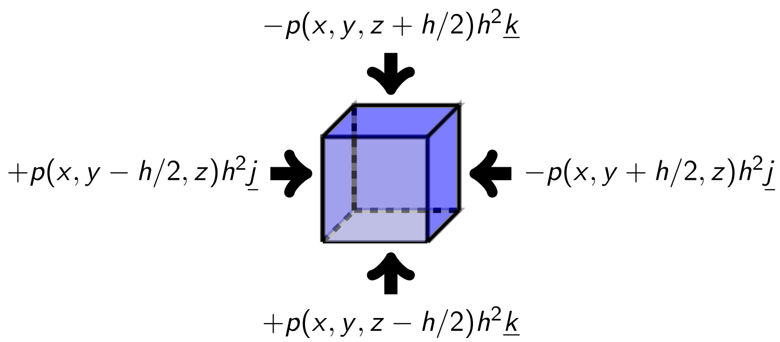
Pressure in any fluid in equilibrium



Equation of hydrostatic pressure
Let Ω be an region with surface S. In equilibrium

$$(\text{force on surface}) + (\text{force due to gravity}) = \underline{0}$$

and with ρ being density this gives

$$\nabla p = \rho \underline{g} = -\rho g \underline{k}.$$

Lagrangean and Eulerian descriptions

Motion of a particle: $\underline{r}(\underline{r}_0, t)$, $t \ge 0$ with $\underline{r}(\underline{r}_0, 0) = \underline{r}_0$.
Velocity:

$$\underline{q} = \underline{q}_L(\underline{r}_0, t) = \frac{\partial}{\partial t}\underline{r}(\underline{r}_0, t).$$

The spatial dependence is in terms the original position $\underline{r}_0$ of the particles in the Lagrangean description.

In the Eulerian description we consider $\underline{q} = \underline{q}_E(\underline{r}, t)$ and the spatial dependence is in terms of the position $\underline{r}$ at time t.

$$\underline{q} = \underline{q}_E(\underline{r}(\underline{r}_0, t), t) = \underline{q}_L(\underline{r}_0, t).$$

Material time derivative

Suppose

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t).$$

The chain rule gives

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\underline{r}, t) = \frac{\partial}{\partial t} f_L(\underline{r}_0, t) = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f.$$

The Lagrangean acceleration of a particle is

$$\underline{a} = \underline{a}_L = \frac{\partial}{\partial t}\underline{q}_L(\underline{r}_0, t) = \frac{\mathrm{D}}{\mathrm{D}t}\underline{q} = \frac{\partial \underline{q}_E}{\partial t} + (\underline{q} \cdot \nabla)\underline{q}_E$$

$$\begin{aligned} \frac{\partial \underline{q}_E}{\partial t} &= \text{local acceleration}, \\ (\underline{q} \cdot \nabla)\underline{q}_E &= \text{convective acceleration}. \end{aligned}$$

With $\underline{q} = \underline{q}_E = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$ we have

$$(\underline{q} \cdot \nabla)\underline{q}_E = (\underline{q} \cdot \nabla u)\underline{i} + (\underline{q} \cdot \nabla v)\underline{j} + (\underline{q} \cdot \nabla w)\underline{k}.$$

Terminology in the examples

1.
$$\underline{r} = \underline{r}(\underline{r}_0, t) = x_0 \mathrm{e}^{\alpha t} + y_0 \mathrm{e}^{-\alpha t}\underline{j} + z_0\underline{k}.$$
$$\underline{q}_E = \underline{q}_E(\underline{r}, t) = \alpha(x\underline{i} - y\underline{j}) = u\underline{i} + v\underline{j},$$
This is steady 2D flow. Particle paths coincide with streamlines.

2.
$$\underline{r} = \underline{r}(\underline{r}_0, t) = x_0(1+t)\underline{i} + y_0\mathrm{e}^{-t}\underline{j}, \quad t \ge 0.$$
$$\underline{q}_E(\underline{r}, t) = \left(\frac{x}{1+t}\right)\underline{i} - y\underline{j} = u\underline{i} + v\underline{j}, \quad u = \left(\frac{x}{1+t}\right), \quad v = -y.$$
This is unsteady 2D flow. Particle paths and streamlines differ.