Exercises involving contour integrals and trig integrals

At the end of the first week on chapter 5 about contour integrals questions 1 and 2 can be attempted. Many of the others which just need knowledge of an anti-derivative can also be considered although there will be more discussion about anti-derivatives and path independence in the lectures in the second week. In the second week there will also be material about loop integrals with rational functions.

Questions 3, 4, and 6 need a result about deforming the loop in a loop integral and the lecture material relevant to this will probably be after the revision for the class test.

The "trig integral" questions starting from question 15 may not be covered too much in the lectures until after the revision for the class test.

1. Let

$$\begin{aligned} \Gamma_1 &= \left\{ \mathrm{e}^{it} : -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}, \\ \Gamma_2 &= \left\{ \mathrm{e}^{-it} : \frac{\pi}{2} \le t \le \frac{3\pi}{2} \right\} \end{aligned}$$

with the direction of both arcs corresponding to the parameter t increasing and thus both arcs are circular and start at $-i$ and end at i. Compute

$$\int_{\Gamma_1} \frac{\mathrm{d}z}{z} \quad \text{and} \quad \int_{\Gamma_2} \frac{\mathrm{d}z}{z}.$$

Solution

In the case of Γ_1 we have

$$z(t) = \mathrm{e}^{it}, \quad z'(t) = i\,\mathrm{e}^{it}$$

and

$$\int_{\Gamma_1} \frac{\mathrm{d}z}{z} = \int_{-\pi/2}^{\pi/2} (\mathrm{e}^{it})^{-1}\, i\,\mathrm{e}^{it}\,\mathrm{d}t = i \int_{-\pi/2}^{\pi/2} \mathrm{d}t = i\,\pi.$$

In the case of Γ_2 we have

$$z(t) = \mathrm{e}^{-it}, \quad z'(t) = -i\,\mathrm{e}^{-it}$$

and

$$\int_{\Gamma_2} \frac{\mathrm{d}z}{z} = \int_{\pi/2}^{3\pi/2} (\mathrm{e}^{-it})^{-1}\,(-i)\,\mathrm{e}^{-it}\,\mathrm{d}t = -i \int_{\pi/2}^{3\pi/2} \mathrm{d}t = -i\,\pi.$$

This demonstrates that in this example the value of the integral depends on the path taken from $-i$ to $+i$.

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2. Let z_1 and z_2 be points such that the straight line segment Γ from z_1 to z_2 does not contain the point 0. Also let Γ' be the straight line segment from $-z_1$ to $-z_2$.

Explain why

$$\int_{\Gamma} \frac{\mathrm{d}z}{z} = \int_{\Gamma'} \frac{\mathrm{d}z}{z}.$$

Show that

$$\int_{\Gamma} \frac{\mathrm{d}z}{z} = \begin{cases} \mathrm{Log}\, z_2 - \mathrm{Log}\, z_1, & \text{if } \Gamma \text{ does not cross the negative axis,} \\ \mathrm{Log}\,(-z_2) - \mathrm{Log}\,(-z_1), & \text{otherwise} \end{cases}$$

where Log denotes the principal valued logarithm.

Solution

We are given that the line segment does not contain the origin and thus the integrand is bounded and the integral exists.

Parametrizations of the line segments Γ and Γ' can be as follows.

$$\Gamma = \{z_1 + t(z_2 - z_1): \ 0 \le t \le 1\}, \quad \Gamma' = \{-z_1 + t(-z_2 + z_1): \ 0 \le t \le 1\}.$$

Hence

$$\int_\Gamma \frac{\mathrm{d}z}{z} = \int_0^1 \frac{z_2 - z_1}{z_1 + t(z_2 - z_1)}\,\mathrm{d}t$$

and

$$\int_{\Gamma'} \frac{\mathrm{d}z}{z} = \int_0^1 \frac{-z_2 + z_1}{-z_1 + t(-z_2 + z_1)}\,\mathrm{d}t = \int_0^1 \frac{z_2 - z_1}{z_1 + t(z_2 - z_1)}\,\mathrm{d}t.$$

Both integrals have the same value.

If the straight line segment from z_1 to z_2 does not cross the negative real axis then $1/z$ has the anti-derivative $F(z) = \operatorname{Log} z$ in a region containing the segment and we have

$$\int_\Gamma \frac{\mathrm{d}z}{z} = F(z_2) - F(z_1) = \operatorname{Log} z_2 - \operatorname{Log} z_1.$$

If z_1 to z_2 is part of the negative real axis then Γ' is part of the positive real axis. If z_1 to z_2 crosses the negative real axis then the intersection of the segment and the axis is just one point and the corresponding segment from $-z_1$ to $-z_2$ crosses the positive real axis at the corresponding point on the positive real axis. In all cases there is a region containing Γ' at which $1/z$ has the anti-derivative $F(z) = \operatorname{Log}(-z)$ and we have

$$\int_\Gamma \frac{\mathrm{d}z}{z} = \int_{\Gamma'} \frac{\mathrm{d}z}{z} = \operatorname{Log}(-z_2) - \operatorname{Log}(-z_1).$$

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3. Let

$$f(z) = \frac{z^{10}}{z^2 - 1} \quad \text{and} \quad g(z) = \frac{z^9}{z^2 - 1}$$

and let C_2 denote the circle with centre at 0 and radius 2 traversed once in the anti-clockwise sense. Compute

$$\oint_{C_2} f(z)\,\mathrm{d}z \quad \text{and} \quad \oint_{C_2} g(z)\,\mathrm{d}z.$$

Solution

The rational function $f(z)$ can be represented in the form

$$f(z) = \frac{z^{10}}{z^2-1} = (\text{polynomial}) + \frac{A}{z-1} + \frac{B}{z+1}$$

and both $+1$ and -1 are inside the circle C_2. The loop integral of the polynomial part is 0 and hence

$$\oint_{C_2} f(z)\,\mathrm{d}z = 2\pi i(A+B).$$

Similar reasoning applies to $g(z)$ and we have

$$g(z) = \frac{z^{9}}{z^2-1} = (\text{polynomial}) + \frac{C}{z-1} + \frac{D}{z+1}$$

and

$$\oint_{C_2} g(z)\,\mathrm{d}z = 2\pi i(C+D).$$

For the coefficients A, B, C and D we have

$$\begin{aligned}
A &= \lim_{z\to 1} z^{10}\left(\frac{z-1}{z^2-1}\right) = \lim_{z\to 1}\left(\frac{z^{10}}{z+1}\right) = \frac{1}{2},\\
B &= \lim_{z\to -1} z^{10}\left(\frac{z+1}{z^2-1}\right) = \lim_{z\to -1}\left(\frac{z^{10}}{z-1}\right) = -\frac{1}{2},\\
C &= \lim_{z\to 1} z^{9}\left(\frac{z-1}{z^2-1}\right) = \lim_{z\to 1}\left(\frac{z^{9}}{z+1}\right) = \frac{1}{2},\\
D &= \lim_{z\to -1} z^{9}\left(\frac{z+1}{z^2-1}\right) = \lim_{z\to -1}\left(\frac{z^{9}}{z-1}\right) = \frac{1}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\oint_{C_2} f(z)\,\mathrm{d}z &= 0,\\
\oint_{C_2} g(z)\,\mathrm{d}z &= 2\pi i.
\end{aligned}$$

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4. Let

$$f(z) = \frac{1}{z^n-1}, \quad n \ge 1,\ n \text{ an integer}$$

and let

$$C_R = \left\{Re^{it}:\ 0 \le t \le 2\pi\right\}$$

the circle of radius R with centre at 0 traversed once in the anti-clockwise sense. Also let

$$I_R = \oint_{C_R} f(z)\,\mathrm{d}z.$$

(i) What is I_R when $R < 1$?.

(ii) If $R > 1$ and $n = 1$ then what is I_R?

(iii) If $R > 1$ then explain why I_R is independent of R.

(iv) If $R > 1$ and $n \ge 2$ then use the ML inequality to show that $I_R = 0$.

Solution

(i) When $R < 1$ the integrand $f(z)$ is analytic inside C_R and thus by the Cauchy integral theorem

$$I_R = 0.$$

(ii) In this case $f(z) = 1/(z-1)$ which has one simple pole at $z = 1$ which is inside C_R and

$$I_R = 2\pi i.$$

(iii) If $\tilde{R} > 1$ then $f(z)$ is analytic between C_R and $C_{\tilde{R}}$ and hence

$$I_R = \oint_{C_R} f(z)\,\mathrm{d}z = \oint_{C_{\tilde{R}}} f(z)\,\mathrm{d}z = I_{\tilde{R}}.$$

(iv) To bound $|f(z)|$ on C_R we just need to observe that $|z^n - 1|$ takes its smallest value when $z = 1$ and thus on C_R we have

$$|f(z)| \le \frac{1}{R^n - 1}.$$

The length of C_R is $2\pi R$ and thus by the ML inequality we have

$$I_R \le \frac{2\pi R}{R^n - 1} = \frac{(2\pi/R^{n-1})}{1-(1/R)^n} \to 0 \quad \text{as } R \to \infty.$$

As I_R is independent of R for $R > 1$ we have $I_R = 0$.

This was not part of the question but an alternative proof that $I_R = 0$ for $R > 1$ is obtained if $f(z)$ is written in partial fraction form.

$$z^n - 1 = (z-1)(z-\omega)\cdots(z-\omega^{n-1}), \quad \omega = \exp(2\pi i/n).$$

The partial fraction decomposition is

$$f(z) = \sum_{k=0}^{n-1} \frac{A_k}{z-\omega^k}$$

with

$$A_k = \lim_{z\to\omega^k} \frac{z-\omega^k}{z^n-1} = \frac{1}{n(\omega^k)^{n-1}} = \frac{\omega^k}{n(\omega^k)^n} = \frac{\omega^k}{n}.$$

The value of the integral is

$$I_R = \oint_{C_R} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=0}^{n-1} A_k = \left(\frac{2\pi i}{n}\right) \sum_{k=0}^{n-1} \omega^k.$$

As we have a finite geometric series we have

$$\sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega} = 0.$$

5. *The following question was in the Jan 2021 class test and it was worth 8 of the 100 marks in the 90 minute test.*

In the following your function $f(z)$ depends on the 5th digit of your 7-digit student id..

If the 5th digit is one of $0, 2, 4, 6, 8$ then

$$f(z) = \frac{1}{z^3} + 1 + 3z.$$

If the 5th digit is one of $1, 3, 5, 7, 9$ then

$$f(z) = \frac{1}{z^2} + 1 + 2z^2.$$

Give an anti-derivative of your version of $f(z)$ and by any means determine

$$\int_\Gamma f(z)\,\mathrm{d}z, \quad \text{where } \Gamma = \left\{\mathrm{e}^{it} : 0 \le t \le \pi/2\right\}$$

and where the direction of integration is from 1 to i.

Solution

The version if the 5th digit is one of the digits $0, 2, 4, 6, 8$.

$$F(z) = \frac{1}{-2z^2} + z + \frac{3}{2}z^2.$$

$$F(1) = -\frac{1}{2} + 1 + \frac{3}{2} = 2.$$

$$F(i) = \frac{-1}{2i^2} + i + \frac{3}{2}i^2 = \frac{1}{2} + i - \frac{3}{2} = -1 + i.$$

The value of the integral is

$$F(i) - F(1) = -3 + i.$$

The version if the 5th digit is one of the digits $1, 3, 5, 7, 9$.

$$F(z) = \frac{-1}{z} + z + \frac{2}{3}z^3.$$

$$F(1) = -1 + 1 + \frac{2}{3} = \frac{2}{3}.$$

$$F(i) = i + i + \frac{2}{3}i^3 = \frac{4}{3}i$$

The value of the integral is

$$F(i) - F(1) = -\frac{2}{3} + i\frac{4}{3}.$$

6. In the December 2018 class test there was a question about determining the partial fraction representation of the rational functions.

$$f_1(z) = \frac{z+11}{(z-1)(z+2)} \quad \text{and} \quad f_2(z) = \frac{4z(2z-1)}{(z-1)^2(z+1)}.$$

Let C_R denote the circle of radius R with centre at 0 traversed once in the anti-clockwise sense.

(a) Use the ML inequality to show that for $R > 2$

$$\oint_{C_R} f_1(z)\,\mathrm{d}z = \oint_{C_R} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

(b) Use the ML inequality to show that for $R > 1$

$$\oint_{C_R} f_2(z)\,\mathrm{d}z = \oint_{C_R} \frac{8}{z}\,\mathrm{d}z = 16\pi i.$$

Solution

(a) Let

$$g_1(z) = f_1(z) - \frac{1}{z} = \frac{(z+11)z - (z-1)(z+2)}{(z-1)(z+2)z} = \frac{10z+2}{(z-1)(z+2)z}$$

By the residue theorem for rational functions the value of the loop integral of g_1 only depends on the residues inside C_R and hence this value is the same for all $R > 2$ as in all cases we have the same points. The length of C_R is $2\pi R$ and on C_R,

$$|g_1(z)| \le \frac{10R+2}{(R-1)(R+2)R}.$$

By the ML inequality

$$\left|\oint_{C_R} g_1(z)\,\mathrm{d}z\right| \le 2\pi R\frac{10R+2}{(R-1)(R+2)R}. = 2\pi\frac{10R+2}{(R-1)(R-2)} \to 0 \quad \text{as } R \to \infty.$$

Hence

$$\oint_{C_R} g_1(z)\,\mathrm{d}z = 0.$$

Thus

$$\oint_{C_R} f_1(z)\,\mathrm{d}z = \oint_{C_R} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

(b) Let

$$g_2(z) = f_2(z) - \frac{8}{z} = \frac{(4z^2(2z-1) - 8(z-1)^2(z+1)}{(z-1)^2(z+1)z}.$$

The numerator is

$$\begin{aligned}
4z^2(2z-1) - 8(z-1)^2(z+1) &= (8z^3 - 4z^2) - 8(z^2 - 2z + 1)(z+1) \\
&= (8z^3 - 4z^2) - 8(z^3 - 2z^2 + z + z^2 - 2z + 1) \\
&= 4z^2 + 8z - 8.
\end{aligned}$$

By the residue theorem for rational functions the value of the loop integral of g_2 only depends on the residues inside C_R and hence this value is the same for all $R > 1$ as in all cases we have the same points. The length of C_R is $2\pi R$ and on C_R,

$$|g_2(z)| \le \frac{4R^2 + 8R + 8}{(R-1)^3 R}.$$

By the ML inequality

$$\left| \oint_{C_R} g_2(z) \, \mathrm{d}z \right| \le 2\pi R \frac{4R^2 + 8R + 8}{(R-1)^3 R} = 2\pi \frac{4R^2 + 8R + 8}{(R-1)^3} \to 0 \quad \text{as } R \to \infty.$$

Hence

$$\oint_{C_R} g_2(z) \, \mathrm{d}z = 0.$$

Thus

$$\oint_{C_R} f_2(z) \, \mathrm{d}z = \oint_{C_R} \frac{8}{z} \, \mathrm{d}z = 16\pi i.$$

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7. *The following were part of question 2 on the May 2020 exam and was worth 8 of the 20 marks*

   Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.

   Suppose that $f(z)$ and the domain D are such that an anti-derivative F exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. It can be shown that

$$\int_\Gamma f(z) \, \mathrm{d}z = F(z_2) - F(z_1).$$

   This result can be used in the questions below.

   Let Γ be the straight line segment from $-i$ to i. Evaluate the following integrals I_1, I_2, I_3 and I_4 giving the answer in Cartesian form. To get the marks you must indicate the method used and show appropriate intermediate workings.

$$I_1 = \int_\Gamma z \, \mathrm{d}z, \qquad I_2 = \int_\Gamma \frac{1}{1+z} \, \mathrm{d}z,$$

$$I_3 = \int_\Gamma \frac{1}{(1+z)^2} \, \mathrm{d}z, \qquad I_4 = \int_\Gamma \mathrm{e}^{z+i} \, \mathrm{d}z.$$

Solution

An anti-derivative F of the function f is a function such that

$$f(z) = F'(z).$$

An anti-derivative for $f(z) = z$ is $F(z) = z^2/2$. As $F(i) = F(-i)$ we have $I_1 = 0$.

An anti-derivative for $f(z) = 1/(1+z)$ on Γ is $F(z) = \mathrm{Log}(1+z)$.

$$\begin{aligned} F(i) &= \mathrm{Log}(1+i) = \ln(\sqrt{2}) + i\frac{\pi}{4}, \\ F(-i) &= \mathrm{Log}(1-i) = \ln(\sqrt{2}) - i\frac{\pi}{4}, \end{aligned}$$

and thus

$$I_2 = i\frac{\pi}{2}.$$

An anti-derivative for $f(z) = 1/(1+z)^2$ is

$$F(z) = \frac{-1}{1+z}$$

and thus

$$I_3 = \frac{-1}{1+i} + \frac{1}{1-i} = \frac{-(1-i)+(1+i)}{2} = i.$$

An anti-derivative for $f(z) = \mathrm{e}^{z+i}$ is $F(z) = f(z)$ and thus

$$I_4 = \mathrm{e}^{2i} - \mathrm{e}^0 = (\cos(2) - 1) + i\sin(2).$$

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8. *The following were part of question 2 on the May 2019 exam and was worth 14 of the 20 marks*

   (a) Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.
   Let
   $$\Gamma = \{z(t) : 0 \le t \le 1\}$$
   denote a curve in D, where $z(t)$ is continuous on $[0, 1]$ and continuously differentiable in $(0, 1)$. Also let $z_0 = z(0)$ and $z_1 = z(1)$. If a function $f(z)$ defined in D has an anti-derivative $F(z)$ in D, then explain why
   $$\int_\Gamma f(z)\,\mathrm{d}z = F(z_1) - F(z_0),$$
   where the direction of integration on Γ corresponds to t increasing.

   (b) Let Γ_1 denote the straight line segment from $-2i$ to 2, and let Γ_2 denote the straight line segment from 2 to $2i$. Evaluate each of the following, giving your answer in cartesian form.

      i.
      $$\int_{\Gamma_1 \cup \Gamma_2} (2 + z^2)\,\mathrm{d}z.$$
      ii.
      $$\int_{\Gamma_2} \frac{\mathrm{d}z}{z}.$$
      iii.
      $$\int_{\Gamma_1} \mathrm{e}^{\pi z}\,\mathrm{d}z.$$

   (c) Define the principal value complex power z^α, where z and α are complex numbers and $z \ne 0$.
   Let Γ be the circle $z(t) = 2\mathrm{e}^{it}$, $-\pi < t \le \pi$ of radius 2. By any means, determine
   $$\int_\Gamma z^{1/4}\,\mathrm{d}z,$$
   where $z^{1/4}$ denotes the principal value root function and where the direction of integration is anti-clockwise.

Solution

(a) An anti-derivative F of the function f is a function such that
$$f(z) = F'(z).$$
When f has an anti-derivative we have
$$f(z(t))z'(t) = F'(z(t))z'(t) = \frac{\mathrm{d}}{\mathrm{d}t}F(z(t)).$$
Hence
$$\int_\Gamma f(z)\,\mathrm{d}z = \int_0^1 f(z(t))z'(t)\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))\,\mathrm{d}t = F(z_1) - F(z_0)$$
by the fundamental theorem of calculus.

(b) i. With $f(z) = 2 + z^2$ an anti-derivative is $F(z) = 2z + z^3/3$. As the integral, which we denote by I, only depends on the end points

$$I = F(2i) - F(-2i) = 2F(2i)$$

as $F(z)$ is an odd function.

$$F(2i) = 2(2i) + (2i)^3/3 = i(4 - 8/3) = i(4/3).$$

Thus

$$I = \frac{8i}{3}.$$

ii. With $f(z) = 1/z$ an anti-derivative is $F(z) = \mathrm{Log}(z)$. The integral I is

$$I = F(2i) - F(2) = (\ln(2) + i\pi/2) - \ln(2) = i\pi/2.$$

iii. With $f(z) = \mathrm{e}^{\pi z}$ an anti-derivative is $F(z) = \mathrm{e}^{\pi z}/\pi$. The integral I is

$$I = F(2) - F(-2i) = \frac{1}{\pi}\left(\mathrm{e}^{2\pi} - \mathrm{e}^{-2\pi i}\right) = \frac{1}{\pi}\left(\mathrm{e}^{2\pi} - 1\right).$$

(c) The principal value complex power is

$$z^\alpha = \exp(\alpha \mathrm{Log}\, z),$$

where $\mathrm{Log}\, z$ is the principal value logarithm, which is given by

$$\mathrm{Log}\, z = \ln|z| + i\mathrm{Arg}\, z$$

where $\mathrm{Arg}\, z$ is the principal argument.
With $f(z) = z^{1/4}$ an anti-derivative is $F(z) = 4z^{5/4}/5$ on the contour Γ starting from $t = -\pi+$ and ending at $t = \pi$. Now

$$-2 = 2\mathrm{e}^{i\pi}, \quad (-2)^{5/4} = 2^{5/4}\mathrm{e}^{5i\pi/4}, \quad \lim_{\epsilon\downarrow 0}(-2 - i\epsilon) = 2^{5/4}\mathrm{e}^{-5i\pi/4}.$$

The value of the integral is thus

$$\begin{aligned} I &= \left(\frac{4}{5}\right) 2^{5/4}\left(\mathrm{e}^{5i\pi/4} - \mathrm{e}^{-5i\pi/4}\right) \\ &= \left(\frac{4}{5}\right) 2^{5/4}\left(2i\sin(5\pi/4)\right) \\ &= -\left(\frac{4}{5}\right) 2^{5/4}\left(2i\sin(\pi/4)\right) = -\left(\frac{4}{5}\right) 2^{7/4} i. \end{aligned}$$

9. *The following were part of question 2 on the May 2023 exam and was worth 10 marks.*

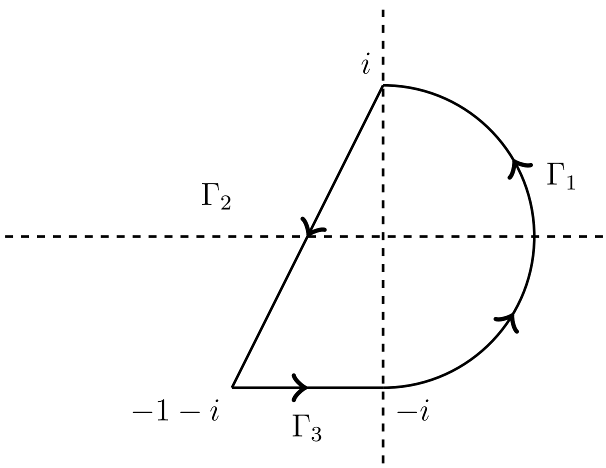
Let $f(z)$ be a function which is analytic in a domain D. Suppose that $f(z)$ and the domain D are such that an anti-derivative F of f exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. We have the following result

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1),$$

which you can use in this question.

(a) As usual let $\mathrm{Log}(z)$ denote the principal value logarithm. Describe the part of the complex plane where $\mathrm{Log}(-z)$ is continuous and give the function which has it as an anti-derivative.

(b) The right half circle Γ_1 and the straight line segments Γ_2 and Γ_3 are as shown in the diagram below. The direction of the arcs are such that the loop $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a loop in the anti-clockwise direction. The half circle Γ_1 has centre 0 and radius 1 and is in the right half plane (i.e. the part with positive real part). Straight line segment Γ_2 joins i and $-1-i$. Straight line segment Γ_3 joins $-1-i$ and $-i$.



By any means evaluate the following 6 integrals and give the value of each integral in cartesian form. You need to justify your workings.

$$\int_{\Gamma_2} \mathrm{d}z, \qquad \int_{\Gamma_2} z\,\mathrm{d}z, \qquad \int_{\Gamma_1} \mathrm{e}^{3z}\,\mathrm{d}z,$$

$$\int_{\Gamma_2} \frac{1}{z}\,\mathrm{d}z, \qquad \int_{\Gamma_3\cup\Gamma_1} \frac{1}{z+1}\,\mathrm{d}z, \qquad \int_{\Gamma_3\cup\Gamma_1} \frac{1}{(z+1)^2}\,\mathrm{d}z.$$

Solution

(a) $\mathrm{Log}(-z)$ is continuous in

$$\left\{ z = r\mathrm{e}^{i\theta} : r > 0, \quad 0 < \theta < 2\pi \right\}.$$

The function is also analytic in this domain and the derivative is $1/z$.

(b) For the first 3 integrals the anti-derivative of 1 is z, of z it is $z^2/2$, and of e^{3z} it is $\mathrm{e}^{3z}/3$.

$$\int_{\Gamma_2} \mathrm{d}z = (-1-i) - (i) = -1-2i.$$

$$\int_{\Gamma_2} z\,\mathrm{d}z = \frac{1}{2}\left((-1-i)^2 - (i)^2\right) = \frac{1}{2}\left(2i-(-1)\right) = \frac{1}{2} + i.$$

$$\int_{\Gamma_1} \mathrm{e}^{3z}\,\mathrm{d}z = \frac{1}{3}\left(\mathrm{e}^{3i} - \mathrm{e}^{-3i}\right) = \frac{2i}{3}\sin(3).$$

An anti-derivative of $1/z$ on Γ_2 is $F(z) = \mathrm{Log}(-z)$ as the branch cut does not cross the contour.

$$\begin{aligned}\int_{\Gamma_2} \frac{1}{z}\,\mathrm{d}z &= F(-1-i) - F(i) = \mathrm{Log}(1+i) - \mathrm{Log}(-i) \\ &= \ln(\sqrt{2}) + i\frac{\pi}{4} - \left(-i\frac{\pi}{2}\right) \\ &= \ln(\sqrt{2}) + i\frac{3\pi}{4}.\end{aligned}$$

An anti-derivative of $1/(z+1)$ on $\Gamma_3 \cup \Gamma_1$ is $F(z) = \mathrm{Log}(z+1)$ as the branch cut does not cross the contour.

$$\begin{aligned}\int_{\Gamma_3\cup\Gamma_1} \frac{1}{z+1}\,\mathrm{d}z &= F(i) - F(-1-i) = \mathrm{Log}(1+i) - \mathrm{Log}(-i) \\ &= \left(\ln(\sqrt{2}) + \frac{i\pi}{4}\right) - \left(-\frac{i\pi}{2}\right) = \ln(\sqrt{2}) + i\frac{3\pi}{4}.\end{aligned}$$

An anti-derivative of $1/(z+1)^2$ is $F(z) = -1/(z+1)$.

$$\int_{\Gamma_3\cup\Gamma_1} \frac{1}{(z+1)^2}\,\mathrm{d}z = F(i) - F(-1-i) = -\frac{1}{1+i} - \left(-\frac{1}{-i}\right) = -\frac{1-i}{2} + i = \frac{-1+3i}{2}.$$

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10. *The following were part of question 2 on the May 2022 exam and was worth 10 marks.*

Let $f(z)$ be a function which is analytic in a domain D. Suppose that $f(z)$ and the domain D are such that an anti-derivative F of f exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. We have the following result

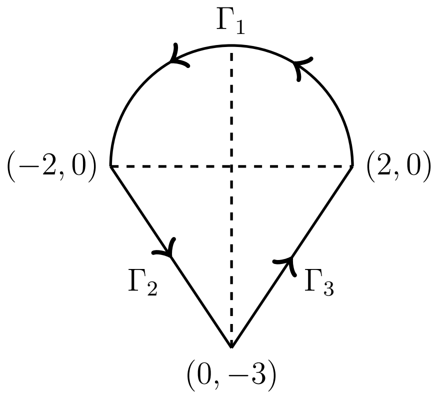
$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1),$$

which you can use in this question.

The half circle Γ_1 and the straight line segments Γ_2 and Γ_3 are as shown in the diagram below. The direction of the arcs are such that the loop $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a loop in the anti-clockwise direction. The half circle Γ_1 has centre 0 and radius 2 and is in the upper half plane. Γ_2 joins -2 and $-3i$. Γ_3 joins $-3i$ and 2.

Let $f_1(z)$ and $f_2(z)$ be functions given by

$$f_1(z) = 2z - z^2 \quad \text{and} \quad f_2(z) = \frac{1}{z}.$$



By any means evaluate the following 6 integrals and give the value of each integral in cartesian form. You need to justify your workings.

$$\int_{\Gamma_1} f_1(z)\,\mathrm{d}z, \qquad \int_{\Gamma_2} f_1(z)\,\mathrm{d}z, \qquad \int_{\Gamma_3} f_1(z)\,\mathrm{d}z,$$

$$\int_{\Gamma_1} f_2(z)\,\mathrm{d}z, \qquad \int_{\Gamma_2} f_2(z)\,\mathrm{d}z, \qquad \int_{\Gamma_3} f_2(z)\,\mathrm{d}z.$$

Solution

With $f_1(z)=2z-z^2$ an anti-derivative is

$$F_1(z)=z^2-\frac{z^3}{3}.$$

$$\int_{\Gamma_1} f_1(z)\,\mathrm{d}z = F_1(-2)-F_1(2)=\left(4+\frac{8}{3}\right)-\left(4-\frac{8}{3}\right)=\frac{16}{3}.$$

$$\begin{aligned}\int_{\Gamma_2} f_1(z)\,\mathrm{d}z &= F_1(-3i)-F_1(-2)=\left(-9-\frac{(-3i)^3}{3}\right)-\left(4+\frac{8}{3}\right)\\ &= (-9-9i)-\left(4+\frac{8}{3}\right)=-\frac{47}{3}-9i.\end{aligned}$$

$$\int_{\Gamma_3} f_1(z)\,\mathrm{d}z = F_1(2)-F_1(-3i)=\left(4-\frac{8}{3}\right)-(-9-9i)=\frac{31}{3}+9i.$$

As a check, the 3 values add to 0. The loop integral of $f_1(z)$ is 0.

With $f_2(z)=1/z$ an anti-derivative is

$$F_2(z)=\mathrm{Log}(z)$$

with care in the evaluation at -2.

$$\int_{\Gamma_1} f_2(z)\,\mathrm{d}z = F_2(-2)-F_2(2)=(\ln(2)+i\pi)-\ln(2)=i\pi.$$

$$\int_{\Gamma_3} f_2(z)\,\mathrm{d}z = F_2(2)-F_2(-3i)=\ln(2)-\left(\ln(3)-i\frac{\pi}{2}\right)=\ln(2/3)+i\frac{\pi}{2}.$$

Let $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ denote the loop.

$$\begin{aligned}\int_{\Gamma_2} f_2(z)\,\mathrm{d}z &= \oint_{\Gamma} f_2(z)\,\mathrm{d}z-\int_{\Gamma_1\cup\Gamma_3} f_2(z)\,\mathrm{d}z\\ &= 2\pi i-\left(\ln(2/3)+i\frac{\pi}{2}+i\pi\right)=\ln(3/2)+i\frac{\pi}{2}.\end{aligned}$$

11. *The following were part of question 2 on the May 2021 exam and was worth 8 marks.*

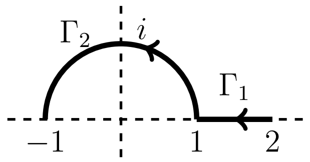
(a) Let $f(z)$ be a function which is analytic in a domain D. Suppose that $f(z)$ and the domain D are such that an anti-derivative F of f exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. We have the following result

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1)$$

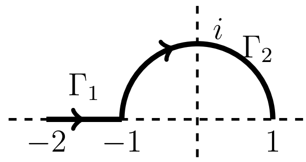
which you can use in the question below where the details depend on the 6th digit of your 7-digit student id..

If the 6th digit of your 7-digit student id. is one of the digits 0, 1, 2, 3, 4 then the straight line segment Γ_1 from 2 to 1 and part of the unit circle Γ_2 from 1 to -1 anti-clockwise are as shown in the diagram on the left hand side.

If the 6th digit of your 7-digit student id. is one of the digits 5, 6, 7, 8, 9 then the straight line segment Γ_1 from -2 to -1 and the part of the unit circle Γ_2 from -1 to 1 clockwise are as shown in the diagram on the right hand side.



Γ_1: line segment from 2 to 1.
Γ_2: upper half unit circle from 1 to -1.
Version for 6th digit of 0, 1, 2, 3, 4.



Γ_1: line segment from -2 to -1.
Γ_2: upper half unit circle from -1 to 1.
Version for 6th digit of 5, 6, 7, 8, 9.

Evaluate the following, appropriate to your version, and give the value of each integral in cartesian form.

If the 6th digit of your 7-digit student id. is one of the digits 0, 1, 2, 3, 4 then you do the following integrals.

$$\int_{\Gamma_2} z^2\,\mathrm{d}z, \qquad \int_{\Gamma_1\cup\Gamma_2} \frac{1}{(z-i/2)^2}\,\mathrm{d}z, \qquad \int_{\Gamma_1\cup\Gamma_2} \left(z+\frac{1}{z}\right)\,\mathrm{d}z.$$

If the 6th digit of your 7-digit student id. is one of the digits 5, 6, 7, 8, 9 then you do the following integrals.

$$\int_{\Gamma_2} z^2\,\mathrm{d}z, \qquad \int_{\Gamma_1\cup\Gamma_2} \frac{1}{(z+i/2)^2}\,\mathrm{d}z, \qquad \int_{\Gamma_1\cup\Gamma_2} \left(3z-\frac{1}{z}\right)\,\mathrm{d}z.$$

Solution

(a) This is the version for a 6th digit of 0, 1, 2, 3, 4.

For $f(z) = z^2$ we can take $F(z) = z^3/3$. Γ_2 starts at 1 and ends at -1 giving the value of the integral as

$$F(-1) - F(1) = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}.$$

$$f(z) = (z - i/2)^{-2}, \quad F(z) = -(z - i/2)^{-1} = \frac{-1}{z - i/2}.$$

$\Gamma_1 \cup \Gamma_2$ starts at 2 and ends at -1 giving the value of the integral as

$$\begin{aligned} F(-1) - F(2) &= \frac{-1}{-1 - i/2} + \frac{1}{2 - i/2}, \\ &= -\frac{-1 + i/2}{5/4} + \frac{2 + i/2}{17/4}, \\ &= \left(\frac{4}{5} + \frac{8}{17}\right) + i\left(-\frac{2}{5} + \frac{2}{17}\right) = \frac{108}{85} - i\frac{24}{85}. \end{aligned}$$

$$f(z) = z + \frac{1}{z}, \quad F(z) = \frac{z^2}{2} + \mathrm{Log}(z).$$

The path does not cross the branch of $\mathrm{Log}(z)$. With this $F(z)$

$$F(-1) = \frac{1}{2} + i\pi, \quad F(2) = 2 + \ln(2).$$

The value of the integral is

$$F(-1) - F(2) = \left(-\frac{3}{2} - \ln(2)\right) + i\pi.$$

This is the version for a 6th digit of 5, 6, 7, 8, 9.

For $f(z) = z^2$ we can take $F(z) = z^3/3$. Γ_2 starts at -1 and ends at 1 giving the value of the integral as

$$F(1) - F(-1) = \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3}.$$

$$f(z) = (z + i/2)^{-2}, \quad F(z) = -(z + i/2)^{-1} = \frac{-1}{z + i/2}.$$

$\Gamma_1 \cup \Gamma_2$ starts at -2 and ends at 1 giving the value of the integral as

$$\begin{aligned} F(1) - F(-2) &= \frac{-1}{1 + i/2} + \frac{1}{-2 + i/2}, \\ &= -\frac{1 - i/2}{5/4} + \frac{-2 - i/2}{17/4}, \\ &= -\left(\frac{4}{5} + \frac{8}{17}\right) + i\left(\frac{2}{5} - \frac{2}{17}\right) = -\frac{108}{85} + i\frac{24}{85}. \end{aligned}$$

$$f(z) = 3z - \frac{1}{z}, \quad F(z) = \frac{3z^2}{2} - \mathrm{Log}(z).$$

The path does not cross the branch of $\mathrm{Log}(z)$. With this $F(z)$

$$F(1) = \frac{3}{2}, \quad F(-2) = \left(\frac{3}{2}\right) 4 - \ln(2) - i\pi.$$

The value of the integral is

$$F(1) - F(-2) = \left(-\frac{9}{2} + \ln(2)\right) + i\pi.$$

12. *The following were part of question 2 on the May 2018 exam.*

Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.

Suppose that $f(z)$ and the domain D are such that an anti-derivative F exists on D. Let Γ denote a simple arc in D with a parametric description $\{z(t) : a \le t \le b\}$ and let $z_1 = z(a)$ and $z_2 = z(b)$. Explain why

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1).$$

Let Γ denote the straight line segment from $-i$ to i on the imaginary axis. Evaluate each of the following giving your answer in cartesian form.

(a)
$$\int_\Gamma \frac{\mathrm{d}z}{(z+1)^2}.$$

(b)
$$\int_\Gamma \frac{\mathrm{d}z}{z+1}.$$

(c)
$$\int_\Gamma \frac{\mathrm{d}z}{z+2}.$$

(d)
$$\int_\Gamma \frac{\mathrm{d}z}{z^2+3z+2}.$$

Solution

An anti-derivative F of the function f is a function such that

$$f(z) = F'(z).$$

In terms of the parametrization the contour integral is

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

As F is an anti-derivative of f we have by the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t).$$

Thus by the fundamental theorem of calculus

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))\,\mathrm{d}t = F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

(a) With $f(z) = 1/(z+1)^2$ an anti-derivative is

$$F(z) = \frac{-1}{z+1}.$$

The value of the integral is

$$F(i) - F(-i) = \left(\frac{-1}{i+1}\right) - \left(\frac{-1}{-i+1}\right) = \frac{i-1+i+1}{2} = i.$$

(b) With $f(z) = 1/(z+1)$ an anti-derivative is $\mathrm{Log}(1+z)$ as the line segment does not cross the branch cut. The value of the integral is

$$F(i) - F(-i) = \mathrm{Log}(1+i) - \mathrm{Log}(1-i) = i\frac{\pi}{2}.$$

(c) $f(z) = 1/(z+2)$ an anti-derivative is $F(z) = \mathrm{Log}(z+2)$. The value of the integral is

$$F(i) - F(-i) = \mathrm{Log}(2+i) - \mathrm{Log}(2-i) = i2\tan^{-1}(1/2).$$

(d) Now $z^2 + 3z + 2 = (z+1)(z+2)$ and by partial fractions we have a representation

$$\frac{1}{z^2+3z+2} = \frac{A}{z+1} + \frac{B}{z+2}.$$

$$A = \lim_{z\to -1} \frac{z+1}{z^2+3z+2} = \frac{1}{-2+3} = 1, \quad B = \lim_{z\to -2} \frac{z+2}{z^2+3z+2} = \frac{1}{-4+3} = -1.$$

Using the previous two results the value of the integral is

$$i\left(\frac{\pi}{2} - 2\tan^{-1}(1/2)\right).$$

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13. *The following were part of question 2 on the May 2017 exam.*

    (a) Let $f(z)$ be a function which is analytic in a domain D. Explain what is meant by an anti-derivative $F(z)$ of $f(z)$.

    Let $F_1(z) = \mathrm{Log}(z)$ and let $F_2(z) = \mathrm{Log}(-z)$ where Log denotes the principal valued logarithm. Give expressions for $F_1'(z)$ and $F_2'(z)$ for values of z where the functions are differentiable. Hence, or otherwise, give an expression for an anti-derivative of $f(z) = 1/z$ which is valid in the half plane with positive real part, and also give an expression for an anti-derivative of $f(z) = 1/z$ which is valid in the half plane with negative real part.

    (b) Suppose that $f(z)$ and the domain D are such that an anti-derivative F exists on D. Let Γ denote a simple arc in D starting at z_1 and ending at z_2. It can be shown that

    $$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1).$$

    This result can be used in the questions below.

    Let Γ_1 be the straight line segment from 2 to $1+i$ and let Γ_2 be the line segment from $-1-i$ to $-1+i$. Evaluate the following integrals giving the answer in Cartesian form.

    i.
    $$\int_{\Gamma_1} z\,\mathrm{d}z.$$

    ii.
    $$\int_{\Gamma_1} \frac{\mathrm{d}z}{z}.$$

    iii.
    $$\int_{\Gamma_2} \frac{\mathrm{d}z}{z}.$$

    iv.
    $$\int_{\Gamma_2} \frac{\mathrm{d}z}{z^2}.$$

Solution

(a) An anti-derivative F of the function f is a function such that

$$f(z) = F'(z).$$

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} = \frac{1}{z}, \quad \frac{\mathrm{d}F_2}{\mathrm{d}z} = \frac{-1}{-z} = \frac{1}{z}.$$

$F_1(z)$ is a an anti-derivative of $f(z)$ throughout the half plane with positive real part. $F_2(z)$ is a an anti-derivative of $f(z)$ throughout the half plane with negative real part.

(b) i. With $f(z) = z$ an anti-derivative is $F(z) = z^2/2$. The value of the integral is

$$F(1+i) - F(2) = \left(\frac{1}{2}\right)((1+i)^2 - 4) = \left(\frac{1}{2}\right)(2i - 4) = -2 + i.$$

ii. With $f(z) = 1/z$ and with the segment being in the half plane with positive real part an anti-derivative is $F_1(z)$. The value of the integral is

$$\begin{aligned} F_1(1+i) - F_1(2) &= \mathrm{Log}(\sqrt{2}) + i\frac{\pi}{4} - \mathrm{Log}(2) \\ &= (\mathrm{Log}(\sqrt{2}) - \mathrm{Log}(2)) + i\frac{\pi}{4} = -\frac{\mathrm{Log}(2)}{2} + i\frac{\pi}{4}. \end{aligned}$$

iii. With $f(z) = 1/z$ and with the segment being in the half plane with negative real part an anti-derivative is $F_2(z)$. The value of the integral is

$$F_2(-1+i) - F_2(-1-i) = \mathrm{Log}(1-i) - \mathrm{Log}(1+i) = -i\frac{\pi}{4} - i\frac{\pi}{4} = -i\frac{\pi}{2}.$$

iv. With $f(z) = 1/z^2$ an anti-derivative is $F(z) = -1/z$. The value of the integral is

$$\begin{aligned} F(-1+i) - F(-1-i) &= \frac{1}{1-i} - \frac{1}{1+i} \\ &= \frac{(1+i) - (1-i)}{2} = i. \end{aligned}$$

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14. *The following were part of question 2 on the May 2016 exam.*

Let Γ_1 denote the straight line segment from 1 to $2i$ and let Γ_2 denote the straight line segment from $2i$ to -1. Evaluate the following integrals justifying your answer in each case. You need to express your answer in cartesian form.

(a)
$$\int_{\Gamma_1} (1 + 2z + 3z^2)\,\mathrm{d}z.$$

(b)
$$\int_{\Gamma_1} \frac{\mathrm{d}z}{2-z}.$$

(c)
$$\int_{\Gamma_2} \frac{\mathrm{d}z}{2-z}.$$

(d)
$$\int_{\Gamma_1 \cup \Gamma_2} \frac{\mathrm{d}z}{2-z}.$$

Solution

(a) With $f(z) = 1 + 2z + 3z^2$ an anti-derivative is $F(z) = z + z^2 + z^3$. The value of the integral is

$$F(2i) - F(1) = (2i + 4i^2 + 8i^3) - (1 + 1 + 1) = (-4 - 6i) - 3 = -7 - 6i.$$

(b) With $f(z) = 1/(2 - z)$ an anti-derivative involves the principal valued logarithm and is $F(z) = -\mathrm{Log}(2 - z)$. The value of the integral is

$$F(2i) - F(1) = -\mathrm{Log}(2 - 2i) = -\mathrm{Log}(\sqrt{8}) + i\frac{\pi}{4}.$$

(c) As in the last part the value of the integral is

$$F(-1) - F(2i) = -\mathrm{Log}(3) + \mathrm{Log}(\sqrt{8}) - i\frac{\pi}{4}.$$

(d)
$$\int_{\Gamma_1 \cup \Gamma_2} \frac{\mathrm{d}z}{2-z} = \int_{\Gamma_1} \frac{\mathrm{d}z}{2-z} + \int_{\Gamma_2} \frac{\mathrm{d}z}{2-z} = -\mathrm{Log}(3).$$

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15. *These were parts question 4a and 4b of the May 2020 MA3614 paper and were worth 11 marks together.*

(a) Let $f(z)$ be a function which is analytic in a domain D except for isolated singularities at points $z_1, z_2, \ldots, z_n$. Let Γ denote a simple closed loop in D traversed once in the anti-clockwise direction such that none of the points $z_1, z_2, \ldots, z_n$ lie on Γ. State the Cauchy Residue theorem involving the closed loop Γ.

(b) Let $0 < \alpha < \pi/2$. By using the substitution $z = \mathrm{e}^{i\theta}$, show that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - \cos(\alpha)\sin(\theta)} = \frac{2\pi}{\sin(\alpha)}.$$

In your answer you can use the following inequality

$$0 < \frac{1-\sin(\alpha)}{\cos(\alpha)} < 1 < \frac{1+\sin(\alpha)}{\cos(\alpha)} \quad \text{when } 0 < \alpha < \frac{\pi}{2}$$

where appropriate.

Solution

(a) Suppose the isolated singularities which are inside Γ can be labelled as $z_1, \ldots, z_m$. The Cauchy Residue theorem is

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^m \mathrm{Res}(f,\, z_k),$$

where $\mathrm{Res}(f,\, z_k)$ is the residue of $f(z)$ at $z = z_k$.

(b)

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \sin\,\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{1}{1-\cos(\alpha)\sin(\theta)}\right) = \frac{1}{i}F(z),$$

with

$$F(z) = \frac{1}{z}\left(\frac{1}{1 - \left(\frac{\cos(\alpha)}{2i}\right)\left(z - \frac{1}{z}\right)}\right) = \frac{2i}{2iz - \cos(\alpha)(z^2-1)}.$$

$$\cos(\alpha)z^2 - 2iz - \cos(\alpha) = 0 \quad \text{when } z = \frac{2i \pm \sqrt{-4 + 4\cos^2(\alpha)}}{2\cos(\alpha)}.$$

As $-1 + \cos^2(\alpha) = -\sin^2(\alpha)$ the two roots are

$$z_1 = \frac{(1-\sin(\alpha))i}{\cos(\alpha)} \quad \text{and} \quad z_2 = \frac{(1+\sin(\alpha))i}{\cos(\alpha)}.$$

z_1 is inside the unit circle and z_2 is outside the unit circle. $F(z)$ has a simple pole at these points and, by L'Hopital's rule, the residue at z_1 is

$$\mathrm{Res}(F,\, z_1) = \lim_{z\to z_1}(z - z_1)F(z) = \frac{2i}{2i - 2\cos(\alpha)z_1} = \frac{i}{i - \cos(\alpha)z_1}.$$

From the expression for z_1 we have

$$i - \cos(\alpha)z_1 = \sin(\alpha)i \quad \text{and} \quad \frac{i}{i-\cos(\alpha)z_1} = \frac{1}{\sin(\alpha)}.$$

By the residue theorem the value of the integral is

$$2\pi\mathrm{Res}(F,\, z_1) = \frac{2\pi}{\sin(\alpha)}.$$

16. *This was question 4a of the 2018 MA3614 paper and was worth 10 marks.*

By first using the substitution $z = \mathrm{e}^{i\theta}$, evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + 8\cos^2\theta}.$$

Solution

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}).$$

The interval $0 \le \theta \le 2\pi$ maps to the unit circle C traversed once in the anti-clockwise direction and we have that the integral is

$$I = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z,$$

where

$$F(z) = \frac{1}{z}\left(\frac{1}{1 + 2(z + 1/z)^2}\right) = \frac{z}{z^2 + 2(z^2+1)^2} = \frac{z}{2z^4 + 5z^2 + 2}.$$

The denominator vanishes when

$$2z^4 + 5z^2 + 2 = 0.$$

This is a quadratic in z^2 and the smaller in magnitude root z_1 satisfies

$$\begin{aligned} z_1^2 &= \frac{-5 + \sqrt{25 - 16}}{4} \\ &= \frac{-5+3}{4} = -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \mathrm{Res}(F,\, z_1) &= \lim_{z \to z_1} (z - z_1) F(z) = z_1 \lim_{z \to z_1} \frac{z - z_1}{2z^4 + 5z^2 + 2} \\ &= \frac{z_1}{8z_1^3 + 10 z_1} \\ &= \frac{1}{8z_1^2 + 10} = \frac{1}{-4+10} = \frac{1}{6} \end{aligned}$$

by using L'Hopital's rule and the expression for z_1. By similar workings the residue at $-z_1$ is the same value. As both z_1 and $-z_1$ are inside C the residue theorem gives

$$I = 2\pi(\mathrm{Res}(F,\, z_1) + \mathrm{Res}(F,\, -z_1)) = 2\pi\left(\frac{1}{6} + \frac{1}{6}\right) = \frac{2\pi}{3}.$$

17. *This was question 4a of the 2023 MA3614 paper and was worth 10 marks.*

By using the substitution $z = \mathrm{e}^{i\theta}$ determine the value of the integral I given below.

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 2\sin(\theta)}.$$

Given that for all $a \in \mathbb{R}$ we have

$$\int_a^{a+2\pi} \frac{\mathrm{d}\theta}{5 - 2\sin(\theta)} = I$$

explain why for all integers $m \ge 1$ we have

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 2\sin(m\theta)} = I.$$

Solution

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \sin\,\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{1}{5 - 2\sin(\theta)}\right) = \frac{1}{i}F(z),$$

$$F(z) = \left(\frac{1}{z}\right)\left(\frac{1}{5 - \frac{1}{i}\left(z - \frac{1}{z}\right)}\right) = \frac{1}{5z + i(z^2 - 1)}.$$

$F(z)$ has simple poles when

$$iz^2 + 5z - i = 0, \quad 2iz = -5 \pm \sqrt{25 + 4i^2} = -5 \pm \sqrt{21}.$$

The point

$$z_1 = \frac{-5 + \sqrt{21}}{2i}$$

is inside the unit circle. By the residue theorem and L'Hopital's rule

$$I = 2\pi \mathrm{Res}(F,\, z_1) = 2\pi \lim_{z \to z_1} (z - z_1)F(z) = \frac{2\pi}{5 + 2iz_1} = \frac{2\pi}{\sqrt{21}}.$$

Let $t = m\theta$.

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = m, \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{m}, \quad \left(\frac{1}{5 - 2\sin(m\theta)}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(\frac{1}{5 - 2\sin(t)}\right)\frac{1}{m}.$$

In terms of t the limits are $t = 0$ to $t = 2m\pi$.

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 2\sin(m\theta)} &= \frac{1}{m}\int_0^{2m\pi} \frac{\mathrm{d}t}{5 - 2\sin(t)} \\
&= \frac{1}{m}\left(\int_0^{2\pi} + \int_{2\pi}^{4\pi} + \cdots + \int_{2(m-1)\pi}^{2m\pi}\right)\frac{\mathrm{d}t}{5 - 2\sin(t)} \\
&= \frac{1}{m}\left(I + \cdots + I\right) = I.
\end{aligned}$$

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18. *This was question 4a of the 2022 MA3614 paper and was worth 10 marks.*

You have two integrals to determine as shown below. Using the substitution $z = \mathrm{e}^{i\theta}$ determine the value of the first integral that you consider. Any method can be used for the other integral. The integrals to determine are the following.

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{6-5\cos(\theta)} \quad \text{and} \quad \int_0^{2\pi} \frac{5\cos(\theta)}{6-5\cos(\theta)}\,\mathrm{d}\theta.$$

Solution

Let

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{6-5\cos(\theta)}.$$

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\,\theta = \frac{1}{2}\left(z+\frac{1}{z}\right).$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{1}{6-5\cos(\theta)}\right) = \frac{1}{i}F(z),$$

$$F(z) = \left(\frac{1}{z}\right)\frac{1}{6-\frac{5}{2}\left(z+\frac{1}{z}\right)} = \frac{2}{12z-5(z^2+1)}.$$

$$5z^2-12z+5=0 \quad \text{when } z = \frac{12\pm\sqrt{144-100}}{10} = \frac{6\pm\sqrt{11}}{5}.$$

The simple pole of $F(z)$ which is inside the unit circle C is at

$$z_1 = \frac{6-\sqrt{11}}{5}.$$

By the residue theorem

$$I = \oint_C \frac{1}{i}F(z)\,\mathrm{d}z = 2\pi\mathrm{Res}(F,\,z_1).$$

$$\mathrm{Res}(F,\,z_1) = \lim_{z\to z_1}(z-z_1)F(z) = 2\lim_{z\to z_1}\frac{z-z_1}{12z-5(z^2+1)} = \frac{2}{12-10z_1} = \frac{1}{6-5z_1}.$$

$$5z_1-6 = -\sqrt{11}, \quad \mathrm{Res}(F,\,z_1) = \frac{1}{\sqrt{11}}.$$

$$I = \frac{2\pi}{\sqrt{11}}.$$

$$5\cos(\theta) = 6-(6-5\cos(\theta))$$

and thus the integrand in the second integral is

$$\frac{5\cos(\theta)}{6-5\cos(\theta)} = \frac{6}{6-5\cos(\theta)} - 1$$

and

$$\int_0^{2\pi}\frac{5\cos(\theta)}{6-5\cos(\theta)}\,\mathrm{d}\theta = 6I-2\pi = 2\pi\left(\frac{6}{\sqrt{11}}-1\right).$$

19. *This was question 4a of the 2017 MA3614 paper and was worth 10 marks.*

By first using the substitution $z = \mathrm{e}^{i\theta}$ determine the value of the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{13 + 12\cos\theta}.$$

By using the value just obtained, or otherwise, determine the value of the following integral.

$$\int_0^{2\pi} \frac{12\cos\theta}{13 + 12\cos\theta}\,\mathrm{d}\theta.$$

Solution

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad 2\cos\theta = z + z^{-1}.$$

The interval $0 \le \theta \le 2\pi$ maps to the unit circle C traversed once in the anti-clockwise direction and we have that the integral is

$$I = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z,$$

where

$$F(z) = \frac{1}{z}\left(\frac{1}{13 + 6\left(z + \frac{1}{z}\right)}\right) = \frac{1}{6z^2 + 13z + 6}.$$

By inspection the quadratic in the denominator of $F(z)$ can be factored as

$$6z^2 + 13z + 6 = (3z+2)(2z+3)$$

and $F(z)$ has one simple pole inside C at $z_1 = -2/3$.

The value of the integral is

$$I = 2\pi \mathrm{Res}(F, z_1).$$

The residue of F at the pole z_1 is

$$\begin{aligned}
\mathrm{Res}(F, z_1) &= \lim_{z \to z_1} (z - z_1) F(z) \\
&= \lim_{z \to z_1} \frac{z - z_1}{6z^2 + 13z + 6} = \frac{1}{12z_1 + 13} = \frac{1}{5}.
\end{aligned}$$

Hence

$$I = \frac{2\pi}{5}.$$

$$12\cos\theta = (13 + 12\cos\theta) - 13.$$

Thus

$$\frac{12\cos\theta}{13 + 12\cos\theta} = 1 - \frac{13}{13 + 12\cos\theta}.$$

By using the answer to the first part we hence have

$$\int_0^{2\pi} \frac{12\cos\theta}{13 + 12\cos\theta}\,\mathrm{d}\theta = 2\pi - 13\left(\frac{2\pi}{5}\right) = -\frac{16\pi}{5}.$$

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20. *This was question 4a of the May 2021 MA3614 paper and was worth 10 marks*

In the following which integral you consider depends on the 4th digit of your 7-digit student id.. If your 4th digit is one of 0, 1, 2, 3, 4 then you consider I_1 and if it is one of the digits 5, 6, 7, 8, 9 then you consider I_2. By first using the substitution $z = \mathrm{e}^{i\theta}$, determine the value of I_1 or I_2 depending on your version.

$$I_1 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{4 - \cos(\theta) - 3\sin(\theta)}, \qquad I_2 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 4\cos(\theta) - \sin(\theta)}.$$

Solution

This is the version for a 4th digit of 0, 1, 2, 3, 4.

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\theta = \frac{1}{2i}(z - z^{-1}).$$

The interval $0 \le \theta \le 2\pi$ maps to the unit circle C traversed once in the anti-clockwise direction and we have that the integral is

$$I_1 = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z,$$

where

$$\begin{aligned} F(z) &= \left(\frac{1}{z}\right)\left(\frac{1}{4 - \frac{1}{2}\left(z + \frac{1}{z}\right) - \frac{3}{2i}\left(z - \frac{1}{z}\right)}\right) \\ &= \frac{2}{8z - (z^2+1) + 3i(z^2-1)} = \frac{2}{(-1+3i)z^2 + 8z + (-1-3i)}. \end{aligned}$$

The denominator has zeros when

$$\begin{aligned} z &= \frac{-8 \pm \sqrt{64 - 4(-1+3i)(-1-3i)}}{2(-1+3i)} = \frac{-4 \pm \sqrt{16 - (-1+3i)(-1-3i)}}{-1+3i} \\ &= \frac{-4 \pm \sqrt{6}}{-1+3i}. \end{aligned}$$

If these are labelled as z_1 and z_2 with $|z_1| < |z_2|$ then the product of the roots satisfies $|z_1 z_2| = 1$ and

$$z_1 = \frac{-4 + \sqrt{6}}{-1 + 3i}$$

is inside the unit circle. By the residue theorem

$$I_1 = 2\pi \mathrm{Res}(F,\, z_1).$$

By L'Hopital's rule

$$\mathrm{Res}(F,\, z_1) = \lim_{z \to z_1} (z - z_1) F(z) = \frac{2}{2(-1+3i)z_1 + 8} = \frac{1}{(-1+3i)z_1 + 4}.$$

$$(-1+3i)z_1 + 4 = \sqrt{6} \quad \text{and thus } I_1 = \frac{2\pi}{\sqrt{6}}.$$

This is the version for a 4th digit of 5, 6, 7, 8, 9.

$$z = \mathrm{e}^{i\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\mathrm{e}^{i\theta} = iz, \quad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\theta = \frac{1}{2i}(z - z^{-1}).$$

The interval $0 \le \theta \le 2\pi$ maps to the unit circle C traversed once in the anti-clockwise direction and we have that the integral is

$$I_1 = \oint_C \frac{1}{i} F(z)\,\mathrm{d}z,$$

where

$$\begin{aligned} F(z) &= \left(\frac{1}{z}\right)\left(\frac{1}{5 - 2\left(z + \frac{1}{z}\right) - \frac{1}{2i}\left(z - \frac{1}{z}\right)}\right) \\ &= \frac{2}{10z - 4(z^2+1) + i(z^2-1)} = \frac{2}{(-4+i)z^2 + 10z + (-4-i)}. \end{aligned}$$

The denominator has zeros when

$$\begin{aligned} z &= \frac{-10 \pm \sqrt{100 - 4(-4+i)(-4-i)}}{2(-4+i)} = \frac{-5 \pm \sqrt{25 - (-4+i)(-4-i)}}{-4+i} \\ &= \frac{-5 \pm \sqrt{8}}{-4+i}. \end{aligned}$$

If these are labelled as z_1 and z_2 with $|z_1| < |z_2|$ then the product of the roots satisfies $|z_1 z_2| = 1$ and

$$z_1 = \frac{-5 + \sqrt{8}}{-4+i}$$

is inside the unit circle. By the residue theorem

$$I_2 = 2\pi \mathrm{Res}(F,\, z_1).$$

By L'Hopital's rule

$$\mathrm{Res}(F,\, z_1) = \lim_{z \to z_1} (z - z_1) F(z) = \frac{2}{2(-4+i)z_1 + 10} = \frac{1}{(-4+i)z_1 + 5}.$$

$$(-4+i)z_1 + 5 = \sqrt{8} \quad \text{and thus } I_1 = \frac{2\pi}{\sqrt{8}} = \frac{\pi}{\sqrt{2}}.$$

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21. Let $a > 0$ and $b > 0$. Show that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta} = \frac{2\pi}{ab}.$$

Solution

Let

$$I=\int_0^{2\pi}\frac{\mathrm{d}\theta}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

First note that when $a=b$ the integrand is the constant $1/a^2$ and the result is correct and thus we just need to consider the case when $a\neq b$.

We can simplify the expression a bit before we make a substitution by using

$$\sin^2\theta=1-\cos^2\theta$$

so that

$$a^2\sin^2\theta+b^2\cos^2\theta=(b^2-a^2)\cos^2\theta+a^2.$$

As in the other questions let $z=\mathrm{e}^{i\theta}$ and note that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta}=i\,\mathrm{e}^{i\theta}=iz,\quad \frac{\mathrm{d}\theta}{\mathrm{d}z}=\frac{1}{iz}$$

and

$$\cos^2\theta=\left(\frac{z+1/z}{2}\right)^2=\frac{z^2+2+1/z^2}{4}.$$

With this substitution

$$I=\oint_C\frac{1}{i}F(z)\,\mathrm{d}z$$

where C is the unit circle traversed once in the anti-clockwise sense and where

$$\begin{aligned}F(z)&=\left(\frac{1}{z}\right)\frac{1}{\left(\frac{b^2-a^2}{4}\right)\left(z^2+2+\frac{1}{z^2}\right)+a^2}\\&=\frac{4z}{(b^2-a^2)(z^4+2z^2+1)+4a^2z^2},\\&=\frac{4z}{(b^2-a^2)(z^4+1)+2(b^2+a^2)z^2}.\end{aligned}$$

The denominator is a quadratic in z^2 and is 0 when

$$\begin{aligned}z_i^2&=-\frac{2(b^2+a^2)\pm\sqrt{4(b^2+a^2)^2-4(b^2-a^2)^2}}{2(b^2-a^2)}\\&=-\frac{(b^2+a^2)\pm\sqrt{4a^2b^2}}{(b^2-a^2)}\\&=-\frac{(b^2+a^2)\pm 2ab}{(b^2-a^2)}.\end{aligned}$$

The two values are

$$z_1^2=-\frac{(b-a)^2}{b^2-a^2}\quad\text{and}\quad z_2^2=-\frac{(b+a)^2}{b^2-a^2}.$$

The product of the roots is 1 and z_1^2 is the smaller in magnitude as $a>0$ and $b>0$. The function $F(z)$ has poles at $\pm z_1$ inside C. We can get the residue at the pole by using

L'Hopitals rule.

$$\begin{aligned}
\mathrm{Res}(F,\ z_1) &= \lim_{z \to z_1} (z - z_1) F(z), \\
&= 4z_1 \lim_{z \to z_1} \frac{z - z_1}{(b^2 - a^2)(z^4 + 1) + 2(b^2 + a^2) z^2}, \\
&= \frac{4z_1}{4(b^2 - a_2) z_1^3 + 4(b^2 + a^2) z_1} \\
&= \frac{1}{(b^2 - a_2) z_1^2 + (b^2 + a^2)} \\
&= \frac{1}{-(b - a)^2 + (b^2 + a^2)} \\
&= \frac{1}{2ab}.
\end{aligned}$$

Similarly

$$\mathrm{Res}(F,\ z_1) = \mathrm{Res}(F,\ -z_1).$$

and

$$I = 2\pi(\mathrm{Res}(F,\ z_1) + \mathrm{Res}(F,\ -z_1)) = \frac{2\pi}{ab}.$$