Gradient, divergence and curl in cartesian coordinates

Gradient of ϕ.

$$\nabla\phi = \frac{\partial \phi}{\partial x}\,\underline{i} + \frac{\partial \phi}{\partial y}\,\underline{j} + \frac{\partial \phi}{\partial z}\,\underline{k},$$

Divergence of $\underline{F} = F_i\underline{i} + F_2\underline{j} + F_3\underline{k}$.

$$\nabla\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

Curl of $\underline{F} = F_i\underline{i} + F_2\underline{j} + F_3\underline{k}$.

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}.$$

Directional derivative, divergence and Stokes' theorems

Directional derivative of ϕ in the direction of $\underline{n}$.

$$\frac{\partial\phi}{\partial n}(\underline{r}) = \frac{\partial}{\partial s}\phi(\underline{r}+s\underline{n})\Big|_{s=0} = \underline{n}\cdot\nabla\phi.$$

Divergence theorem.

$$\int_\Omega \nabla\cdot\underline{F}\,\mathrm{d}v = \int_S \underline{F}\cdot\underline{n}\,\mathrm{d}s,$$

$$\int_\Omega \nabla p\,\mathrm{d}v = \int_S p\underline{n}\,\mathrm{d}s.$$

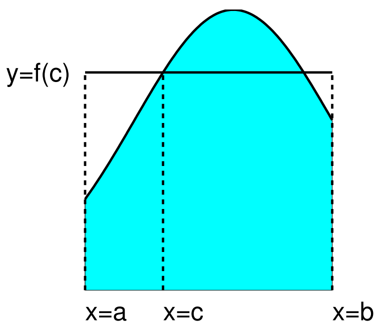
Stokes' theorem.

$$\int_S (\nabla\times\underline{F})\cdot\underline{n}\,\mathrm{d}s = \oint_C \underline{F}\cdot\mathrm{d}\underline{r}.$$

In the above $\underline{n}$ denotes an appropriate unit vector in each case.

The divergence and curl defined as a limit

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(c)$$



Divergence as a limit: V is the volume inside surface S and point $\underline{P}$ is inside the surface which we shrink to $\underline{P}$.

$$\nabla\cdot\underline{F}(\underline{P}) = \lim_{V\to 0}\frac{1}{V}\int_S \underline{F}\cdot\underline{n}\,\mathrm{d}s.$$

Curl as a limit: A is the area inside loop C and point $\underline{P}$ is inside the loop which we shrink to $\underline{P}$.

$$(\nabla\times\underline{F}(\underline{P}))\cdot\underline{n} = \lim_{A\to 0}\frac{1}{A}\oint_C \underline{F}\cdot\mathrm{d}\underline{r}.$$

Some vector identities involving $\times$ and curl

For the cross product

$$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a} \quad \text{which implies} \quad \underline{a}\times\underline{a} = \underline{0}.$$

$\underline{a}\times\underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$.

For the base vectors

$$\underline{i}\times\underline{j} = \underline{k},\quad \underline{j}\times\underline{k} = \underline{i},\quad \underline{k}\times\underline{i} = \underline{j}.$$

$\nabla\times\underline{F}$ is divergence free as

$$\nabla\cdot(\nabla\times\underline{F}) = 0.$$

$\nabla\phi$ is irrotational in that

$$\nabla\times\nabla\phi = \underline{0}.$$