), \quad k = 0, 1, \ldots, 5.$$

All 4 points in the set given have magnitude of $\sqrt{2}$ and the angles are $-3\pi/4$, $-\pi/4$, $\pi/4$ and $3\pi/4$.

The angle of z_2 is

$$\frac{\pi}{12} + 2\frac{\pi}{3} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

and hence this is in the set. $z_5 = -z_2$ is also in the set. The other 4 solutions are not in the set.

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10. *This was Q2 of the December 2021 class test.*

    In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 2, 4, 6, 8 then you do part (a) whilst if it is one of the digits 1, 3, 5, 7, 9 then you do part (b).

    (a) This is the version if the last digit is one of the digits 0, 2, 4, 6, 8.
    Give in polar form all solutions of

    $$z^{11} = 1 + i,$$

    which are in the quadrant corresponding to $z = x + iy$ with $x > 0$ and $y > 0$.

    (b) This is the version if the last digit is one of the digits of 1, 3, 5, 7, 9.
    Give in polar form all solutions of

    $$z^9 = 1 - i,$$

    which are in the quadrant corresponding to $z = x + iy$ with $x > 0$ and $y > 0$.

Solution

(a) This is the version if the last digit is one of the digits of 0, 2, 4, 6, 8.
With $z = r\mathrm{e}^{i\theta}$, $r > 0$ and $\theta \in \mathbb{R}$, we first put the right hand side in polar form so that we have

$$z^{11} = r^{11}\mathrm{e}^{11i\theta} = 1 + i = \sqrt{2}\mathrm{e}^{i\pi/4}.$$

One solution is

$$z_0 = 2^{1/22}\mathrm{e}^{i\pi/44}.$$

z_0 is in the first quadrant. The spacing in the angles is $2\pi/11$. In total there are three solutions in the first quadrant which are

$$z_k = 2^{1/22}\exp\left(i\left(\frac{\pi}{44} + \frac{2k\pi}{11}\right)\right), \quad k = 0, 1, 2.$$

(b) This is the version if the last digit is one of the digits of 1, 3, 5, 7, 9.
With $z = r\mathrm{e}^{i\theta}$, $r > 0$ and $\theta \in \mathbb{R}$, we first put the right hand side in polar form so that we have

$$z^{9} = r^{9}\mathrm{e}^{9i\theta} = 1 - i = \sqrt{2}\mathrm{e}^{-i\pi/4}.$$

One solution is

$$z_0 = 2^{1/18}\mathrm{e}^{-i\pi/36}.$$

z_0 is not in the first quadrant. The spacing in the angles is $2\pi/9$. In total there are two solutions in the first quadrant which are

$$z_k = 2^{1/18}\exp\left(i\left(-\frac{\pi}{36} + \frac{2k\pi}{9}\right)\right), \quad k = 1, 2.$$

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11. *This was Q2 of the January 2021 class test.*

In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits $0, 2, 4, 6, 8$ then you do part (a) whilst if it is one of the digits $1, 3, 5, 7, 9$ then you do part (b).

(a) The version when your last digit is one of $0, 2, 4, 6, 8$.
Give in polar form all solutions of

$$z^5 = i.$$

In each case give the principal argument of each solution.
In this problem if ζ is a solution then $-\overline{\zeta}$ is also a solution. In all cases where these two points are different indicate this correspondence for all the solutions given in the previous part.

(b) The version when your last digit is one of $1, 3, 5, 7, 9$.
Give in polar form all solutions of

$$z^6 = i.$$

In each case give the principal argument of each solution.
In this problem if ζ is a solution then $-\zeta$ is also a solution. Indicate the correspondence for all the solutions given in the previous part.

Solution

(a) This is the version for a last digit of $0, 2, 4, 6, 8$.

$$i = \exp(i\pi/2).$$

One solution of $z^5 = i$ is $z_0 = \exp(i\pi/10)$. All solutions are given by

$$z_k = \exp(i\pi/10 + i2k\pi/5), \quad k = -2, -1, 0, 1, 2.$$

The principal arguments of the solutions are

$$\frac{\pi}{10} - \frac{4\pi}{5} = -\frac{7\pi}{10}, \quad \frac{\pi}{10} - \frac{2\pi}{5} = -\frac{3\pi}{10}, \quad \frac{\pi}{10}, \quad \frac{5\pi}{10} = \frac{\pi}{2}, \quad \frac{9\pi}{10}.$$

If $\zeta = a + ib$, $a, b \in \mathbb{R}$ then $-\overline{\zeta} = -a + ib$ is the reflection in the imaginary axis.
$z_1 = i$ is on the imaginary axis.
z_2 is the reflection of z_0, i.e. $z_2 = -\overline{z_0}$.
z_{-1} is the reflection of z_{-2}, i.e. $z_{-1} = -\overline{z_{-2}}$.

(b) This is the version for a last digit of $1, 3, 5, 7, 9$.

$$i = \exp(i\pi/2).$$

One solution of $z^6 = i$ is $z_0 = \exp(i\pi/12)$. All solutions are given by

$$z_k = \exp(i\pi/12 + i2k\pi/6), \quad k = -3, -2, -1, 0, 1, 2.$$

The principal arguments of the solutions are

$$\frac{\pi}{12} - \frac{6\pi}{6} = -\frac{11\pi}{12}, \quad \frac{\pi}{12} - \frac{4\pi}{6} = -\frac{7\pi}{12}, \quad -\frac{3\pi}{12} = -\frac{\pi}{4}, \quad \frac{\pi}{12}, \quad \frac{5\pi}{12}, \quad \frac{9\pi}{12} = \frac{3\pi}{4}.$$

$z_{-3} = -z_0$, $z_{-2} = -z_1$, $z_{-1} = -z_2$.

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12. *This was Q2 of the December 2019 class test.*

Give in polar form all solutions of $z^8 = 256$.

Give in cartesian form all solutions of $z^8 = 256$ which have a negative real part.

Solution

$z = 2$ is one solution of $z^8 = 256$. All the solutions are

$$2\exp(2k\pi i/8), \quad k = 0, 1, \ldots, 7.$$

The solutions with negative real part, which includes the point -2, correspond to $= 3, 4, 5$. $k = 4$ is the point -2. The other 2 points are

$$\begin{aligned} 2\exp(6\pi i/8) &= 2\cos(3\pi/4) + i2\sin(3\pi/4) = \sqrt{2}(-1+i), \\ 2\exp(10\pi i/8) &= 2\cos(5\pi/4) + i2\sin(5\pi/4) = \sqrt{2}(-1-i). \end{aligned}$$

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13. *This was Q2 of the December 2018 class test.*

Determine in polar form the 9 solutions of

$$z^9 = i$$

and indicate which solutions, if any, are real or purely imaginary. Also give the number of solutions which have a principal argument in the interval $(-\pi, -\pi/2)$.

Solution

In polar form

$$i = e^{i\pi/2}.$$

One solution is $e^{i\pi/18}$. The 9 solutions are uniformly spaced around the unit circle and in polar form are described by

$$z_k = \exp\left(\frac{\pi}{18} + \frac{2k\pi}{9}\right), \quad k = 0, 1, \ldots, 8.$$

A purely imaginary solution by inspection is $z_2 = i$. There are no other real or purely imaginary solutions.

The number of solutions in the interval $(-\pi, -\pi/2)$ is 2. In terms of the labelling above these are z_5 and z_6.

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14. *This was Q2 of the December 2016 class test.*

    Find all solution to
    $$z^7 = -128$$
    and indicate which solutions, if any, are real.

Solution

In polar form
$$-128 = 2^7(\cos(\pi) + i\sin(\pi)).$$
The 7 solutions are
$$z_k = 2(\cos(\theta_k) + i\sin(\theta_k)), \quad \theta_k = \frac{\pi}{7} + \frac{2k\pi}{7}, \quad k = 0, 1, 2, 3, 4, 5, 6.$$
There is one real solution which is $z_3 = -2$.

---

15. Let
    $$f(z) = \frac{z-i}{z+i}.$$
    Determine $f(-1)$, $f(0)$ and $f(1)$ and determine $|f(z)|$ when z is real.

Solution

$$f(-1) = \frac{-1-i}{-1+i} = \frac{(-1-i)(-1-i)}{2} = \frac{(1+i)^2}{2} = i.$$
$$f(0) = \frac{-i}{i} = -1.$$
$$f(1) = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = -i.$$
If x is real then $x - i$ is the complex conjugate of $x + i$ and thus both values have the same magnitude and thus $|f(x)| = 1$.

---

16. Let z, z_1 and z_2 be complex numbers. Prove the following involving the complex conjugate operation.
    (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
    (b) $\overline{z_1 z_2} = (\overline{z_1})\,(\overline{z_2})$.
    (c) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$.
    (d) $\overline{z^n} = (\overline{z})^n$.

Solution

Let $z = r\mathrm{e}^{i\theta}$ with r and θ being real and similarly let $z_1 = x_1 + iy_1 = r_1\mathrm{e}^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2\mathrm{e}^{i\theta_2}$ with x_1, y_1, r_1, θ_1 and x_2, y_2, r_2, θ_2 all being real.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= r_1\mathrm{e}^{i\theta_1} r_2\mathrm{e}^{i\theta_2} = r_1 r_2 \mathrm{e}^{i(\theta_1+\theta_2)}, \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\mathrm{e}^{i(\theta_1-\theta_2)}, \\ z^n &= r^n \mathrm{e}^{in\theta}. \end{aligned}$$

Thus

$$\begin{aligned} \overline{z_1 + z_2} &= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}, \\ \overline{z_1 z_2} &= = r_1 r_2 \mathrm{e}^{-i(\theta_1+\theta_2)} = r_1\mathrm{e}^{-i\theta_1} r_2\mathrm{e}^{-i\theta_2} = (\overline{z_1})\,(\overline{z_2}), \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{r_1}{r_2}\mathrm{e}^{-i(\theta_1-\theta_2)} = \frac{r_1\mathrm{e}^{-i\theta_1}}{r_2\mathrm{e}^{-i\theta_2}} = \frac{\overline{z_1}}{\overline{z_2}}, \\ \overline{z^n} &= r^n\mathrm{e}^{-in\theta} = (r\mathrm{e}^{-i\theta})^n = (\overline{z})^n. \end{aligned}$$

---

17. Let $p_n(z)$ be a polynomial of degree n with real coefficients, i.e.

$$\begin{gathered} p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \\ a_k \in \mathbb{R},\ k = 0, 1, \ldots, n \quad a_n \neq 0. \end{gathered}$$

Prove that

$$p_n(\overline{z}) = \overline{p_n(z)}.$$

What does this imply about the non-real roots of this polynomial?

Solution

$$\begin{aligned} \overline{p_n(z)} &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0}, && \text{by question 16a}, \\ &= a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1\overline{z} + \overline{a_0}, && \text{by question 16b and 16d and } a_k \text{ is real}, \\ &= p_n(\overline{z}). \end{aligned}$$

If $p_n(\zeta) = 0$ then $p_n(\overline{\zeta}) = 0$ and hence non-real roots must occur in complex conjugate pairs.

---

18. Given that $z = 1 - i$ is one root of the polynomial equation

$$z^4 + 4z^3 - 8z + 20 = 0,$$

find all other roots in $\mathbb{C}$.

Solution

The polynomial has real coefficients and as $\zeta = 1 - i$ is a root the complex conjugate $\overline{\zeta} = 1 + i$ must also be a root and hence the following is a factor

$$(z - \zeta)(z - \overline{\zeta}) = (z - 1 + i)(z - 1 - i) = z^2 - 2z + 2.$$

The other factor must be of the form $z^2 + az + 10$ in order to match the z^4 and constant term, i.e.

$$z^4 + 4z^3 - 8z + 20 = (z^2 + az + 10)(z^2 - 2z + 2).$$

By equating the z^3 coefficients gives $4 = -2 + a$, i.e. $a = 6$.

$$z^2 + 6z + 10 = 0 \quad \text{gives } z = -3 \pm i.$$

For information, there is a command in Matlab for finding the roots of a polynomial which in this case just involves the following statements.

```
a=[1, 4, 0, -8, 20];
roots(a)
```

The output generated is as follows

```
ans =
  -3.0000 + 1.0000i
  -3.0000 - 1.0000i
   1.0000 + 1.0000i
   1.0000 - 1.0000i
```

confirming that the above is correct.

---

19. Do the following.

   (a) Determine in $x + iy$ form $(x, y \in \mathbb{R})$ the 6 solutions of $z^6 = 64$.

   (b) Determine in polar form the 6 solutions of $z^6 = 1 + i$.

Solution

(a) Possibly the easiest way to start this is to note that the right hand side is 2^6 and thus the 6 solutions are just

$$2, \quad 2\omega, \quad 2\omega^2, \quad 2\omega^3, \quad 2\omega^4, \quad 2\omega^5$$

where $\omega = \exp(2\pi i/6)$ is a root of unity. To put in $x + iy$ form you can then note that $\omega^3 = -1$ and the non-real roots occur in complex conjugate pairs, i.e. the solutions are

$$2, \quad 2\omega, \quad 2\omega^2, \quad -2, \quad 2\overline{\omega}^2, \quad 2\overline{\omega}.$$

You might also note that if z is a solution then so is $-z$ and thus in particular

$$\omega^2 = -\omega^5 = -\overline{\omega}.$$

Now

$$\omega = \exp(2\pi i/6) = \cos(2\pi/6) + i\sin(2\pi/6) = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Thus the 6 solutions are uniformly spaced around the circle $|z| = 2$ and are given concisely in the form

$$\pm 2, \quad \pm(1 \pm i\sqrt{3}).$$

(b) The first step is to express the number $1 + i$ in polar form.

$$1 + i = re^{i\theta}, \quad r = \sqrt{2} = 2^{1/2}, \quad \theta = \frac{\pi}{4}.$$

The 6 solutions are

$$2^{1/12}e^{i\theta_k}, \quad \theta_k = \frac{\pi}{24} + \frac{2k\pi}{6}, \quad k = 0, 1, 2, 3, 4, 5.$$

---

20. Let z_0 and z be complex numbers and let

$$w = \frac{z - z_0}{1 - \overline{z_0}z}.$$

Show that if $|z_0| \neq 1$ and $|z| = 1$ then $|w| = 1$. [Hint: Make use of the operations using the complex conjugate and the absolute value.]

Solution

Now as $|z|^2 = z\overline{z} = 1$,

$$z - z_0 = z\left(1 - \frac{1}{z}z_0\right) = z\left(1 - \overline{z}z_0\right).$$

Taking the absolute value and using $|z| = 1$ gives

$$|z - z_0| = |1 - \overline{z}z_0|.$$

If $\zeta = 1 - \overline{z}z_0$ then $|\zeta| = |\overline{\zeta}|$ and thus

$$|z - z_0| = |1 - z\overline{z_0}|.$$

Hence $|w| = 1$.

---

21. Let z_1 and z_2 be non-zero complex numbers with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$. Show that in this case

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg} z_1 + \mathrm{Arg} z_2.$$

Is this true for any points z_1 and z_2 in $\mathbb{C}$? Explain your answer.

Solution

Let $z_1 = r_1 \mathrm{e}^{i\theta_1}$ and $z_2 = r_2 \mathrm{e}^{i\theta_2}$ with $\theta_1 = \mathrm{Arg} z_1$, $\theta_2 = \mathrm{Arg} z_2$.

$$z_1 z_2 = r_1 r_2 \exp(i(\theta_1 + \theta_2)).$$

As $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$ we have $|\theta_1| < \pi/2$ and $|\theta_2| < \pi/2$. Thus $-\pi < \theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_2$ is the principal argument of $z_1 z_2$.

The result is not true for all points z_1 and z_2 in $\mathbb{C}$. For example, if we take $z_1 = z_2 = -1 + i$ then $\mathrm{Arg} z_1 = \mathrm{Arg} z_2 = 3\pi/4$. In this case

$$\mathrm{Arg} z_1 + \mathrm{Arg} z_2 = \frac{3\pi}{2} > \pi$$

but

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(-2i) = -\frac{\pi}{2}.$$

$\mathrm{Arg} z_1 + \mathrm{Arg} z_2$ and $\mathrm{Arg}(z_1 z_2)$ differ by 2π.

---

22. Let z_1 and z_2 be complex numbers. The triangle inequality is

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

When do we have $||z_1| - |z_2|| = |z_1 + z_2|$ and when do we have $|z_1 + z_2| = |z_1| + |z_2|$?

Solution

Let $z_1 = r_1 \mathrm{e}^{i\theta_1}$ and $z_2 = r_2 \mathrm{e}^{i\theta_2}$. As was shown in the lectures

$$|z_1 + z_2|^2 = r_1^2 + 2 r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2.$$

Now as

$$-1 \le \cos(\theta_1 - \theta_2) \le 1$$

the extreme values occur when the cosine term is -1 or 1. The largest value is when $\theta_1 - \theta_2 = 0$ (or any multiple of 2π), i.e. when z_1 and z_2 are on the same radial line, and the smallest value is when $\theta_1 - \theta_2 = \pi$, i.e. when z_1 and z_2 are on opposite radial lines. When $\cos(\theta_1 - \theta_2) = 1$

$$|z_1 + z_2|^2 = r_1^2 + 2 r_1 r_2 + r_2^2 = (r_1 + r_2)^2 = (|z_1| + |z_2|)^2.$$

When $\cos(\theta_1 - \theta_2) = -1$

$$|z_1 + z_2|^2 = r_1^2 - 2 r_1 r_2 + r_2^2 = (r_1 - r_2)^2 = (|z_1| - |z_2|)^2.$$

---

23. Compute the following.

    (a) $(2+i)(3+i)$.

    (b) $(1+i)(5-i)^4$.

    Hence show that

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right),\\ &= 4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right).\end{aligned}$$

Solution

$$(2+i)(3+i) = 5+5i = 5(1+i).$$

All the numbers have positive real parts and by the previous question

$$\mathrm{Arg}(2+i)+\mathrm{Arg}(3+i) = \mathrm{Arg}(5+5i) = \frac{\pi}{4}.$$

When we have

$$z = x+iy, \quad \mathrm{Arg} z = \tan^{-1}\frac{y}{x}.$$

Hence

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

For the part involving $(5-i)^4$ note that

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

For the calculation consider the real and imaginary parts of $(5-i)^4$ separately.

$$\begin{aligned}\mathrm{Re}(5-i)^4 &= 5^4+6(5^2)(-1)+1 = 525-150+1 = 476,\\ \mathrm{Im}(5-i)^4 &= 4(5^3(-1)+5(1)) = -20(24).\end{aligned}$$

Thus

$$(5-i)^4 = 476-i(20)(24) = 4(119-i120)$$

and

$$(1+i)(5-i)^4 = 4(239-i).$$

All the numbers have positive real parts and by the previous question if we take the principal argument of this expression then we get

$$\frac{\pi}{4}-4\tan^{-1}\left(\frac{1}{5}\right) = -\tan^{-1}\left(\frac{1}{239}\right)$$

and the result follows. In the above we have used

$$\mathrm{Arg}(5-i)^4 = 4\mathrm{Arg}(5-i) = -4\mathrm{Arg}(5+i) = -4\tan^{-1}\left(\frac{1}{5}\right).$$

For information, Matlab handles complex numbers and the statements

```
format compact
0.2*(2+i)*(3+i)
0.25*(1+i)*(5-i)^4
```

generates the following output.

```
ans =
   1.0000 + 1.0000i
ans =
   2.3900e+02 - 1.0000e+00i
```

---

24. In previous study one of the standard integrals that you meet is

$$\int \mathrm{e}^{kx}\,\mathrm{d}x = \frac{\mathrm{e}^{kx}}{k} + \text{const.}$$

Given that this is true for $k \in \mathbb{C}$ (as well as $k \in \mathbb{R}$) and that

$$\mathrm{e}^{x+iy} = \exp(x+iy) = \exp(x)(\cos y + i \sin y)$$

obtain expressions for

$$\int \mathrm{e}^{px} \cos(qx)\,\mathrm{d}x \quad \text{and} \quad \int \mathrm{e}^{px} \sin(qx)\,\mathrm{d}x$$

where p and q are real.

Solution

To use the result given in the question we take $k = p + iq$ and to shorten the expressions we let $c = \cos(qx)$, $s = \sin(qx)$. Then

$$\frac{\mathrm{e}^{kx}}{k} = \mathrm{e}^{px}\left(\frac{c+is}{p+iq}\right) = \mathrm{e}^{px}\left(\frac{(p-iq)(c+is)}{p^2+q^2}\right) = \mathrm{e}^{px}\left(\frac{(pc+qs)+i(ps-qc)}{p^2+q^2}\right).$$

If we take the real and imaginary part then we get

$$\begin{aligned}
\int \mathrm{e}^{px} \cos(qx)\,\mathrm{d}x &= \mathrm{e}^{px}\left(\frac{p\cos(qx)+q\sin(qx)}{p^2+q^2}\right) + \text{constant}, \\
\int \mathrm{e}^{px} \sin(qx)\,\mathrm{d}x &= \mathrm{e}^{px}\left(\frac{p\sin(qx)-q\cos(qx)}{p^2+q^2}\right) + \text{constant}.
\end{aligned}$$

---

25. *Series is done in term 2 and what is asked here is very similar to what will be asked in term 2 in some exercises. Thus some of this will nearly be repeated. You only need techniques learned in year 2 to answer this question.*

Determine the largest open interval for x such that the following power series converges. In each case you must justify your answer.

(a)
$$\sum_{n=0}^{\infty} n^2 x^n.$$

(b)
$$\sum_{n=0}^{\infty} \frac{2n+1}{n!}(x+3)^n.$$

(c)
$$\sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n.$$

(d)
$$\sum_{n=0}^{\infty}(2+\sin(n))x^n.$$

Solution

(a) Let $a_n = n^2 x^n$.

$$\frac{a_{n+1}}{a_n} = x\frac{(n+1)^2}{n^2} = x\left(1+\frac{1}{n}\right)^2 \to x \quad \text{as } n \to \infty.$$

By the ratio test the series converges in $|x| < 1$ and diverges in $|x| > 1$ and thus $(-1, 1)$ is the largest open interval where the series converges.

(b) Let $a_n = (2n+1)/n!$ and let $b_n = a_n(x+3)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{2n+1}\left(\frac{1}{n+1}\right) = \frac{2+3/n}{2+1/n}\left(\frac{1}{n+1}\right) \to 0 \quad \text{as } n \to \infty.$$

Hence for all x
$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(x+3) \to 0 \quad \text{as } n \to \infty.$$
By the ratio test the series converges for all x, i.e. it converges in $(-\infty, \infty)$.

(c) Let now $a_n = n/2^n$ and let $b_n = a_n(x-1)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\left(\frac{1}{2}\right) \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Hence
$$\frac{|b_{n+1}|}{|b_n|} \to \frac{|x-1|}{2} \quad \text{as } n \to \infty.$$

By the ratio test the series converges when $|x-1| < 2$ and diverges when $|x-1| > 2$ and thus the largest open interval where it converges is $(-1, 3)$.

(d) Let now $a_n = 2 + \sin(n)$ and let $b_n = a_n x^n$. As $1 \le a_n \le 3$ we have

$$1 \le a_n^{1/n} \le 3^{1/n} \to 1 \quad \text{as } n \to \infty.$$

Hence

$$|b_n|^{1/n} \to |x| \quad \text{as } n \to \infty.$$

By the root test the series converges when $|x| < 1$ and diverges when $|x| > 1$ and thus the largest open interval is $(-1, 1)$.