The material time derivative

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t} f(r,t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\
&= \frac{\partial f}{\partial t} + \underline{q}\cdot\nabla f.
\end{aligned}
$$

The equation of mass conservation

This can be written as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{q}) = 0$$

or equivalently

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\underline{q} = 0.$$

The flow is incompressible if

$$\nabla\cdot\underline{q} = 0.$$

Equation of hydrostatic pressure

Let Ω be an region with surface S. In equilibrium

$$\text{(force on surface)} + \text{(force due to gravity)} = \underline{0}$$

Euler's equations of motion

$$\text{(force due to acc.)} + \text{(force on surface)} + \text{(force due to gravity)} = \underline{0}.$$

$$\rho\frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = -\nabla(p + \rho g z).$$

With steady flow

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial\underline{q}}{\partial t} = \underline{0} \quad \text{and} \quad \frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = (\underline{q}\cdot\nabla)\underline{q} = \nabla\left(\frac{1}{2}|\underline{q}|^2\right) + (\nabla\times\underline{q})\times\underline{q}.$$

The term

$$\underline{\omega} = \nabla\times\underline{q}$$

is known as the **vorticity**. We can connect the velocity $\underline{q}$ with the pressure p along streamlines. With several 2D flows $\underline{\omega} = \underline{0}$.

2D steady incompressible flow and the stream function ψ

Assuming that the xy-plane is the 2D region the incompressibility condition

$$\nabla\cdot\underline{q} = 0$$

implies the existence of a stream function ψ such that

$$\underline{q} = (\nabla\psi)\times\underline{k}.$$

In Cartesian's $\psi = \psi(x,y)$ and

$$\underline{q} = \frac{\partial\psi}{\partial y}\underline{i} - \frac{\partial\psi}{\partial x}\underline{j}$$

and in polars $\psi = \psi(r,\theta)$ and

$$\underline{q} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta.$$

Curves of the form $\psi =$const are the streamlines of the flow.

Vorticity in the 2D incompressible case

$$
\begin{aligned}
\underline{\omega} = \nabla\times\underline{q} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u(x,y) & v(x,y) & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k} \\
&= -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\underline{k}.
\end{aligned}
$$

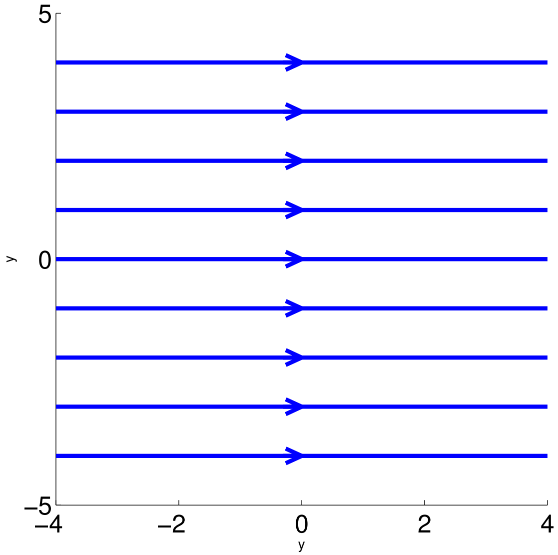
Irrotational flow is when $\underline{\omega} = \nabla\times\underline{q} = \underline{0}$ and this is when ψ satisfies Laplace's equation. In this 2D case we then also have a velocity potential ϕ such that

$$\underline{q} = \nabla\phi = (\nabla\psi)\times\underline{k}.$$

ϕ also satisfies Laplace's equation.

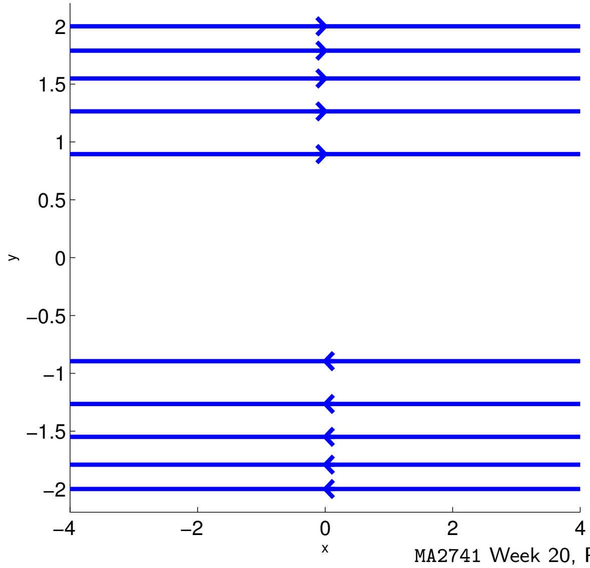
Streamlines for uniform flow in the $\underline{i}$ direction

$$\psi = -a_2 x + a_1 y \quad \text{and} \quad \underline{q} = a_1\, \underline{i} + a_2\, \underline{j}.$$



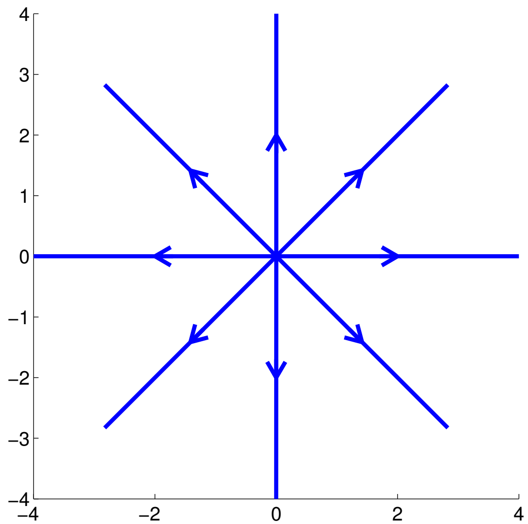
Streamlines for a simple shear flow

$$\psi = \frac{1}{2}\beta y^2 \quad \text{and} \quad \underline{q} = \beta y\, \underline{i}.$$



Streamlines for a line source at $r = 0$

$$\psi = A\theta \quad \text{and} \quad \underline{q} = \frac{A}{r}\, \underline{e}_r.$$



Streamlines for a line vortex at $r = 0$

$$\psi(r) = -\frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right) \quad \text{and} \quad \underline{q} = \left(\frac{\Gamma}{2\pi}\right)\frac{1}{r}\, \underline{e}_\theta.$$