Exercises related to the introduction chapter

The following are a collection of questions obtained from various sources which include some past MA3614 class tests that I have set, the book of Saff and Snider as well as questions that I have created myself. One of the hours each week will usually just be exercises and we can decide which is the most popular hour to use when we meet. All parts of this exercise sheet can be attempted from the start of the module although some parts may seem hard until you become a bit more familiar with with manipulations with complex numbers in cartesian form, polar form, properties of $e^{i\theta}$ and simplifications when you only want the magnitude just to mention a few things. You will note that some questions are from past class tests which were either in week 11 or were in the winter exam weeks.

1. Let $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $w = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ be complex numbers. Compute zw and z/w. Express your answer in the form $x+iy$ $(x, y \in \mathbb{R})$ and in polar form.

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2. Determine the real and imaginary parts of the following

$$\text{(a) } \frac{1+i}{1-i}, \qquad \text{(b) } \frac{1-i}{1+i}, \qquad \text{(c) } (1+i)^8, \qquad \text{(d) } \left(\frac{1}{1+i} - \frac{1}{1-i}\right)^2.$$

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3. *This was Q1 of the December 2022 class test. It was work 15 of the 100 marks on the class test.*

   Let
   $$f(z) = \frac{4}{4-z}.$$

   (a) Determine $f(1)$, $f(i)$ and $f(-i)$ expressing the value in cartesian form.

   (b) Let $\theta \in \mathbb{R}$. Explain why
   $$1 + \left(\frac{e^{i\theta}}{4}\right) + \left(\frac{e^{i\theta}}{4}\right)^2 + \cdots + \left(\frac{e^{i\theta}}{4}\right)^n + \cdots = f\left(e^{i\theta}\right).$$

   Further explain why the real part of $f\left(e^{i\theta}\right)$ is
   $$\frac{16 - 4\cos(\theta)}{17 - 8\cos(\theta)}.$$

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4. *This was Q1 of the December 2021 class test. The test was on-campus but close to the time of the test there was a possibility that we may have had to switch an online at-home exam.*

   In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 1, 2, 3, 4 then

$$f(z) = \frac{2z-1}{z-1}$$

   whilst if the last digit is one of the digits 5, 6, 7, 8, 9 then

$$f(z) = \frac{3z+1}{z+1}.$$

   (i) For your version of $f(z)$ indicate which of the following is not defined and give in cartesian form the complex number in the other three cases.

$$f(1), \quad f(-1), \quad f(i) \quad \text{and} \quad f(-i).$$

   (ii) For your version of $f(z)$ show that the real part is a constant for values of the form $z = \mathrm{e}^{it}$, $0 \le t < 2\pi$ for which you are able to evaluate $f(z)$.

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5. *This was Q1 of the January 2021 class test. The exam was an online at-home exam due to covid.*

   In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits $0, 1, 2, 3, 4$ then

$$f(z) = \frac{2z-1}{z-2}$$

   whilst if the last digit is one of the digits $5, 6, 7, 8, 9$ then

$$f(z) = \frac{1+3z}{3+z}.$$

   (a) For your version of $f(z)$ give in cartesian form the following complex numbers. $f(1)$, $f(-1)$, $f(i)$ and $f(-i)$.

   (b) For your version of $f(z)$ determine z such that

$$f(z) = i.$$

   You need to express z in cartesian form.

   (c) For your version of $f(z)$ show that $|f(\mathrm{e}^{i\theta})| = 1$ when $\theta \in \mathbb{R}$.

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6. *This was Q1 of the December 2019 class test.*

   For $z \in \mathbb{C}$ with $z \neq i$ let $f(z)$ be defined by

$$f(z) = \frac{z-1}{i-z}.$$

   (a) Give in cartesian form the following complex numbers: $f(1)$, $f(-1)$ and $f(-i)$.

   (b) Determine z in the form $z = x + iy$, $x, y \in \mathbb{R}$, such that $f(z) = 1 - i$.

   (c) Let $t \in \mathbb{R}$. If $f(z) = t(1-i)$ then show that $|z| = 1$.

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7. *This was Q1 of the December 2018 class test.*

   For $z \in \mathbb{C}$ and $z \neq i$ let the function f be defined by

$$f(z) = \frac{i}{i-z}.$$

   Determine the real and imaginary parts of the following.

   (a) $f(1)$.

   (b) $f(-1)$.

   (c) $f(-i)$.

   For $0 < \theta < 2\pi$ show that

$$f(\mathrm{e}^{i\theta + i\pi/2}) = \frac{1}{2}\left(1 + i\cot\left(\frac{\theta}{2}\right)\right).$$

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8. *This was Q1 of the December 2017 class test.*

   Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = 2 + \frac{3}{z-2}.$$

   (a) Express in cartesian form each of the following.

      i. $f(1)$.

      ii. $f(i)$.

      iii. $f(-i)$.

      iv. $f(\mathrm{e}^{i\theta})$, where $\theta \in \mathbb{R}$.

   (b) Show that

$$(5 - 4\cos\theta)^2 = (4 - 5\cos\theta)^2 + 9\sin^2\theta.$$

   Hence, or otherwise, determine $|f(\mathrm{e}^{i\theta})|$.

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9. *This was Q2 of the December 2022 class test. It was work 6 of the 100 marks on the class test.*

   Give in polar form all solutions of

   $$z^6 = 8i.$$

   Indicate all the solutions which are also in the following set of 4 complex numbers.

   $$\{-1-i,\, 1-i,\, 1+i,\, -1+i\}\,.$$

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10. *This was Q2 of the December 2021 class test.*

    In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits 0, 2, 4, 6, 8 then you do part (a) whilst if it is one of the digits 1, 3, 5, 7, 9 then you do part (b).

    (a) This is the version if the last digit is one of the digits 0, 2, 4, 6, 8.
    Give in polar form all solutions of

    $$z^{11} = 1 + i,$$

    which are in the quadrant corresponding to $z = x + iy$ with $x > 0$ and $y > 0$.

    (b) This is the version if the last digit is one of the digits of 1, 3, 5, 7, 9.
    Give in polar form all solutions of

    $$z^9 = 1 - i,$$

    which are in the quadrant corresponding to $z = x + iy$ with $x > 0$ and $y > 0$.

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11. *This was Q2 of the January 2021 class test.*

    In this question the version that you do depends on the last digit of your 7-digit student id.. If the last digit is one of the digits $0, 2, 4, 6, 8$ then you do part (a) whilst if it is one of the digits $1, 3, 5, 7, 9$ then you do part (b).

    (a) The version when your last digit is one of $0, 2, 4, 6, 8$.
    Give in polar form all solutions of

    $$z^5 = i.$$

    In each case give the principal argument of each solution.
    In this problem if ζ is a solution then $-\overline{\zeta}$ is also a solution. In all cases where these two points are different indicate this correspondence for all the solutions given in the previous part.

    (b) The version when your last digit is one of $1, 3, 5, 7, 9$.
    Give in polar form all solutions of

    $$z^6 = i.$$

    In each case give the principal argument of each solution.
    In this problem if ζ is a solution then $-\zeta$ is also a solution. Indicate the correspondence for all the solutions given in the previous part.

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12. *This was Q2 of the December 2019 class test.*

    Give in polar form all solutions of $z^8 = 256$.

    Give in cartesian form all solutions of $z^8 = 256$ which have a negative real part.

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13. *This was Q2 of the December 2018 class test.*

    Determine in polar form the 9 solutions of

    $$z^9 = i$$

    and indicate which solutions, if any, are real or purely imaginary. Also give the number of solutions which have a principal argument in the interval $(-\pi, -\pi/2)$.

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14. *This was Q2 of the December 2016 class test.*

    Find all solution to

    $$z^7 = -128$$

    and indicate which solutions, if any, are real.

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15. Let

    $$f(z) = \frac{z - i}{z + i}.$$

    Determine $f(-1)$, $f(0)$ and $f(1)$ and determine $|f(z)|$ when z is real.

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16. Let z, z_1 and z_2 be complex numbers. Prove the following involving the complex conjugate operation.

    (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

    (b) $\overline{z_1 z_2} = (\overline{z_1})\,(\overline{z_2})$.

    (c) $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$.

    (d) $\overline{z^n} = (\overline{z})^n$.

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17. Let $p_n(z)$ be a polynomial of degree n with real coefficients, i.e.

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$
$$a_k \in \mathbb{R},\ k = 0, 1, \ldots, n \quad a_n \neq 0.$$

Prove that

$$p_n(\overline{z}) = \overline{p_n(z)}.$$

What does this imply about the non-real roots of this polynomial?

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18. Given that $z = 1 - i$ is one root of the polynomial equation

$$z^4 + 4z^3 - 8z + 20 = 0,$$

find all other roots in $\mathbb{C}$.

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19. Do the following.

    (a) Determine in $x + iy$ form $(x, y \in \mathbb{R})$ the 6 solutions of $z^6 = 64$.

    (b) Determine in polar form the 6 solutions of $z^6 = 1 + i$.

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20. Let z_0 and z be complex numbers and let

$$w = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Show that if $|z_0| \neq 1$ and $|z| = 1$ then $|w| = 1$. [Hint: Make use of the operations using the complex conjugate and the absolute value.]

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21. Let z_1 and z_2 be non-zero complex numbers with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$. Show that in this case

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg} z_1 + \mathrm{Arg} z_2.$$

Is this true for any points z_1 and z_2 in $\mathbb{C}$? Explain your answer.

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22. Let z_1 and z_2 be complex numbers. The triangle inequality is

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

When do we have $||z_1| - |z_2|| = |z_1 + z_2|$ and when do we have $|z_1 + z_2| = |z_1| + |z_2|$?

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23. Compute the following.

    (a) $(2+i)(3+i)$.

    (b) $(1+i)(5-i)^4$.

    Hence show that

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right), \\ &= 4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right).\end{aligned}$$

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24. In previous study one of the standard integrals that you meet is

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + \text{const}.$$

Given that this is true for $k \in \mathbb{C}$ (as well as $k \in \mathbb{R}$) and that

$$e^{x+iy} = \exp(x+iy) = \exp(x)(\cos y + i\sin y)$$

obtain expressions for

$$\int e^{px}\cos(qx)\,dx \quad \text{and} \quad \int e^{px}\sin(qx)\,dx$$

where p and q are real.

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25. *Series is done in term 2 and what is asked here is very similar to what will be asked in term 2 in some exercises. Thus some of this will nearly be repeated. You only need techniques learned in year 2 to answer this question.*

    Determine the largest open interval for x such that the following power series converges. In each case you must justify your answer.

    (a)
$$\sum_{n=0}^{\infty} n^2 x^n.$$

    (b)
$$\sum_{n=0}^{\infty} \frac{2n+1}{n!}(x+3)^n.$$

    (c)
$$\sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n.$$

    (d)
$$\sum_{n=0}^{\infty} (2+\sin(n))x^n.$$
