The material time derivative

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}f(r,t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + \underline{q}\cdot\nabla f.\end{aligned}$$

The equation of mass conservation

This can be written as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{q}) = 0$$

or equivalently

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\underline{q} = 0.$$

The flow is incompressible if

$$\nabla\cdot\underline{q} = 0.$$

Equation of hydrostatic pressure

Let Ω be an region with surface S. In equilibrium

$$(\text{force on surface}) + (\text{force due to gravity}) = \underline{0}$$

Euler's equations of motion

$$(\text{force due to acc.}) + (\text{force on surface}) + (\text{force due to gravity}) = \underline{0}.$$

$$\rho \frac{D\underline{q}}{Dt} = -\nabla(p + \rho g z).$$

With steady flow

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \underline{q}}{\partial t} = \underline{0} \quad \text{and} \quad \frac{D\underline{q}}{Dt} = (\underline{q}\cdot\nabla)\underline{q} = \nabla\left(\frac{1}{2}|\underline{q}|^2\right) + (\nabla\times\underline{q})\times\underline{q}.$$

The term

$$\underline{\omega} = \nabla \times \underline{q}$$

is known as the **vorticity**. We can connect the velocity $\underline{q}$ with the pressure p along streamlines. With several 2D flows $\underline{\omega} = \underline{0}$.

2D steady incompressible flow and the stream function ψ

Assuming that the xy-plane is the 2D region the incompressibility condition

$$\nabla \cdot \underline{q} = 0$$

implies the existence of a stream function ψ such that

$$\underline{q} = (\nabla \psi) \times \underline{k}.$$

In Cartesian's $\psi = \psi(x, y)$ and

$$\underline{q} = \frac{\partial \psi}{\partial y}\underline{i} - \frac{\partial \psi}{\partial x}\underline{j}$$

and in polars $\psi = \psi(r, \theta)$ and

$$\underline{q} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_r - \frac{\partial \psi}{\partial r}\underline{e}_\theta.$$

Curves of the form $\psi =$const are the streamlines of the flow.

Vorticity in the 2D incompressible case

$$
\begin{aligned}
\underline{\omega} = \nabla \times \underline{q} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u(x,y) & v(x,y) & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{k} \\
&= -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \underline{k}.
\end{aligned}
$$

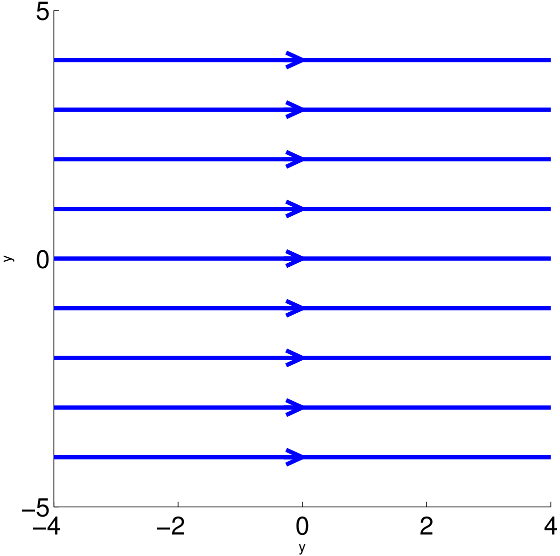
Irrotational flow is when $\underline{\omega} = \nabla \times \underline{q} = \underline{0}$ and this is when ψ satisfies Laplace's equation. In this 2D case we then also have a velocity potential ϕ such that

$$
\underline{q} = \nabla \phi = (\nabla \psi) \times \underline{k}.
$$

ϕ also satisfies Laplace's equation.

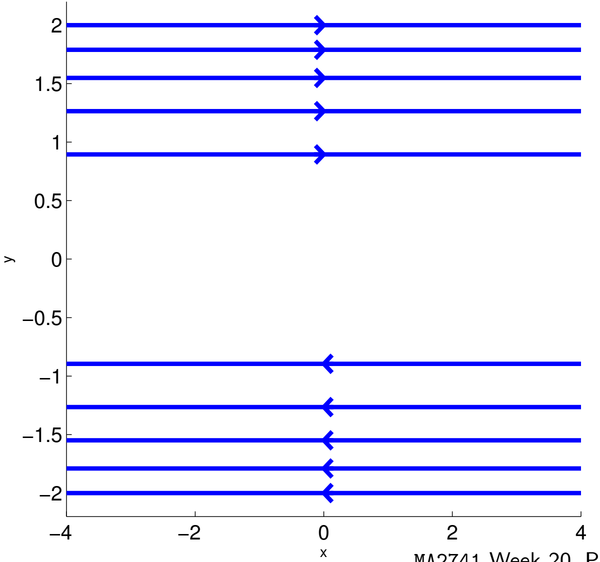
Streamlines for uniform flow in the $\underline{i}$ direction

$$\psi = -a_2 x + a_1 y \quad \text{and} \quad \underline{q} = a_1\,\underline{i} + a_2\,\underline{j}.$$



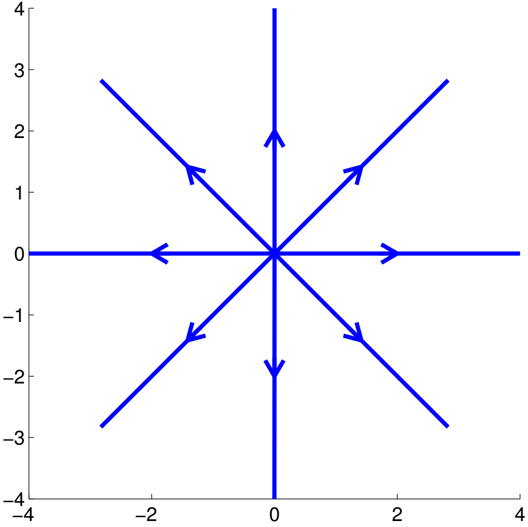
Streamlines for a simple shear flow

$$\psi = \frac{1}{2}\beta y^2 \quad \text{and} \quad \underline{q} = \beta y\, \underline{i}.$$



Streamlines for a line source at $r = 0$

$$\psi = A\theta \quad \text{and} \quad \underline{q} = \frac{A}{r}\,\underline{e}_r.$$



Streamlines for a line vortex at $r = 0$

$$\psi(r) = -\frac{\Gamma}{2\pi} \ln\left(\frac{r}{a}\right) \quad \text{and} \quad \underline{q} = \left(\frac{\Gamma}{2\pi}\right) \frac{1}{r} \, \underline{e}_\theta .$$