Exercises involving elementary functions

1. *This question was in the class test in Dec 2022 and was worth 24 marks.*

   Determine the partial fraction representation of each of the following 3 functions and state the residue at any pole.

$$f_1(z) = \frac{1}{(2z+1)(z-1)},$$

$$f_2(z) = \frac{-3z^2+4z+3}{z^2(z+1)},$$

$$f_3(z) = \frac{z^3}{(z+2)^5}.$$

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2. *This question was in the class test in Dec 2021 and was worth 25 marks.*

   In this question the functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ that you consider depends on the 5th digit of your 7-digit student id..

   If the 5th digit is one of the digits 0, 1, 2, 3, 4 then your functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ are as follows.

$$f_1(z) = \frac{-13-z}{z^2+z-6}, \quad f_2(z) = \frac{9}{z^2(3+z)}, \quad f_3(z) = \frac{(z+2)^3}{(z-1)^3}.$$

   If the 5th digit is one of the digits 5, 6, 7, 8, 9 then your functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ are as follows.

$$f_1(z) = \frac{6z-19}{z^2-3z-4}, \quad f_2(z) = \frac{1}{z(z-1)^2}, \quad f_3(z) = \frac{(z+1)^3}{(z-3)^3}.$$

   (i) For your version of $f_1(z)$ and $f_2(z)$ determine the partial fraction representation in each case and state the residue at each pole.

   (ii) For your version of $f_3(z)$ determine the residue at the pole.

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3. *This question was in the class test in Jan 2021 and was worth 22 marks.*

   In the following there are three rational functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ that you need to consider and your version of these depends on the 6th digit of your 7-digit student id. as follows.

   If the 6th digit is one of $0, 3, 6, 9$ then you have

$$f_1(z) = \frac{5z-i}{z^2+1}, \quad f_2(z) = \frac{1}{z^3-8}, \quad f_3(z) = \frac{z^4-2z^3}{(z+1)^3}.$$

If the 6th digit is one of $1, 4, 7$ then you have

$$f_1(z) = \frac{7z - 2i}{z^2 + 4}, \quad f_2(z) = \frac{1}{z^3 + 8}, \quad f_3(z) = \frac{z^5}{(z-1)^3}.$$

If the 6th digit is one of $2, 5, 8$ then you have

$$f_1(z) = \frac{-z - 9i}{z^2 + 9}, \quad f_2(z) = \frac{1}{z^4 - 1}, \quad f_3(z) = \frac{-z^4 + 3z^3}{(z-1)^3}.$$

(a) For your version of $f_1(z)$ determine the partial fraction representation and state the residue at any pole.

(b) For your version of $f_2(z)$ state in cartesian form the points where it has simple poles and determine the residue at one of these points.

(c) For your version of $f_3(z)$ determine the residue at the pole of the function.

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4. *This question was in the class test in 2019/2020 and was worth 25 marks.*

Let

$$f_1(z) = \frac{2z}{z^2 - 3z + 2}, \quad f_2(z) = \frac{16}{z^2(z-4)} \quad \text{and} \quad f_3(z) = \frac{z^4}{(z-2)^3}.$$

(a) Determine the partial fraction representation of $f_1(z)$ and state the residue at each pole.

(b) Determine the partial fraction representation of $f_2(z)$ and state the residue at each pole.

(c) Determine the residue of $f_3(z)$ at $z = 2$.

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5. *This question was in the class test in 2018/2019 and was worth 20 marks.*

Let f_1 and f_2 denote the following rational functions.

$$f_1(z) = \frac{z + 11}{(z-1)(z+2)}, \quad f_2(z) = \frac{4z(2z-1)}{(z-1)^2(z+1)}.$$

In each case determine the partial fraction representation and state the residue at any pole.

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6. Let

$$R(z) = \frac{p(z)}{(z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_n)^{r_n}}$$

denote a rational function in which $z_1, \ldots, z_n$ are distinct points, where each $r_k \ge 1$ is an integer and where $p(z)$ is a polynomial which is non-zero at these n points. What can you say about the order of the poles of $R'(z)$ and $R''(z)$ and what can you say about the residues of the function $R'(z)$?

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7. Let

$$q(z) = (z - z_1)^{r_1} (z - z_2)^{r_2} \cdots (z - z_n)^{r_n}$$

where $z_1, \ldots, z_n$ are distinct points. What can you say about the multiplicity of the zeros of $q'(z)$ at the points $z_1, \ldots, z_n$? Using a derivation based on partial fractions show that

$$\frac{q'(z)}{q(z)} = \frac{r_1}{z - z_1} + \frac{r_2}{z - z_2} + \cdots + \frac{r_n}{z - z_n}.$$

(Observe that the result is consistent with the result in an exericise of the previous exercise sheet which involved a proof by induction.)

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8. *This question was in the class test in Dec 2022 and was worth 16 marks.*

   (a) Give in cartesian form the principal values of the following.

      i. $\mathrm{Log}(1 + i\sqrt{3})$.

      ii. $i^{1/2}$.

   In the above Log denotes the principal value logarithm.

   In your answer you must give an exact representation of the real and imaginary parts which may involve the square root of a positive number, π and the natural logarithm of a positive number.

   (b) Let $\theta \in \mathbb{R}$. Give in cartesian form the principal value of i^α when $\alpha = \mathrm{e}^{i\theta}$. Give all values of $\theta \in (-\pi, \pi]$ such that i^α is pure imaginary.

   Full reasoning must be given to get all the marks.

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9. *This question was in the class test in Dec 2021 and was worth 9 marks.*

This question is for all student numbers.

Let $f(z)$ and $g(z)$ be defined as follows.

$$f(z) = \tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}},$$

$$g(z) = \coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{\mathrm{e}^z - \mathrm{e}^{-z}}.$$

Give the location of all the zeros and all the poles of both $f(z)$ and $g(z)$.

Consider the straight line segment

$$\Gamma = \left\{ iy : \ \frac{\pi}{4} \le y \le \frac{\pi}{3} \right\}.$$

Describe as concisely as possible the image of Γ under the function $f(z)$.

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10. *This question was in the class test in Jan 2021 and was worth 8 marks.*

Let $z = x+iy$ with $x, y \in \mathbb{R}$. The definition of e^z, $\cosh(z)$ and $\sinh(z)$ are respectively

$$e^z = e^x(\cos(y) + i\sin(y)), \quad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right), \quad \sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right).$$

In the following you need to show an identity and which one you need to show depends on the 6th digit of your 7-digit student id..

If the 6th digit is one of the digits $0, 2, 4, 6, 8$ then show that

$$\sinh(z + i\pi/3) + \sinh(z - i\pi/3) = \sinh(z).$$

If the 6th digit is one of the digits $1, 3, 5, 7, 9$ then show that

$$\cosh(z + i\pi/3) + \cosh(z - i\pi/3) = \cosh(z).$$

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11. *The following was in the class test in 2019/2020 and was worth 13 marks.*

Let $z = x + iy$ with $x, y \in \mathbb{R}$. Given that

$$e^z = e^x(\cos(y) + i\sin(y)) \quad \text{and} \quad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

show that

$$\cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y).$$

Let $a > 0$ and let

$$z_1 = a + i\frac{\pi}{2}, \quad z_2 = i\frac{\pi}{2}, \quad z_3 = 0, \quad \text{and} \quad z_4 = a.$$

State the image of each point z_1, z_2, z_3 and z_4 under the mapping $\cosh(z)$. Describe, as concisely as possible, the image of the polygonal path z_1 to z_2, z_2 to z_3 and z_3 to z_4 under the mapping $\cosh(z)$. You need to justify your answer.

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12. Show that if $y \in \mathbb{R}$ then

$$\left|\tan\left(\frac{\pi}{4} + iy\right)\right| = 1.$$

Describe in words the set

$$G = \left\{\tan\left(\frac{\pi}{4} + iy\right) \,:\, -\infty < y < \infty\right\}.$$

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13. *The following was in the class test in 2018/2019 and was worth 16 marks.*

The complex cos, sin, cosh and sinh functions are defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cosh(z) = \frac{e^{z} + e^{-z}}{2}, \quad \sinh(z) = \frac{e^{z} - e^{-z}}{2}.$$

(a) By using these definitions and properties of the exponential function show that for all complex numbers z_1 and z_2

$$2\sin\left(\frac{z_1 + z_2}{2}\right)\sin\left(\frac{z_1 - z_2}{2}\right) = \cos(z_2) - \cos(z_1).$$

(b) By making use of the identities above (including part (a)), or otherwise, explain why

$$\cos(z) - \cosh(z) = 0$$

has solutions $z = (1 - i)k\pi$ and $z = (1 + i)k\pi$ for all integers k.

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14. *The following was in the class test in 2017/2018 and was worth 8 marks.*

It can be shown that $\tan z$ can be written as

$$\tan z = (-i)\left(\frac{1 - e^{-2iz}}{1 + e^{-2iz}}\right).$$

By using this expression, or otherwise, describe in words the following sets.

$$S_1 = \{\tan(iy) : y \in \mathbb{R}\},$$
$$S_2 = \{\tan(\pi/2 + iy) : y > 0\}.$$

In your answer you need to indicate if the set is part of the real axis, or part of the imaginary axis or any other line segment.

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15. Give the definition of the principal value of z^α and show that

$$\frac{d}{dz} z^\alpha = \alpha\, z^{\alpha - 1}.$$

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16. *This question was in the class test in Dec 2021 and was worth 9 marks.*

This question is for all student numbers.

Let $z = re^{i\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$.

(i) In terms of r and θ give the real and imaginary parts and the magnitude of $(\mathrm{Log}(z))^2$. Here $\mathrm{Log}(z)$ means the principal valued logarithm.

(ii) Show that the principal value of z^{1+i} is real when r and θ satisfy $re^{\theta} = e^{\pi}$.

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17. *This question was in the class test in Jan 2021 and was worth 12 marks.*

In the following you should attempt either part (a) or part (b) depending on the 5th digit of your 7-digit student id..

(a) If the 5th digit is one of the numbers $0, 1, 2, 3, 4$ then you do this case.

(i) Let $z_1 = \sqrt{3} + i$ and $z_2 = z_1^7$. With Log denoting the principal value logarithm determine $\mathrm{Log}(z_1)$ and $\mathrm{Log}(z_2)$ stating your answer in cartesian form.

(ii) Let

$$\Gamma = \left\{ z = r\mathrm{e}^{2\pi i/3} : \ 0 < r < \infty \right\}.$$

Give in polar form the image set $\{w = f(z) : z \in \Gamma\}$ when $f(z)$ denotes the principal value complex power

$$f(z) = z^{2+i},$$

i.e. the complex exponent is $2 + i$.
Give any value of $r \in (0, \infty)$ such that the value is pure imaginary.

(b) If the 5th digit is one of the numbers $5, 6, 7, 8, 9$ then you do this case.

(i) Let $z_1 = 1 + \sqrt{3}i$ and $z_2 = z_1^5$. With Log denoting the principal value logarithm determine $\mathrm{Log}(z_1)$ and $\mathrm{Log}(z_2)$ stating your answer in cartesian form.

(ii) Let

$$\Gamma = \left\{ z = r\mathrm{e}^{5\pi i/6} : \ 0 < r < \infty \right\}.$$

Give in polar form the image set $\{w = f(z) : z \in \Gamma\}$ when $f(z)$ denotes the principal value complex power

$$f(z) = z^{3+i},$$

i.e. the complex exponent is $3 + i$.
Give any value of $r \in (0, \infty)$ such that the value is real.

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18. (a) Give in cartesian form the value of $\mathrm{Log}(2 - 2i)$.

(b) Let $z = r\mathrm{e}^{i\theta}$ with $r > 0$ and with $-\pi < \theta \le \pi$. As concisely as possible, give an expression for the imaginary part of the principal value of z^{2i}. Explain why the imaginary part of the principal value of z^{2i} is 0 when $r = \mathrm{e}^{3\pi/2}$.

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19. Let $z = x + iy = r\mathrm{e}^{i\theta}$ with x, y, r, $\theta \in \mathbb{R}$, $r > 0$ and $\theta \in (-\pi, \pi]$.

(a) In terms of x and y give the real part of the principal value of i^z.

(b) In terms of r and θ give the imaginary part of the principal value of z^i.
