Some terminology used in describing fluid flow in MA2741

Only inviscid fluids were considered, a Eulerian description was used, $\rho = \rho(\underline{r}, t)$ is density, $p = p(r, t)$ is pressure and $\underline{q} = \underline{q}(\underline{r}, t)$ is velocity.

Steady:	$\underline{q} = \underline{q}(\underline{r})$, i.e. no time dependence
Two-dimensional:	$\underline{q} = u(x, y)\,\underline{i} + v(x, y)\,\underline{j}$.
Incompressibility:	$\nabla \cdot \underline{q} = 0$.
Stream function ψ:	$\underline{q} = \nabla \times (\psi \underline{k}) = (\nabla \psi) \times \underline{k}$.
Stagnation points:	$\underline{q}(\underline{r}) = \underline{0}$.
Vorticity:	$\underline{\omega} = \nabla \times \underline{q}$.
Irrotational flow:	$\underline{\omega} = \underline{0}$.

With irrotational flow there exists a velocity potential ϕ such that

$$\underline{q} = \nabla \phi = (\nabla \psi) \times \underline{k} \quad \text{and} \quad \nabla^2 \psi = \nabla^2 \phi = 0.$$

Lagrangian, Eulerian descriptions of the motion

Lagrangian description: $\underline{r}(\underline{r}_0, t)$ with $\underline{r}(\underline{r}_0, 0) = \underline{r}_0$. The spatial description is in terms of a reference configuration.

$$\underline{q}_L = \frac{\partial \underline{r}(\underline{r}_0, t)}{\partial t} = \text{velocity}.$$

$\underline{r}(\underline{r}_0, t)$, $t \geq 0$ describes a **particle path**.

Eulerian description: involves dependence on the position at time t.

$$\text{velocity} = \underline{q}(\underline{r}, t) = \underline{q}_L(\underline{r}_0, t).$$

Particle paths are obtained from

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q}(\underline{r}, t), \quad \text{with } \underline{r}(0) = \underline{r}_0.$$

Streamlines: These depend on $\underline{q}(r, t)$ for some fixed time t. The tangent to a streamline is in the direction of $\underline{q}$.

$$x(s)\underline{i} + y(s)\underline{j} + z(s)\underline{k} \quad \text{where} \quad \frac{x'(s)}{u} = \frac{y'(s)}{v} = \frac{z'(s)}{w}.$$

Streamlines are the same as particle paths when the flow is steady.

Example

2D unsteady flow described in Eulerian form by

$$\underline{q}(\underline{r}, t) = \underline{q}_E(\underline{r}, t) = xt\,\underline{i} - y\,\underline{j}, \quad t \geq 0.$$

Particle paths satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u = xt, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v = -y. \end{aligned}$$

Streamlines at fixed time t satisfy

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} \quad \text{i.e.} \; \frac{\mathrm{d}x}{xt} = \frac{\mathrm{d}y}{-y}$$

Example

2D unsteady flow described in Eulerian form by

$$\underline{q} = U\,\underline{i} + \frac{x}{(1+t)}\,\underline{j}, \quad t \geq 0,$$

Particle paths satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u = U, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{x}{1+t}. \end{aligned}$$

Specifying an initial condition $x(0) = x_0$, $y(0) = y_0$ gives a particular path.

Streamlines at fixed time t satisfy

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} \quad \text{i.e.} \quad \frac{\mathrm{d}x}{U} = \frac{\mathrm{d}y}{\left(\dfrac{x}{1+t}\right)}.$$

Lagrangian, Eulerian descriptions of any function

When $\underline{r}_0$ at time 0 moves to $\underline{r}$ at time t and

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t)$$

the **material time derivative** is

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\underline{r}, t) = \frac{\partial}{\partial t} f_L(\underline{r}_0, t) = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f$$

where $\underline{q} = \underline{q}(\underline{r}, t)$ is the velocity. (The right hand side follows by using the chain rule.) Here f_L is the Lagrangian description and f is the Eulerian description.

Applying this to u, v and w in $\underline{q} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$ gives

$$\begin{aligned} (\text{Lagrangian acc}) &= (\text{Local acc}) + (\text{Convective acc}) \\ \underline{a}_L = \frac{\mathrm{D}}{\mathrm{D}t}\underline{q} &= \frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \nabla)\underline{q}. \end{aligned}$$

$$(\underline{q} \cdot \nabla)\underline{q} = (\underline{q} \cdot \nabla u)\underline{i} + (\underline{q} \cdot \nabla v)\underline{j} + (\underline{q} \cdot \nabla w)\underline{k}$$

Equation of mass conservation and incompressibility

For a region Ω without sources and sinks the change in the amount of mass in Ω is entirely encountered for by the flow of the material through the surface S of Ω.

$$-\int_\Omega \frac{\partial \rho}{\partial t}\,\mathrm{d}v = \int_S \rho(\underline{q}\cdot\underline{n})\,\mathrm{d}S.$$

By using divergence theorem we get

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{q}) = 0$$

or equivalently

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\underline{q} = 0.$$

A material is **incompressible** if

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \text{which is equivalent to} \quad \nabla\cdot\underline{q} = 0.$$

Two-dimensional incompressible flows: $\underline{q} = u\underline{i} + v\underline{j}$

$$\underline{q} = \nabla \times (\psi \underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y}\underline{i} - \frac{\partial \psi}{\partial x}\underline{j} = (\nabla \psi) \times \underline{k}.$$

The velocity in Cartesian and polars coordinates

In Cartesian's $\psi = \psi(x, y)$ and

$$\underline{q} = \left(\frac{\partial \psi}{\partial x}\underline{i} + \frac{\partial \psi}{\partial y}\underline{j} \right) \times \underline{k} = \frac{\partial \psi}{\partial y}\underline{i} - \frac{\partial \psi}{\partial x}\underline{j}$$

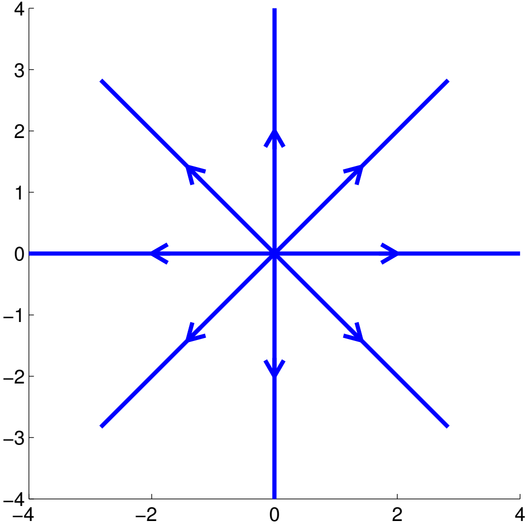
and in polars $\psi = \psi(r, \theta)$ and

$$\underline{q} = \left(\frac{\partial \psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_\theta \right) \times \underline{k} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_r - \frac{\partial \psi}{\partial r}\underline{e}_\theta.$$

Curves of the form $\psi =$const are the streamlines of the flow.

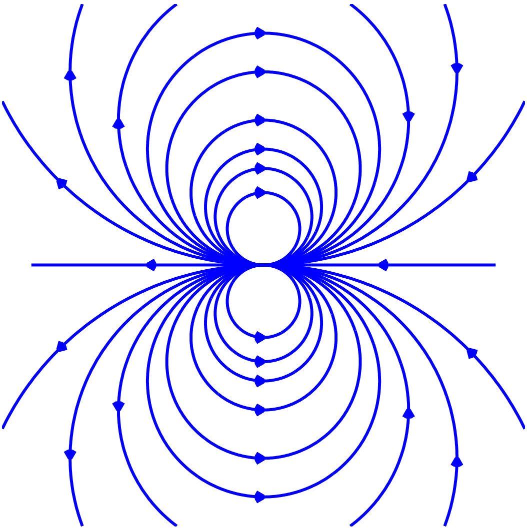
Streamlines for a line source at $r = 0$

$$\psi = A\theta \quad \text{and} \quad \underline{q} = \frac{A}{r}\,\underline{e}_r .$$



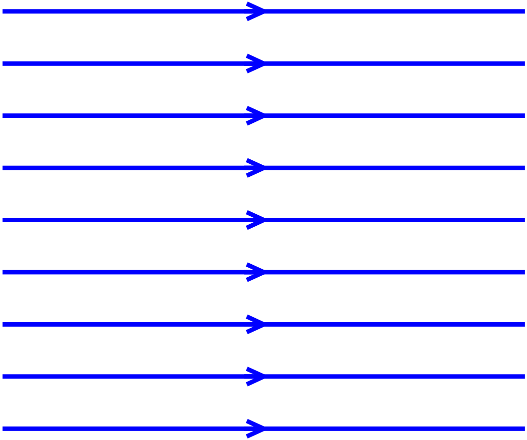
Streamlines for a dipole in the $-\underline{i}$ direction

$$\psi = -\mu \frac{\sin\,\theta}{r}$$



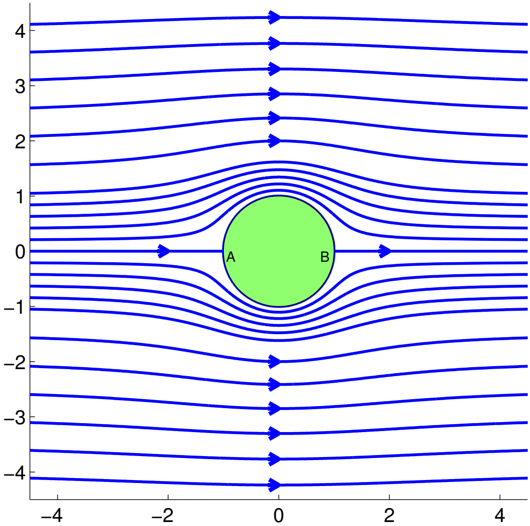
Streamlines for uniform flow in the $\underline{i}$ direction

$\psi = Ur\sin\theta$



Streamlines for flow round a cylinder

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$



The velocity on a streamline and stagnation points

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\,\theta$$

$$\begin{aligned} \frac{\partial\psi}{\partial r} &= U\left(1 + \frac{a^2}{r^2}\right)\sin\,\theta \\ \frac{\partial\psi}{\partial\theta} &= U\left(r - \frac{a^2}{r}\right)\cos\,\theta \end{aligned}$$

Both are zero when $r = a$ and $\sin\,\theta = 0$ (i.e. $\theta = 0$ and $\theta = \pi$).
$r = a$ is the streamline $\psi = 0$. On this streamline

$$\underline{q} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta = -\frac{\partial\psi}{\partial r}\underline{e}_\theta = -2U\sin\,\theta.$$