Exercises involving Cauchy's integral formula

1. *The following were parts of question 2 on the May 2023* MA3614 *exam paper and was worth 9 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by
   $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$
   Use this result to determine I_1 and I_2 in the following where Γ is the circle with centre at 2 and radius 5.
   $$I_1 = \oint_\Gamma \frac{\mathrm{e}^{\pi z}}{(z-i)^3}\,\mathrm{d}z, \qquad I_2 = \oint_\Gamma \frac{\sin(z)}{(z-\pi/2)^2(z+2\pi)}\,\mathrm{d}z.$$

   (b) Let $f(z)$ be a function which is analytic in the disk $\{z : |z| < R\}$, let C denote the circle with centre 0 and radius $r < R$ traversed once in the anti-clockwise sense and further let $0 < h < r$.

      i. Determine the partial fraction representation of
      $$\frac{2z}{z^2-h^2}$$
      and use this to show that
      $$\frac{f(h)+f(-h)}{2} = \frac{1}{2\pi i}\oint_C \left(\frac{z}{z^2-h^2}\right) f(z)\,\mathrm{d}z.$$

      ii. Let $\omega = \mathrm{e}^{2\pi i/5}$. The 5 roots of unity are $1, \omega, \omega^2, \omega^3, \omega^4$. Determine the partial fraction representation of
      $$\frac{5z^4}{z^5-h^5}$$
      and use this to show that
      $$\frac{f(h)+f(\omega h)+f(\omega^2 h)+f(\omega^3 h)+f(\omega^4 h)}{5} = \frac{1}{2\pi i}\oint_C \left(\frac{z^4}{z^5-h^5}\right) f(z)\,\mathrm{d}z.$$
      You need to justify your steps.

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2. *The following were parts of question 2 on the May 2022* `MA3614` *exam paper and was worth 10 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by
   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$
   Use this result to determine I_1 and I_2 in the following where Γ is the circle with centre at -1 and radius 3. In each case you need to express the value in cartesian form and you need to justify your workings.
   $$I_1 = \oint_\Gamma \frac{2z^2+z^4}{(z+2)^4}\,\mathrm{d}z, \qquad I_2 = \oint_\Gamma \frac{\cosh(z)}{z^2(z-6)}\,\mathrm{d}z.$$

   (b) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. When $0 < h < 1$ use the generalised Cauchy integral formula to show that
   $$\frac{f(-h)-2f(0)+f(h)}{h^2} - f''(0) = \frac{h^2}{\pi i} \oint_C \frac{f(z)}{z^3(z^2-h^2)}\,\mathrm{d}z.$$

3. *The following was part of question 2 on the May 2021* MA3614 *exam paper and was worth 7 of the 20 marks of the entire question.*

   Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

   Use this result to evaluate the following where the problems you need to consider depends on the 6th digit of your 7-digit student id.. If the 6th digit is even then you do the loop integrals on the left hand side and if it is odd then you do the loop integrals on the right hand side. In each case you need to express the value in cartesian form and you need to justify your workings. In the case of I_2, Log means the principal value logarithm.

If the 6th digit is one of $0, 2, 4, 6, 8$ then you do these cases. In each case Γ is the circle with centre at $1+2i$ and radius 4.	If the 6th digit is one of $1, 3, 5, 7, 9$ then you do these cases. In each case Γ is the circle with centre at $-2+i$ and radius 5.
$$\begin{aligned} I_1 &= \oint_\Gamma \frac{z^3-2z}{(z-1)^3}\,\mathrm{d}z, \\ I_2 &= \oint_\Gamma \frac{\mathrm{Log}(z-7\sqrt{3}+7i)}{z-\sqrt{3}+i}\,\mathrm{d}z, \\ I_3 &= \oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-6)}\,\mathrm{d}z. \end{aligned}$$	$$\begin{aligned} I_1 &= \oint_\Gamma \frac{z^3+3z}{(z+2)^3}\mathrm{d}z, \\ I_2 &= \oint_\Gamma \frac{\mathrm{Log}(z+9-i9\sqrt{3})}{z+1-i\sqrt{3}}\,\mathrm{d}z, \\ I_3 &= \oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-3)}\,\mathrm{d}z. \end{aligned}$$

4. *The following was part of question 2 on the May 2021* MA3614 *exam paper and was worth 5 of the 20 marks of the entire question.*

   This part of the question is for all student numbers.

   Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. Let $0 < h < 1$ and let $\omega = \mathrm{e}^{2\pi i/3}$. By using the generalised Cauchy integral formula (which is stated in the previous part), or otherwise, show the following.

$$\frac{f(h)+\omega^2 f(\omega h)+\omega f(\omega^2 h)}{3h} - f'(0) = \frac{h^3}{2\pi i}\oint_C \frac{f(z)}{(z^3-h^3)z^2}\,\mathrm{d}z.$$

5. *The following was part of question 2 on the Aug 2020* `MA3614` *exam paper and was worth 11 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

   i.
   $$\oint_\Gamma \frac{\mathrm{e}^{2z}}{(z-1)^3} \, \mathrm{d}z.$$

   ii.
   $$\oint_\Gamma \frac{z^3}{(z+i)^4} \, \mathrm{d}z.$$

   iii.
   $$\oint_\Gamma \frac{\mathrm{Log}(z+4)}{(z+i)^2} \, \mathrm{d}z,$$

   where Log denotes the principal valued logarithm.

   (b) With the same set-up as the previous part let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a closed loop in the domain traversed once in the anti-clockwise direction.

   Consider the following two functions $g_1(z)$ and $g_2(z)$ where z is a point which is not on Γ.

   $$g_1(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0}, & \text{when } z \neq z_0, \\ f'(z_0), & z = z_0, \end{cases} \quad \text{and} \quad g_2(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} \, \mathrm{d}\zeta.$$

   When z is any point inside Γ explain why $g_1(z) = g_2(z)$.

   Give the value of $g_2(z)$ when z is outside of Γ.

6. *The following was part of question 2 on the May 2020* `MA3614` *exam paper and was worth 12 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle $\{z : |z| = 4\}$. In your answer you must show intermediate workings.

   i.
   $$\oint_\Gamma \frac{z\mathrm{e}^z}{(z-1)^3} \, \mathrm{d}z.$$

   ii.
   $$\oint_\Gamma \frac{\sin(\pi z)}{(z-2)^2(z+6)} \, \mathrm{d}z.$$

   (b) Let $f(z)$ be a function which is analytic in a region which contains the unit disk and let C denote the unit circle traversed once in the anti-clockwise direction. By using the generalised Cauchy integral formula (which is stated in the previous part), or otherwise, show that when $0 < h < 1$ we have the following.

   $$\begin{aligned}
   \frac{f(h) - f(-h)}{2h} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^2 - h^2} \, \mathrm{d}z, \\
   \frac{f(h) - f(-h)}{2h} - f'(0) &= \frac{h^2}{2\pi i} \oint_C \frac{f(z)}{z^2(z^2 - h^2)} \, \mathrm{d}z, \\
   \frac{f(h) - if(ih) - f(-h) + if(-ih)}{4h} &= \frac{1}{2\pi i} \oint_C \frac{z^2 f(z)}{z^4 - h^4} \, \mathrm{d}z.
   \end{aligned}$$

7. *The following was part of question 2 on the May 2019* `MA3614` *exam paper and was worth 6 of the 20 marks of the entire question.*

   Let $f(z)$ be a function which is analytic in a domain which contains z_0, and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ, the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle with centre at $1+i$ and radius 2.

   (a)
   $$\oint_\Gamma \frac{\mathrm{Log}(3-z)}{(z-2)^3}\,\mathrm{d}z,$$
   where Log denotes the principal value logarithm.

   (b)
   $$\oint_\Gamma \frac{\mathrm{e}^z}{z^2(z-3)}\,\mathrm{d}z.$$

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8. *The following was part of question 2 on the May 2017* `MA3614` *exam paper and was worth 9 of the 20 marks of the entire question.*

   (a) Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

   Use this result to evaluate the following when Γ is the circle $\{z : \ |z-1| = 2\}$.

   i.
   $$\oint_\Gamma \frac{\mathrm{e}^z}{z(4-z)}\,\mathrm{d}z.$$

   ii.
   $$\oint_\Gamma \frac{z^4}{(z-1)^3}\,\mathrm{d}z.$$

   (b) By using the generalised Cauchy integral formula given in the previous part of the question, show that if Γ is a closed loop traversed once in the anti-clockwise direction, $z_0 \neq 0$ and 0 are inside Γ and f is a function analytic on Γ and inside Γ, then

   $$\frac{z_0^2}{2\pi i} \oint_\Gamma \frac{f(z)}{z^2(z-z_0)}\,\mathrm{d}z = f(z_0) - f(0) - z_0 f'(0).$$

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9. *The following was part of question 2 on the May 2016* `MA3614` *exam paper and was worth 5 of the 20 marks of the entire question.*

   Let C be the unit circle traversed once in the anti-clockwise direction. Let $g(z)$ denote a function which is continuous on C and let

$$G(z) = \oint_C \frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

   Given that for all z inside the unit circle the limit

$$\lim_{h\to 0} \oint_C \frac{g(\zeta)\mathrm{d}\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

   exists explain why $G(z)$ is analytic in $\{z : \ |z| < 1\}$ with derivative

$$\oint_C \frac{g(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta.$$

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10. *The following was part of question 2 on the May 2016* `MA3614` *exam paper and was worth 6 of the 20 marks of the entire question.*

    Let $f(z)$ be a function which is analytic in a domain which contains z_0 and let Γ denote a loop in the domain traversed once in the anti-clockwise direction. When z_0 is inside Γ the generalised Cauchy integral formula is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

    Use this result to evaluate the following when Γ is the circle with centre at 0 and radius 3.

    (a)

$$\oint_\Gamma \frac{z^2}{(z-2)^3}\,\mathrm{d}z.$$

    (b)

$$\oint_\Gamma \frac{z\,\mathrm{e}^{3z}}{(z+2)^2}\,\mathrm{d}z.$$

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11. Suppose that $f(z)$ is analytic in a domain which contains $\{z : \ |z| \le 1\}$ and suppose that $f(0) = 0$. Show that the function

$$F(z) = \begin{cases} \dfrac{f(z)}{z}, & z \ne 0, \\ f'(0), & z = 0 \end{cases}$$

    is analytic at $z = 0$. [Hint use the Cauchy integral formula and the result of question 9.]

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12. Let C be the unit circle traversed once in the anti-clockwise direction, left $f(z)$ be a function analytic inside C and let h be such that $0 < |h| < 1/2$. Show the following involving approximations to $f'(0)$.

$$\begin{aligned}
\frac{-f(2h)+4f(h)-3f(0)}{2h} &= \frac{1}{2\pi i}\oint_C \frac{(z-3h)f(z)}{(z-2h)(z-h)z}\,\mathrm{d}z,\\
\frac{-f(2h)+4f(h)-3f(0)}{2h} - f'(0) &= -\frac{2h^2}{2\pi i}\oint_C \frac{f(z)}{(z-2h)(z-h)z^2}\,\mathrm{d}z.
\end{aligned}$$