Gradient, divergence and curl in cartesian coordinates

Gradient of ϕ.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\underline{i} + \frac{\partial\phi}{\partial y}\,\underline{j} + \frac{\partial\phi}{\partial z}\,\underline{k},$$

Divergence of $\underline{F} = F_i\underline{i} + F_2\underline{j} + F_3\underline{k}$.

$$\nabla\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

Curl of $\underline{F} = F_i\underline{i} + F_2\underline{j} + F_3\underline{k}$.

$$\begin{aligned}\nabla\times\underline{F} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}.\end{aligned}$$

Directional derivative, divergence and Stokes' theorems

Directional derivative of ϕ in the direction of $\underline{n}$.

$$\frac{\partial \phi}{\partial n}(\underline{r}) = \left. \frac{\partial}{\partial s}\phi(\underline{r} + s\underline{n}) \right|_{s=0} = \underline{n} \cdot \nabla \phi.$$

Divergence theorem.

$$\begin{aligned} \int_{\Omega} \nabla \cdot \underline{F} \, \mathrm{d}v &= \int_{S} \underline{F} \cdot \underline{n} \, \mathrm{d}s, \\ \int_{\Omega} \nabla p \, \mathrm{d}v &= \int_{S} p \underline{n} \, \mathrm{d}s. \end{aligned}$$

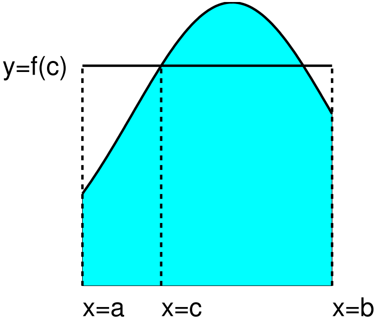
Stokes' theorem.

$$\int_{S} (\nabla \times \underline{F}) \cdot \underline{n} \, \mathrm{d}s = \oint_{C} \underline{F} \cdot \mathrm{d}\underline{r}.$$

In the above $\underline{n}$ denotes an appropriate unit vector in each case.

The divergence and curl defined as a limit

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(c)$$



Divergence as a limit: V is the volume inside surface S and point $\underline{P}$ is inside the surface which we shrink to $\underline{P}$.

$$\nabla \cdot \underline{F}(\underline{P}) = \lim_{V \to 0} \frac{1}{V} \int_S \underline{F} \cdot \underline{n}\,\mathrm{d}s.$$

Curl as a limit: A is the area inside loop C and point $\underline{P}$ is inside the loop which we shrink to $\underline{P}$.

$$(\nabla \times \underline{F}(\underline{P})) \cdot \underline{n} = \lim_{A \to 0} \frac{1}{A} \oint_C \underline{F} \cdot \,\mathrm{d}\underline{r}.$$

Some vector identities involving $\times$ and curl

For the cross product

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a} \quad \text{which implies} \quad \underline{a} \times \underline{a} = \underline{0}.$$

$\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$.

For the base vectors

$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j}.$$

$\nabla \times \underline{F}$ is divergence free as

$$\nabla \cdot (\nabla \times \underline{F}) = 0.$$

$\nabla\phi$ is irrotational in that

$$\nabla \times \nabla\phi = \underline{0}.$$