Terminology: Lagrangean, Eulerian descriptions

When $\underline{r}_0$ at time 0 moves to $\underline{r}$ at time t and

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t)$$

the **material time derivative** is

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\underline{r}, t) = \frac{\partial}{\partial t} f_L(\underline{r}_0, t) = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f$$

where $\underline{q} = \underline{q}(\underline{r}, t)$ is the velocity. Here f_L is the Lagrangean description and f is the Eulerian description.

Applying this to u, v and w in $\underline{q} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$ gives

$$\begin{aligned} (\text{Lagrangean acc}) &= (\text{Local acc}) + (\text{Convective acc}) \\ \underline{a}_L = \frac{\mathrm{D}}{\mathrm{D}t}\underline{q} &= \frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \nabla)\underline{q}. \end{aligned}$$

Terminology describing fluid flows

$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}, \quad \underline{q} = \underline{q}(\underline{r}, t) = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}.$$

Two-dimensional flow in x, y plane:

$$w = 0 \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial \rho}{\partial z} = 0.$$

Steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \underline{q}}{\partial t} = 0.$$

Stagnation Point: This is a point at which $\underline{q} = \underline{0}$ for all time.

Particle Paths: From a Eulerian description we have

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q}(\underline{r}, t), \quad \text{with } \underline{r}(0) = \underline{r}_0.$$

Streamlines:

$$x(s)\underline{i} + y(s)\underline{j} + z(s)\underline{k} \quad \text{where} \quad \frac{x'(s)}{u} = \frac{y'(s)}{v} = \frac{z'(s)}{w}.$$

Streamlines are the same as particle paths when the flow is steady.

Example

2D unsteady flow described in Eulerian form by

$$\underline{q}(\underline{r}, t) = \underline{q}_E(\underline{r}, t) = xt\,\underline{i} - y\,\underline{j}, \quad t \geq 0.$$

Particle paths satisfy

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u = xt, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v = -y. \end{aligned}$$

Streamlines at fixed time t satisfy

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} \quad \text{i.e.} \ \frac{\mathrm{d}x}{xt} = \frac{\mathrm{d}y}{-y}$$

Example

2D unsteady flow described in Eulerian form by

$$\underline{q} = U\,\underline{i} + \frac{x}{(1+t)}\,\underline{j}, \quad t \geq 0,$$

Particle paths satisfy

$$\begin{aligned} \frac{dx}{dt} &= u = U, \\ \frac{dy}{dt} &= \frac{x}{1+t}. \end{aligned}$$

Specifying an initial condition $x(0) = x_0$, $y(0) = y_0$ gives a particular path.

Streamlines at fixed time t satisfy

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{i.e. } \frac{dx}{U} = \frac{dy}{\left(\frac{x}{1+t}\right)}.$$

Equation of mass conservation and incompressibility

For a region Ω without sources and sinks the change in the amount of mass in Ω is entirely encountered for by the flow of the material through the surface S of Ω and this leads to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0$$

or equivalently

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \underline{q} = 0.$$

A material is **incompressible** if

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \text{which is equivalent to} \quad \nabla \cdot \underline{q} = 0.$$

Equation of hydrostatic pressure

Let Ω be an region with surface S. In equilibrium

$$(\text{force on surface}) + (\text{force due to gravity}) = \underline{0}$$

Euler's equations of motion

$$(\text{force due to acc.}) + (\text{force on surface}) + (\text{force due to gravity}) = \underline{0}.$$

$$\rho \frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = -\nabla(p + \rho g z).$$

With steady flow

$$\frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = (\underline{q} \cdot \nabla)\underline{q} = \nabla\left(\frac{1}{2}|\underline{q}|^2\right) + (\nabla \times \underline{q}) \times \underline{q}.$$

The term

$$\underline{\omega} = \nabla \times \underline{q}$$

is known as the vorticity. We can connect the velocity $\underline{q}$ with the pressure p along streamlines. With several 2D flows $\underline{\omega} = \underline{0}$.

Further comments relating to Euler's equations

When the fluid is incompressible the density ρ is constant.

Steady incompressible flow thus involves

$$\rho \frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = \rho(\underline{q} \cdot \nabla)\underline{q} = -\nabla(p + \rho g z).$$

Using the identity gives

$$\rho \nabla \left(\frac{1}{2} |\underline{q}|^2 \right) + \rho(\nabla \times \underline{q}) \times \underline{q} = -\nabla(p + \rho g z).$$

i.e.

$$\nabla \left(\frac{1}{2} |\underline{q}|^2 + \frac{p}{\rho} + g z \right) + (\nabla \times \underline{q}) \times \underline{q} = \underline{0}.$$

On a streamline

$$H = \frac{1}{2} |\underline{q}|^2 + \frac{p}{\rho} + g z$$

is a constant. H is constant everywhere when $\underline{\omega} = \nabla \times \underline{q} = \underline{0}$.

Equations when we have viscous fluids

When there is no viscosity the stress is of the form

$$\boldsymbol{\sigma} = -pI.$$

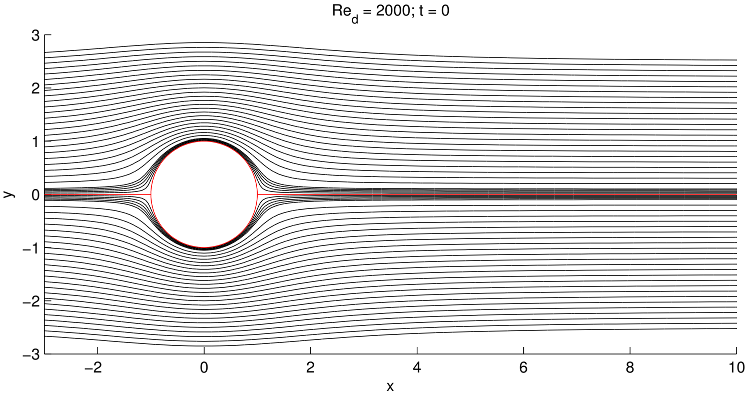
When we have and incompressible Newtonian fluid with viscosity $\mu \neq 0$ we have instead

$$\boldsymbol{\sigma} = -pI + \mu \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} & \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} & 2\dfrac{\partial w}{\partial z} \end{pmatrix}.$$

The equations of motion that this leads to are known as the Navier Stokes' equations. The following plots were obtained by approximately solving these equations using a Matlab program available from the URL: `http://www.cfmbyexample.com/resources/Cylinder_06October2011.zip`

Flow around a cylinder – inviscid case, $\mu = 0$

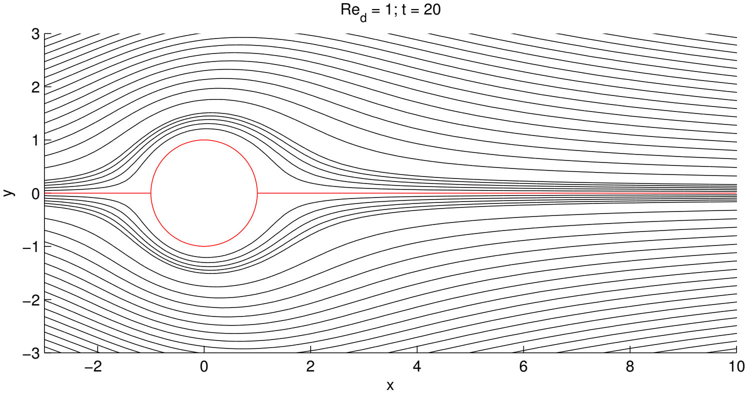
There is frictionless flow around the cylinder. The streamlines for this case are given in the next chapter of the notes.



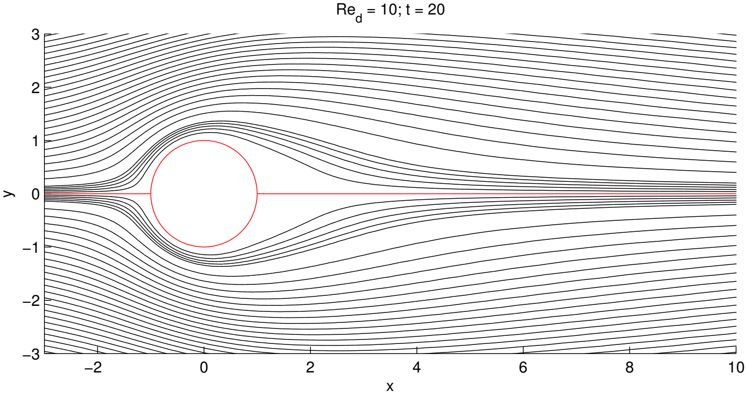
Flow around a cylinder – Reynold's number, Re=1

No fluid has no viscosity. The flow depends on something known as the Reynold's number where

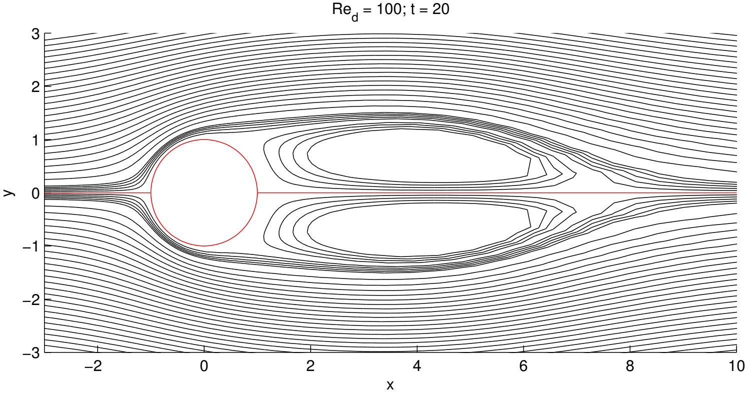
$$\mathrm{Re} \propto \frac{|\text{inertia terms in equation of motion}|}{|\text{viscous terms in equation of motion}|}.$$



Flow around a cylinder – Re=10



Flow around a cylinder – Re=100



An inviscid model is not adequate in this case.