MA2741: 2012/2013 Term 2

MA2741
Vector Calculus and Applications

Lecture Notes by M.K. Warby in 2012/3
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Remaining assessment dates

Final exam: April/May exam period, 2 hours with the assessment code of MA2841.

The final exam will include material from both term 1 and term 2.

Recommended reading and my sources

There is no essential text to obtain for this part of the module. Much of this material is derived from notes from Dr. Jane Lawrie who has taught this in previous years and who will teach the last few weeks of the module.

Chapter 1

Vector calculus – revision, some key identities and polar coordinates

1.1 A revision of some vector calculus

Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ denote the usual cartesian base vectors in respectively the x, y and z directions. The point (x, y, z) is represented by the position vector

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}.$$

A vector field $\underline{F} = \underline{F}(x, y, z)$ can be represented in component form as

$$\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$$

where $F_i = F_i(x, y, z)$ for $i = 1, 2, 3$. We can also more compactly write $\underline{F} = \underline{F}(\underline{r})$ to indicate that we have a vector input (i.e. $\underline{r}$) and we get a vector value (i.e. $\underline{F}(\underline{r})$). Each of the component functions are scalar valued and for the results that follow also let $\phi = \phi(x, y, z)$, or more compactly $\phi = \phi(\underline{r})$, denote another scalar field.

1.1.1 The gradient, divergence and curl in cartesian coordinates

Assuming that all partial derivatives exist when needed we have the following.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}, \quad \text{the } \textbf{gradient} \text{ of } \phi, \tag{1.1.1}$$

$$\nabla\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}, \quad \text{the } \textbf{divergence} \text{ of } \underline{F}, \tag{1.1.2}$$

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}, \tag{1.1.3}$$

the **curl** of $\underline{F}$.

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}, \quad \text{the } \textbf{Laplacian} \text{ of } \phi.$$

If $\underline{n}$ denotes a unit vector then the **directional derivative** of ϕ in the direction of $\underline{n}$ at $\underline{r}$ is

$$\frac{\partial\phi}{\partial n}(\underline{r}) = \lim_{h\to 0}\frac{\phi(\underline{r}+h\underline{n})-\phi(\underline{r})}{h} = \frac{\partial}{\partial s}\phi(\underline{r}+s\underline{n})\bigg|_{s=0} = \underline{n}\cdot\nabla\phi \tag{1.1.4}$$

where the last part follows by using the chain rule of partial differentiation. An implication of (1.1.4) is that if we have a two dimensional set-up with $\phi=\phi(x,y)$ then one of the ways to attempt to visually show how ϕ varies with position is to give a contour plot in which curves are given along which ϕ has a constant value. In everyday life this is what is done on a weather map with ϕ typically either being the atmospheric pressure or being temperature. As ϕ is constant along these curves the directional derivative of ϕ in a direction tangential to the curve is 0 and hence if $\nabla\phi\neq\underline{0}$ then it is a vector which is orthogonal to the tangent. We have a similar situation in a three dimensional set-up with a region where ϕ is constant instead being a surface. To summarise this discussion we have the following.

$$\nabla\phi \quad \text{is normal to regions of the form } \phi(\underline{r}) = \text{const.}$$

1.1.2 The divergence theorem and Stokes' theorem

When we first meet differentiation and integration with functions of one variable we learn at an early stage that in appropriate sense that they are the inverse of each other (the fundamental theorem of calculus), i.e.

$$\int_a^b f'(x)\,\mathrm{d}x = f(b)-f(a).$$

In the case of functions of several variables and with partial derivatives the results of this type in vector calculus are as follows.

For a three dimensional region Ω with surface S **the divergence theorem** is

$$\boxed{\int_\Omega \nabla\cdot\underline{F}\,\mathrm{d}v = \int_S \underline{F}\cdot\underline{n}\,\mathrm{d}s}$$

where $\underline{n}$ denotes the unit outward normal to S.

For a surface S inside a closed loop C considered in the anti-clockwise sense and with the unit normal $\underline{n}$ to S considered in this sense **Stokes' theorem** is

$$\boxed{\int_S (\nabla\times\underline{F})\cdot\underline{n}\,\mathrm{d}s = \oint_C \underline{F}\cdot\mathrm{d}\underline{r}.}$$

When the surface S is in the x, y plane and the unit normal is $\underline{n}=\underline{k}$ this reduces to

$$\iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y$$

which is known as **Green's theorem in the plane**.

1.1.3 A further result derived from the divergence theorem

Let

$$\underline{n} = n_1\,\underline{i} + n_2\,\underline{j} + n_3\underline{k}$$

denote the unit outward normal evaluated on a closed surface S with Ω denoting the region inside S. Now if we take $\underline{F} = p\underline{i}$, where $p = p(x, y, z)$, then the divergence theorem applied to $\underline{F}$ gives

$$\int_\Omega \frac{\partial p}{\partial x}\,\mathrm{d}v = \int_S p n_1\,\mathrm{d}s \tag{1.1.5}$$

since $\underline{F}\cdot\underline{n} = p\,n_1$. Similar results are obtained if we take $\underline{F} = p\underline{j}$ or $\underline{F} = p\underline{k}$ and these are

$$\int_\Omega \frac{\partial p}{\partial y}\,\mathrm{d}v = \int_S p n_2\,\mathrm{d}s \tag{1.1.6}$$

and

$$\int_\Omega \frac{\partial p}{\partial z}\,\mathrm{d}v = \int_S p n_3\,\mathrm{d}s. \tag{1.1.7}$$

In fact the usual proof of the divergence theorem obtains these results as steps in the proof. However, if we now combine the 3 results by taking (1.1.5) times $\underline{i}$ plus (1.1.6) times $\underline{j}$ plus (1.1.7) times $\underline{k}$ then we obtain the vector identity

$$\int_\Omega \nabla p\,\mathrm{d}v = \int_S p\underline{n}\,\mathrm{d}s.$$

This result will be used when the equations governing how a fluid flows are considered and p denotes the pressure in the fluid.

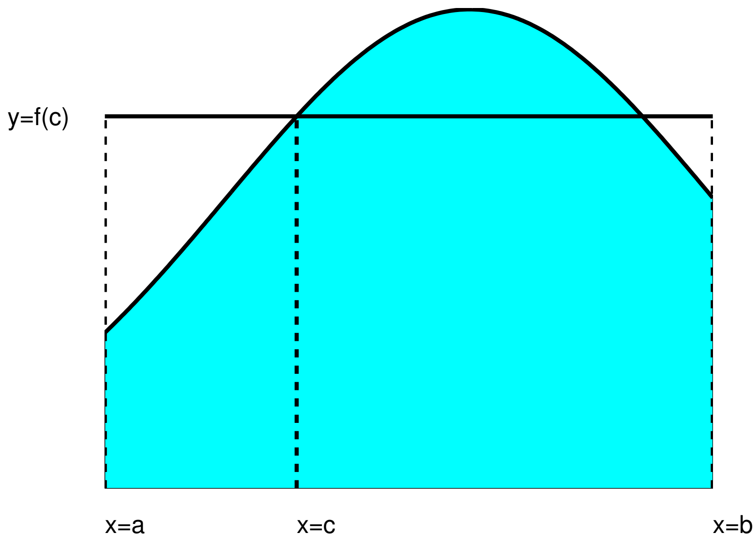
1.1.4 The divergence and curl defined as limits

The divergence theorem and Stokes' theorem enable us to express $\nabla\cdot\underline{F}$ and $\nabla\times\underline{F}$ as limits as follows.

In the case of continuous functions of one variable we have the mean value theorem for integrals which is

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)f(c)$$

for some point $c \in (a, b)$. All this says is that the area under the curve of $f(x)$ between a and b is equal to the area of a rectangle with width $b - a$ and height which is one of the function values and this is illustrated below.



When $b-a$ is small then c is close to both a and b and in the limit we have

$$f(a)=\lim_{b-a\to 0}\int_a^b f(x)\,\mathrm{d}x.$$

A similar result holds for volume integrals and surface integrals where we just need to replace the length of the interval $b-a$ by the volume being considered in the case of the volume integral or the area of the surface in the case of the surface integral and the outcome is as follows.

For the divergence we have

$$\boxed{\nabla\cdot\underline{F}(\underline{P})=\lim_{V\to 0}\frac{1}{V}\int_S \underline{F}\cdot\underline{n}\,\mathrm{d}s} \tag{1.1.8}$$

where in the expression in the limit S is a closed surface containing the point $\underline{P}$ inside the surface which we are shrinking so that it just contains $\underline{P}$ as an interior point. $\underline{n}$ is the outward normal to S and thus it is pointing away from the point $\underline{P}$.

For the curl we have

$$\boxed{(\nabla\times\underline{F}(\underline{P}))\cdot\underline{n}=\lim_{A\to 0}\frac{1}{A}\oint_C \underline{F}\cdot\,\mathrm{d}\underline{r}} \tag{1.1.9}$$

where in the expression in the limit A is the area inside the closed contour C which contains the point $\underline{P}$ and $\underline{n}$ is the unit vector normal to the surface element that C generates with the direction determined by the direction of the line integral. For a particular case of $\underline{n}$ we have $\underline{n}=\underline{k}$ when C is a circle in the x,y plane and the line integral is considered in the anti-clockwise sense, i.e. if $\underline{r}(\theta)=x_0\underline{i}+y_0\underline{j}+\epsilon(\cos\theta\underline{i}+\sin\theta\underline{j})$, $0\le\theta\le 2\pi$ is a parametrisation of such a circle centred at (x_0,y_0) and with radius ϵ then the anti-clockwise sense corresponds to increasing θ.

The representations given in (1.1.8) and (1.1.9) can be taken as defining the divergence of a vector field and the curl of a vector field.

1.1.5 Some vector identities useful in the description of fluids

In term 1 a number of identities involving the ∇ operator would have been considered and we recall some of these here which will be needed later when fluid flow is described.

1.
$$\boxed{\nabla \cdot (\nabla \times \underline{F}) = 0.} \qquad (1.1.10)$$

This follows by using the definition of curl and divergence given in (1.1.3) and (1.1.2) respectively as we have

$$\nabla\cdot(\nabla\times\underline{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right). \quad (1.1.11)$$

When $\underline{F}$ is twice continuously differentiable we can do the partial differentiation in any order and thus

$$\frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x}\right) = 0$$

for two of the six terms in the above expression. Observe that in (1.1.11) each term with a plus sign is matched by a corresponding term with a minus sign and we get 0 as the outcome.

A vector field $\underline{A}$ which is such that $\nabla \cdot \underline{A} = 0$ is said to be divergence free and this shows that $\nabla \times \underline{F}$ is divergence free. In fact it can be shown that if $\nabla \cdot \underline{A} = 0$ then there exists a **vector potential** $\underline{F}$ such that $\underline{A} = \nabla \times \underline{F}$ and we will use this later in the module although we never prove the result. The result is used when incompressible fluid flows are considered with the property characterising such a flow being that $\nabla \cdot \underline{q} = 0$ where $\underline{q}$ denotes the velocity of the fluid. In fact when incompressible flows are considered we will restrict attention to two-dimensional flows in the x, y plane which will lead to $\underline{A}$ being of the form $\underline{A} = \psi \underline{k}$ and we will only have one scalar function $\psi = \psi(x, y)$ to consider and this will be known as the **stream function**.

2.
$$\boxed{\nabla \times \nabla\phi = \underline{0}.} \qquad (1.1.12)$$

This follows from the definition of the gradient and the definition of the curl given in (1.1.1) and (1.1.3) as

$$\begin{aligned}
(\nabla \times \nabla\phi) \cdot \underline{i} &= \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = 0, \\
(\nabla \times \nabla\phi) \cdot \underline{j} &= \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) = 0, \\
(\nabla \times \nabla\phi) \cdot \underline{k} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0
\end{aligned}$$

where the property that partial differentiation can be done in any order is used in all 3 cases.

A vector field $\underline{A}$ such that $\nabla \times \underline{A} = \underline{0}$ is said to be **irrotational** and this shows that $\nabla\phi$ is irrotational. In fact if $\nabla \times \underline{A} = \underline{0}$ then it can be shown that there exists a scalar function ϕ such that $\underline{A} = \nabla\phi$ and this is not too difficult to do as we can take

$$\phi(\underline{r}) = \int_{\underline{r}_0}^{\underline{r}} \underline{A} \cdot \mathrm{d}\underline{r},$$

for any fixed point $\underline{r}_0$. In the expression for $\phi(\underline{r})$ any path between $\underline{r}_0$ and $\underline{r}$ can be used as the value of the line integral only depends on the end points as a consequence of Stokes' theorem applied to $\underline{A}$. In the context of describing fluid flow the case of what is known as irrotational flow is when $\nabla \times \underline{q} = \underline{0}$ where, as above, $\underline{q}$ is velocity and the corresponding function ϕ is known as the **velocity potential**.

3. If

$$\underline{q} = q_1\underline{i} + q_2\underline{j} + q_3\underline{k}$$

then

$$\nabla\left(\frac{1}{2}|\underline{q}|^2\right) - \underline{q} \times (\nabla \times \underline{q}) = (\underline{q} \cdot \nabla q_1)\underline{i} + (\underline{q} \cdot \nabla q_2)\underline{j} + (\underline{q} \cdot \nabla q_3)\underline{k}. \qquad (1.1.13)$$

The right hand side is typically abbreviated so that we can write

$$\nabla\left(\frac{1}{2}|\underline{q}|^2\right) - \underline{q} \times (\nabla \times \underline{q}) = (\underline{q} \cdot \nabla)\underline{q}. \qquad (1.1.14)$$

This is one of the harder results to prove and the result will only be mentioned again when the equations governing how a fluid flows are briefly discussed. In the context of fluids the vector $\underline{q}$ will denote the velocity of the fluid and the vector $\underline{\omega} = \nabla \times \underline{q}$ is called the vorticity and both of these will be mentioned in the module. This is an important identity if you study fluids beyond what is done in `MA2741`.

To prove the result first note that

$$\frac{1}{2}|\underline{q}|^2 = \frac{1}{2}\left(q_1^2 + q_2^2 + q_3^2\right).$$

Now for the gradient of the first term we have

$$\nabla\left(\frac{1}{2}q_1^2\right) = q_1\frac{\partial q_1}{\partial x}\underline{i} + q_1\frac{\partial q_1}{\partial y}\underline{j} + q_1\frac{\partial q_1}{\partial z}\underline{k} = q_1\nabla q_1.$$

We get similar expressions for the q_2 and q_3 parts and combining these we have

$$\nabla\left(\frac{1}{2}|\underline{q}|^2\right) = q_1\nabla q_1 + q_2\nabla q_2 + q_3\nabla q_3.$$

For the curl term we have

$$\nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{vmatrix} = \left(\frac{\partial q_3}{\partial y} - \frac{\partial q_2}{\partial z}\right)\underline{i} - \left(\frac{\partial q_3}{\partial x} - \frac{\partial q_1}{\partial z}\right)\underline{j} + \left(\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y}\right)\underline{k}.$$

For the cross product of $\underline{q}$ and $\nabla \times \underline{q}$ we have

$$\underline{q} \times (\nabla \times \underline{q}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ q_1 & q_2 & q_3 \\ \left(\frac{\partial q_3}{\partial y} - \frac{\partial q_2}{\partial z}\right) & \left(\frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial x}\right) & \left(\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y}\right) \end{vmatrix}.$$

The coefficient of $\underline{i}$ is

$$q_2 \left(\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y}\right) - q_3 \left(\frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial x}\right) = \left(q_2 \frac{\partial q_2}{\partial x} + q_3 \frac{\partial q_3}{\partial x}\right) - \left(q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z}\right).$$

The $\underline{i}$ component on the left hand side of (1.1.13) is hence

$$\begin{aligned} &\left(q_1 \frac{\partial q_1}{\partial x} + q_2 \frac{\partial q_2}{\partial x} + q_3 \frac{\partial q_3}{\partial x}\right) - \left(q_2 \frac{\partial q_2}{\partial x} + q_3 \frac{\partial q_3}{\partial x}\right) + \left(q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z}\right) \\ = \quad & q_1 \frac{\partial q_1}{\partial x} + q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z} = \underline{q} \cdot \nabla q_1. \end{aligned}$$

This proves the result for the $\underline{i}$ component. The details for the $\underline{j}$ and $\underline{k}$ components are similar.

1.1.6 The gradient, divergence and curl in polar coordinates

Later in the module we consider two-dimensional flows and some of these are most easily described using cylindrical polar coordinates and we consider next representations for the gradient, divergence and curl for scalar valued and vector valued functions of r, θ and z. In questions you will always have formulae sheets available to check the precise details and thus it is not essential to memorize all of the details that follow although it usually helps to have some knowledge of how the formulae are obtained.

First to relate x, y and r, θ we have

$$x = r \cos\theta, \quad y = r \sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

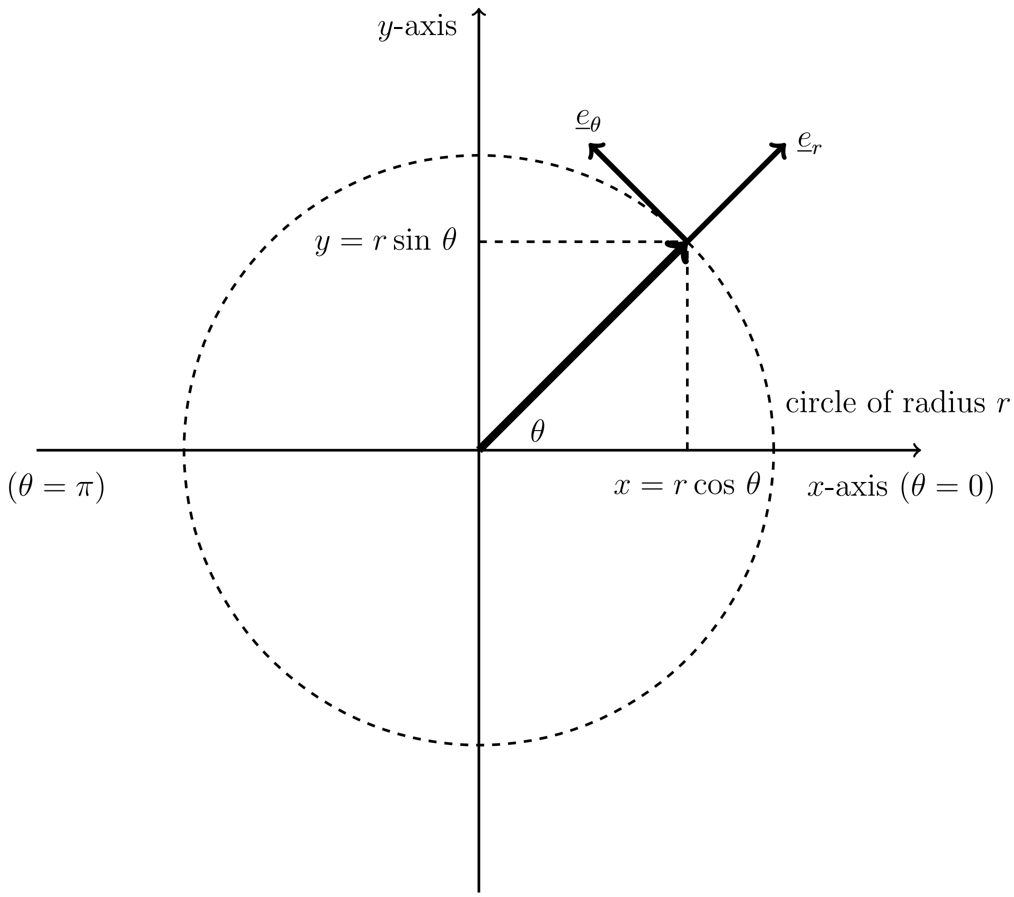
The unit base vectors $\underline{e}_r = \underline{e}_r(\theta)$ and $\underline{e}_\theta = \underline{e}_\theta(\theta)$ are given by

$$\begin{aligned} \underline{e}_r &= \cos\theta \underline{i} + \sin\theta \underline{j}, \\ \underline{e}_\theta &= -\sin\theta \underline{i} + \cos\theta \underline{j} \end{aligned}$$

and for a general point we have

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = r\, \underline{e}_r(\theta) + z\underline{k}$$

and we illustrate such a vector in the x, y plane below.



It is important to note here that $\underline{e}_r$ and $\underline{e}_\theta$ both depend on θ and we need to take this into account when we differentiate with respect to θ as we have

$$\frac{\partial}{\partial\theta}\,\underline{e}_r = \underline{e}_\theta \quad \text{and} \quad \frac{\partial}{\partial\theta}\,\underline{e}_\theta = -\underline{e}_r. \tag{1.1.15}$$

Another point to note is that $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{k}$ are orthogonal unit vectors and in terms of the cross product

$$\underline{e}_r \times \underline{e}_\theta = \underline{k}, \quad \underline{e}_\theta \times \underline{k} = \underline{e}_r, \quad \underline{k} \times \underline{e}_r = \underline{e}_\theta, \tag{1.1.16}$$

i.e. the cross product of any two of the vectors gives plus or minus the other vector.

The gradient

For a scalar field $\phi(x, y, z)$ we define

$$\tilde{\phi}(r, \theta, z) = \phi(r\cos\theta, r\sin\theta, z).$$

By the chain rule

$$\begin{aligned}
\frac{\partial\tilde{\phi}}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial\phi}{\partial x} + \sin\theta\frac{\partial\phi}{\partial y} = \underline{e}_r\cdot\nabla\phi,\\
\frac{\partial\tilde{\phi}}{\partial\theta} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta} = r\left(-\sin\theta\frac{\partial\phi}{\partial x} + \cos\theta\frac{\partial\phi}{\partial y}\right) = r\underline{e}_\theta\cdot\nabla\phi.
\end{aligned}$$

Hence we know the components of $\nabla\phi$ in the $\underline{e}_r$ and $\underline{e}_\theta$ directions and for the gradient itself we have

$$\nabla\phi = (\nabla\phi\cdot\underline{e}_r)\underline{e}_r + (\nabla\phi\cdot\underline{e}_\theta)\underline{e}_\theta + (\nabla\phi\cdot\underline{k})\underline{k} = \frac{\partial\tilde{\phi}}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial\theta}\underline{e}_\theta + \frac{\partial\tilde{\phi}}{\partial z}\underline{k}.$$

As a comment on the notation, the functions ϕ and $\tilde{\phi}$ have the same values in the sense described above and it is common to just write $\phi(r,\theta,z)$ when the context is a polar representation of the function so that we do not have to use slightly more awkward notation of $\tilde{\phi}$. With this understanding and considering now $\phi = \phi(r,\theta,z)$ we have

$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta + \frac{\partial\phi}{\partial z}\underline{k}. \qquad (1.1.17)$$

For the divergence and curl result that follow it is convenient to note that this gives representation for the operator ∇ in this coordinate system in that we have

$$\nabla = \underline{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\underline{e}_\theta\frac{\partial}{\partial\theta} + \underline{k}\frac{\partial}{\partial z}. \qquad (1.1.18)$$

The divergence

Let $\underline{F} = \underline{F}(r,\theta,z)$ denote a vector field which in components is

$$\underline{F} = F_r\underline{e}_r + F_\theta\underline{e}_\theta + F_3\underline{k}.$$

With the operator ∇ as described in (1.1.18) we have

$$\left(\underline{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\underline{e}_\theta\frac{\partial}{\partial\theta} + \underline{k}\frac{\partial}{\partial z}\right)\cdot(F_r\underline{e}_r + F_\theta\underline{e}_\theta + F_3\underline{k}).$$

To understand what follows we need to note that $\underline{e}_r$ and $\underline{e}_\theta$ both vary with θ and hence there are some products to differentiate and also the partial differentiation is done first and the dot product operation is done afterwards. If we consider things term by term we have

$$\begin{aligned}
\underline{e}_r\frac{\partial}{\partial r}\cdot\underline{F} &= \frac{\partial F_r}{\partial r},\\
\frac{1}{r}\underline{e}_\theta\frac{\partial}{\partial\theta}\cdot\underline{F} &= \frac{1}{r}\underline{e}_\theta\cdot\left(\frac{\partial F_r}{\partial\theta}\underline{e}_r + F_r\frac{\partial}{\partial\theta}\underline{e}_r + \frac{\partial F_\theta}{\partial\theta}\underline{e}_\theta + F_\theta\frac{\partial}{\partial\theta}\underline{e}_\theta + \frac{\partial F_3}{\partial\theta}\underline{k}\right)\\
&= \frac{1}{r}\left(F_r + \frac{\partial F_\theta}{\partial\theta}\right) \quad \text{(using (1.1.15))}\\
\underline{k}\frac{\partial}{\partial z}\cdot\underline{F} &= \frac{\partial F_3}{\partial z}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\nabla\cdot\underline{F} &= \left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r}\right) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_3}{\partial z} && (1.1.19)\\
&= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_3}{\partial z}. && (1.1.20)
\end{aligned}$$

The curl

There are more terms in the expression for the curl and thus the details are a bit longer here and the result will eventually be expressed as a determinant but we start with the following which uses the ∇ operator in cylindrical polars.

$$\nabla \times \underline{F} = \left(\underline{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\underline{e}_\theta \frac{\partial}{\partial \theta} + \underline{k}\frac{\partial}{\partial z} \right) \times \left(F_r \underline{e}_r + F_\theta \underline{e}_\theta + F_3 \underline{k} \right).$$

As in the case of the derivation of the expression for the divergence we do the operation in parts and make use of both (1.1.15) and (1.1.16) to give

$$\begin{aligned}
\underline{e}_r \frac{\partial}{\partial r} \times \underline{F} &= \underline{e}_r \times \left(\frac{\partial F_r}{\partial r}\underline{e}_r + \frac{\partial F_\theta}{\partial r}\underline{e}_\theta + \frac{\partial F_3}{\partial r}\underline{k} \right) = \frac{\partial F_\theta}{\partial r}\underline{k} - \frac{\partial F_3}{\partial r}\underline{e}_\theta, \\
\frac{1}{r}\underline{e}_\theta \frac{\partial}{\partial \theta} \times \underline{F} &= \frac{1}{r}\underline{e}_\theta \times \left(\frac{\partial F_r}{\partial \theta}\underline{e}_r + F_r \frac{\partial}{\partial \theta}\underline{e}_r + \frac{\partial F_\theta}{\partial \theta}\underline{e}_\theta + F_\theta \frac{\partial}{\partial \theta}\underline{e}_\theta + \frac{\partial F_3}{\partial \theta}\underline{k} \right) \\
&= \frac{1}{r}\underline{e}_\theta \times \left(\frac{\partial F_r}{\partial \theta}\underline{e}_r + F_r \underline{e}_\theta + \frac{\partial F_\theta}{\partial \theta}\underline{e}_\theta - F_\theta \underline{e}_r + \frac{\partial F_3}{\partial \theta}\underline{k} \right) \\
&= \frac{1}{r}\left(-\frac{\partial F_r}{\partial \theta}\underline{k} + F_\theta \underline{k} + \frac{\partial F_3}{\partial \theta}\underline{e}_r \right), \\
\underline{k}\frac{\partial}{\partial z} \times \underline{F} &= \underline{k} \times \left(\frac{\partial F_r}{\partial z}\underline{e}_r + \frac{\partial F_\theta}{\partial z}\underline{e}_\theta + \frac{\partial F_3}{\partial z}\underline{k} \right) = \frac{\partial F_\theta}{\partial r}\underline{e}_\theta - \frac{\partial F_3}{\partial r}\underline{e}_r,
\end{aligned}$$

Combining the three parts gives

$$\begin{aligned}
\nabla \times \underline{F} &= \left(\frac{1}{r}\frac{\partial F_3}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right)\underline{e}_r - \left(\frac{\partial F_3}{\partial r} - \frac{\partial F_r}{\partial z} \right)\underline{e}_\theta + \left(\frac{\partial F_\theta}{\partial r} + \frac{F_\theta}{r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta} \right)\underline{k} \\
&= \left(\frac{1}{r}\frac{\partial F_3}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial z}(rF_\theta) \right)\underline{e}_r - \left(\frac{\partial F_3}{\partial r} - \frac{\partial F_r}{\partial z} \right)\underline{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial \theta} \right)\underline{k} \\
&= \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.
\end{aligned}$$

In the two-dimensional case when $F_3 = 0$ and F_r and F_θ do not depend on z this reduces to

$$\nabla \times \underline{F} = \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial \theta} \right)\underline{k}. \tag{1.1.21}$$

Chapter 2

An introduction to a continuum model of fluid flow

Matter is discontinuous on the microscopic scale but in everyday life we encounter matter on a much larger scale which is referred to as the macroscopic scale. Continuum models of matter is concerned with describing matter on the macroscopic scale using functions which are continuous and **continuum mechanics** is concerned with the mechanical behaviour using a **continuum model**. Continuum mechanics itself includes a study of all types of matter and hence includes both solids and fluids and **solid mechanics** and **fluid mechanics** are the names given when respectively only solids or only fluids are considered. **Fluid dynamics** is a subset of fluid mechanics when the motion of a fluid is considered and this is what is mostly done in this part of the module. The quantities that we need to describe how solids and fluids behave include the displacement, velocity, density, pressure, stress and various partial derivatives of these. In the case of solids the problem is typically to determine the displacement of a structure to a given load from which the stresses throughout the structure are determined. In the case of fluids the problem is typically to determine the velocity of the fluid. In other applications the problem is to determine how waves travel through the media. As the comments indicate, continuum mechanics covers a wide range of topics and in this module we restrict attention to a few aspects of fluid mechanics involving mainly idealised two dimensional flows (the part given by Michael Warby) followed by the theory and solutions to problems involving the propagation of sound waves through a continuum (the part given by Jane Lawrie). The connection of this material to the earlier material on vectors is that many of the quantities needed to describe the physical situation are vectors and in several places we will need the gradient of a scalar and the divergence and curl of vectors.

2.1 What is a fluid?

For a fluid we usually think of water or air and for a solid we might think of the structure part of a building though it is not easy to give a precise definition of either a solid or a fluid as some materials exhibit both solid-like and fluid-like properties. A fluid material is however usually taken to mean a substance which cannot resists a shear force or a shear stress without moving and if a state of equilibrium is reached then there are no shear stresses. To understand this statement characterising a property of a fluid you need to know what equilibrium means, what stress means and in particular what shear stress means. A state of equilibrium is a state where there are no inertia terms which for this module will be the case when the fluid is not accelerating and we describe what is meant by acceleration of a continuum later in the notes. We start by considering briefly what stress means.

2.2 Force, stress $\boldsymbol{\sigma}$ and pressure p in a continuum

In a three dimensional body stress is something which enables the internal forces in the body to be determined. To be a bit more precise it is actually a tensor which we can represent as a 3×3 symmetric matric and for this module it is sufficient to accept that the components of the internal force on an infinitesimal surface element

$$\delta s\, \underline{n} = \delta s\, (n_1\, \underline{i} + n_2\, \underline{j} + n_3\, \underline{k}), \quad \underline{n} \text{ is a unit vector,}$$

are given by the matrix vector product

$$\delta s \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

Thus, as examples, by taking $\underline{n}$ to correspond to respectively to $\underline{i}$, $\underline{j}$ and $\underline{k}$ we get forces of

$$\begin{aligned} \delta s(\sigma_{11}\underline{i} + \sigma_{21}\underline{j} + \sigma_{31}\underline{k}), &\quad \text{when } \underline{n} = \underline{i}, \\ \delta s(\sigma_{21}\underline{i} + \sigma_{22}\underline{j} + \sigma_{32}\underline{k}), &\quad \text{when } \underline{n} = \underline{j}, \\ \delta s(\sigma_{31}\underline{i} + \sigma_{32}\underline{j} + \sigma_{33}\underline{k}), &\quad \text{when } \underline{n} = \underline{k}. \end{aligned}$$

As we get a force it follows that each stress component has the dimension of force divided by area. If $\boldsymbol{\sigma}$ denotes the matrix, i.e.

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

then for a general surface element $\delta s\, \underline{n}$ the force will have usually have a component in the direction $\underline{n}$ and a component in a direction orthogonal to $\underline{n}$, i.e. we have an outcome of the form

$$\delta s\, \boldsymbol{\sigma}\, \underline{n} = \delta s\, (\alpha \underline{n} + \beta \underline{t})$$

where $\underline{t}$ is a unit vector orthogonal to $\underline{n}$ with possibly a non-zero value of β. The part $\beta \delta s\, \underline{t}$ is known as the shear component of the force. There is no shear term when the vector $\underline{n}$ is an eigenvector of $\boldsymbol{\sigma}$ but there is a shear term in other cases. We get no shear terms only when every direction $\underline{n}$ is an eigenvector of the matrix and this only happens when $\boldsymbol{\sigma}$ is a multiple of the identity matrix I, i.e.

$$\boxed{\boldsymbol{\sigma} = -pI} \tag{2.2.1}$$

where p is known as the hydrostatic pressure. We write $-p$ here instead of p as if we take, for example, a small cube in a fluid then a positive value of p corresponds to the magnitude of the force on the surfaces of the cube due to the neighbouring parts of the fluid as is illustrated in the diagram in figure 2.1

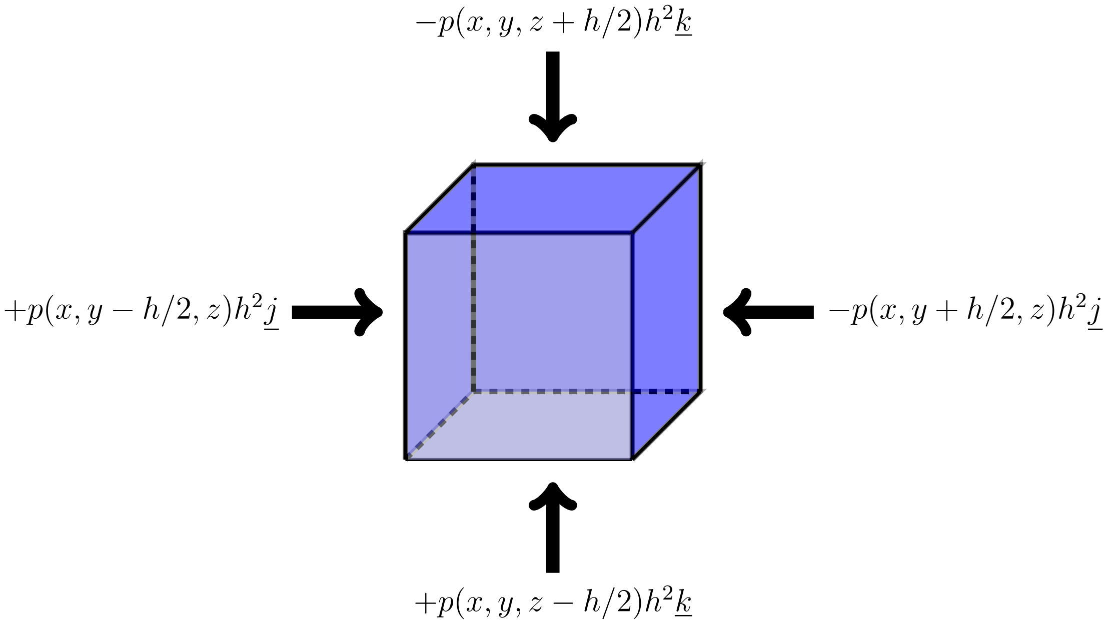


Figure 2.1: A blob of fluid in the shape of a cube of side length equal to h and centred at (x,y,z). There are forces on all 6 sides due to the neighbouring fluid although only 4 arrows corresponding to the $\underline{j}$ and $\underline{k}$ directions are given so as not to clutter the diagram too much. The arrows indicate the forces on the sides. The pressure is evaluated at points on the surface and the magnitude may vary a little depending on which point is being considered. The factor of h^2 in each term is because this is the area of each face of the cube.

For this module we will mostly only consider the situation where the state of stress in a fluid is that of hydrostatic pressure and this is always the case when the fluid is **inviscid**, i.e. the **viscosity** of the fluid is zero. A consequence of this being the only possible state of the fluid is that it has no resistance to shear stress. This can never be exactly true but it can be a good approximation in many situations although the limitations of modelling an actual fluid which will have some viscosity as an fluid with zero viscosity needs to be appreciated and there will be comments on this later in the notes.

2.3 Density ρ and the equilibrium of a fluid when we just have a hydrostatic pressure p

When we just have a hydrostatic pressure the force on a surface element $\delta s\,\underline{n}$ is

$$-p\delta s\,\underline{n}.$$

Thus if we have an infinitesimal blob of fluid occupying a region Ω with volume V and surface S then the force on the surface due to the neighbouring fluid is

$$-\int_S p\underline{n}\,\mathrm{d}s$$

and by one of the identities relating to the divergence theorem given in the previous chapter we have

$$-\int_S p\underline{n}\,\mathrm{d}s = -\int_\Omega \nabla p\,\mathrm{d}v. \tag{2.3.1}$$

When we are in a state of equilibrium there is no inertia force and hence the force in (2.3.1) plus the force due to gravity of the blob must be the zero vector. Now if ρ denotes density then the mass of the blob is

$$\int_\Omega \rho\,\mathrm{d}v$$

and the force due to gravity on the blob is given by

$$\int_\Omega \rho\underline{g}\,\mathrm{d}v.$$

When the direction $\underline{k}$ is the vertical direction the force due to gravity is of the form

$$\underline{g} = -g\underline{k}, \qquad \text{where } g = \text{acceleration due to gravity.}$$

Equilibrium hence requires that

$$-\int_\Omega \nabla p\,\mathrm{d}v + \int_\Omega \rho\underline{g}\,\mathrm{d}v = \int_\Omega (-\nabla p + \rho\underline{g})\,\mathrm{d}v = \underline{0}.$$

As this is must be true for all possible regions Ω it follows that

$$\nabla p = \rho\underline{g} = -\rho g\underline{k} \tag{2.3.2}$$

which is known as the **equation of hydrostatic pressure**.

In the particular case that the density ρ is constant the force due to gravity can be written as a gradient of a scalar involving $-\rho g z$, i.e.

$$-\rho g\underline{k} = -\nabla(\rho g z)$$

and hence in this case

$$\nabla(p + \rho g z) = 0$$

and we get

$$p + \rho g z = \text{const.}$$

This just tells us that as z decreases, i.e. as we consider points deeper in the fluid, the pressure increases due the increasing amount of fluid above.

2.4 The Eulerian and Lagrangean descriptions and the velocity and acceleration of a fluid

Let Ω_0 denote the region of a material at time $t = 0$ which is the initial configuration which we can take as the reference configuration. The motion of a particle which is at position $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ at time $t = 0$ can be described by

$$\underline{r}(\underline{r}_0, t), \quad 0 \le t \quad \text{with } \underline{r}(\underline{r}_0, 0) = \underline{r}_0. \tag{2.4.1}$$

The path that this gives as t varies for a specific point $\underline{r}_0$ is known as a **particle path**. In terms of the components we have

$$\underline{r} = x(\underline{r}_0, t)\underline{i} + y(\underline{r}_0, t)\underline{j} + z(\underline{r}_0, t)\underline{k}.$$

The **velocity** $\underline{q}$ of the particle is

$$\underline{q} = \underline{q}_L(\underline{r}_0, t) = \frac{\partial}{\partial t}\underline{r}(\underline{r}_0, t) = u\underline{i} + v\underline{j} + w\underline{k}. \tag{2.4.2}$$

As a reminder of what the partial differentiation notation means, $\underline{r}_0$ is fixed and just t varies and note that the component functions also depend on space and time although this is not explicitly indicated. Thus at time t the particle starting at $\underline{r}_0$ is at position $\underline{r}(\underline{r}_0, t)$ and has velocity $\underline{q} = \underline{q}_L(\underline{r}_0, t)$ and the subscripted notation $\underline{q}_L$ is used just to emphasise that the spatial dependence involves the initial position $\underline{r}_0$ which is what is done in a **Lagrangean description**. In the case of a fluid this description is often referred to as "following the flow" in the sense that the space dependence at a point refers to the same fluid particle throughout. We need knowledge of this description when the equations of motion are considered.

In the case of describing solid materials the Lagrangean description is mostly used throughout as $\underline{r}(\underline{r}_0, t)$ is never too far from $\underline{r}_0$ and in the case of an elastic body points return to their undeformed position when all loads are removed. The description is however difficult to use in the case of fluids as, for example, just to determine the value of a quantity at a given point $\underline{r}_c$ at time t we need to first solve for $\underline{r}_0$ the equation

$$\underline{r}(\underline{r}_0, t) - \underline{r}_c = 0$$

to determine where the point was at time $t = 0$. Instead, a **Eulerian description** is used with the spatial dependence of quantities being in terms of the current position. In the case of velocity we write $\underline{q}_E(\underline{r}, t)$ where

$$\underline{q} = \underline{q}_E(\underline{r}, t) = \underline{q}_L(r_0, t)$$

which in full means

$$\underline{q}_E(\underline{r}(\underline{r}_0, t), t) = \underline{q}_L(\underline{r}_0, t).$$

In this description when $\underline{r}$ is fixed and t varies we have different particles moving through this position and in this sense we have a description in which "the observer is fixed in space".

As previously stated, for some of the equations that govern the motion we need the time derivative at a point with $\underline{r}_0$ being kept fixed which is what is more easily described with the Lagrangean description and we consider next how to get this for any function f described in an Eulerian way. Hence let $f(\underline{r}, t)$ be the Eulerian description with $f_L(\underline{r}_0, t)$ being the corresponding Lagrangean description with

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t).$$

What we mean by the **material time derivative** of $f(\underline{r}, t)$ is

$$\frac{\mathrm{D}}{\mathrm{D}t} f(\underline{r}, t) = \frac{\partial}{\partial t} f_L(\underline{r}_0, t). \tag{2.4.3}$$

To express this instead in terms of partial derivatives of $f(\underline{r}, t)$ we have, by the chain rule, that

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t} f(r, t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + \underline{q}\cdot\nabla f. \end{aligned} \tag{2.4.4}$$

The **acceleration** of a particle is the time derivative of the velocity of the particle and this is also what is meant by the acceleration of particles which are part of a continuum. This hence corresponds to

$$\underline{a} = \underline{a}_L = \frac{\partial}{\partial t}\underline{q}_L(\underline{r}_0, t)$$

where the notation $\underline{a}_L$ is just to emphasise that it is the "Lagrangean acceleration". We can get this vector from an Eulerian description in which the velocity components are $u(\underline{r}, t)$, $v(\underline{r}, t)$ and $w(\underline{r}, t)$ by using (2.4.4). We have

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t}u &= \frac{\partial u}{\partial t} + \underline{q}\cdot\nabla u, \\ \frac{\mathrm{D}}{\mathrm{D}t}v &= \frac{\partial v}{\partial t} + \underline{q}\cdot\nabla v, \\ \frac{\mathrm{D}}{\mathrm{D}t}w &= \frac{\partial w}{\partial t} + \underline{q}\cdot\nabla w \end{aligned}$$

and writing these 3 scalar relations as a vector gives

$$\underline{a}_L = \frac{\mathrm{D}}{\mathrm{D}t}\underline{q} = \frac{\partial \underline{q}_E}{\partial t} + (\underline{q}\cdot\nabla)\underline{q}_E \tag{2.4.5}$$

with the understanding that

$$\underline{q}\cdot\nabla\underline{q}_E = (\underline{q}\cdot\nabla u)\underline{i} + (\underline{q}\cdot\nabla v)\underline{j} + (\underline{q}\cdot\nabla w)\underline{k}.$$

These terms have names with

$$\begin{aligned} \frac{\partial \underline{q}_E}{\partial t} &= \text{local acceleration or Eulerian acceleration,} \\ (\underline{q}\cdot\nabla)\underline{q}_E &= \text{convective acceleration} \end{aligned}$$

and thus

$$(\text{Lagrangean acceleration}) = (\text{local acceleration}) + (\text{convective acceleration}).$$

In what follows, when we just write $\underline{q}$ we will mean $\underline{q}(\underline{r},t) = \underline{q}_E(\underline{r},t)$, i.e. the velocity with the spatial dependence with respect to the current position. Also as a note back to the previous chapter we had the vector identity in (1.1.13) which we repeat here as

$$(\underline{q}\cdot\nabla)\underline{q} = (\nabla\times\underline{q})\times\underline{q} + \nabla\left(\frac{1}{2}|\underline{q}|^2\right).$$

This is needed if the topic is studied beyond what will be done in this module although comments about this will be made when the equations of motion are considered for an inviscid fluid. The vector

$$\underline{\omega} = \nabla\times\underline{q}$$

is known as the **vorticity** and this will be considered again in these notes and fluid flows are said to be **irrotational** when $\underline{\omega} = \underline{0}$.

Examples involving the Lagrangean and Eulerian description

In the following examples we consider cases where we have the Lagrangean description of a fluid motion and we show how to obtain the Eulerian description as well as verify that the Lagrangean acceleration is indeed the sum of the local acceleration and convective acceleration as defined above.

1. Suppose that in a particular fluid motion the position vector of a particle which starts at $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ has the position at time t given by

   $$\underline{r} = \underline{r}(\underline{r}_0,t) = x_0\mathrm{e}^{\alpha t} + y_0\mathrm{e}^{-\alpha t}\underline{j} + z_0\underline{k}$$

   where α is a constant. The Lagrangean velocity and the Lagrangean acceleration are obtained by partially differentiating with respect to t giving

   $$\underline{q}_L = \alpha\left(x_0\mathrm{e}^{\alpha t}\underline{i} - y_0\mathrm{e}^{-\alpha t}\underline{j}\right), \qquad (2.4.6)$$
   $$\underline{a}_L = \alpha^2\left(x_0\mathrm{e}^{\alpha t}\underline{i} + y_0\mathrm{e}^{-\alpha t}\underline{j}\right). \qquad (2.4.7)$$

   To describe the same velocity in an Eulerian way we need an expression in terms of (x,y) instead of (x_0,y_0). In this case we get this immediately as

   $$x = x_0\mathrm{e}^{\alpha t}, \quad y = y_0\mathrm{e}^{-\alpha t} \qquad (2.4.8)$$

   and terms appear exactly in this form in the expression for $\underline{q}_L$ and hence

   $$\underline{q}_E = \underline{q}_E(\underline{r},t) = \alpha(x\underline{i} - y\underline{j}) = u\underline{i} + v\underline{j}, \quad \text{with } u = \alpha x, \quad v = -\alpha y.$$

   To demonstrate the result about the acceleration note that as the expression for $\underline{q}_E$ does not involve t we have

   $$\text{local acceleration} = \frac{\partial\underline{q}_E}{\partial t} = 0.$$

To get the convective acceleration we need the gradients of u and v and these are

$$\nabla u = \alpha \underline{i}, \quad \nabla v = -\alpha \underline{j}.$$

Thus we have

$$\begin{aligned} \text{convective acceleration} &= (\underline{q} \cdot \nabla)\underline{q} \\ &= (\underline{q} \cdot \nabla u)\underline{i} + (\underline{q} \cdot \nabla v)\underline{j}, \\ &= \alpha u \underline{i} - \alpha v \underline{j} = \alpha^2 (x\underline{i} + y\underline{j}). \end{aligned}$$

Hence

$$\frac{\partial \underline{q}_E}{\partial t} + (\underline{q} \cdot \nabla)\underline{q} = \alpha^2 (x\underline{i} + y\underline{j}). \tag{2.4.9}$$

The expression in (2.4.9) is the same vector as in (2.4.7) because of the relations in (2.4.8).

2. Suppose that a fluid particle moves in two dimensional space with position vector

$$\underline{r} = \underline{r}(\underline{r}_0, t) = x_0(1+t)\underline{i} + y_0 \mathrm{e}^{-t}\underline{j}, \quad t \ge 0,$$

where $\underline{r}_0 = x_0\underline{i} + y_0\underline{j}$ is the initial position of the particle.

As we know the initial positions we can get the Lagrangean velocity and the Lagrangean acceleration by just partially differentiating with respect to t and we get

$$\underline{q}_L = \frac{\partial}{\partial t}\underline{r}(\underline{r}_0, t) = x_0\underline{i} - y_0 \mathrm{e}^{-t}\underline{j}, \tag{2.4.10}$$

$$\underline{a}_L = \frac{\partial}{\partial t}\underline{q}_L(\underline{r}_0, t) = y_0 \mathrm{e}^{-t}\underline{j}. \tag{2.4.11}$$

To describe the same velocity in an Eulerian way we need to replace x_0, y_0 by expressions involving x, y and t.

$$x = x_0(1+t) \quad \text{gives } x_0 = \frac{x}{1+t}$$

and

$$y = y_0 \mathrm{e}^{t} \quad \text{gives } y_0 = y\mathrm{e}^{-t}.$$

As $\underline{q}_E(\underline{r}, t) = \underline{q}_L(\underline{r}_0, t)$ it follows that the Eulerian velocity is given by

$$\underline{q}_E(\underline{r}, t) = \frac{x}{1+t}\underline{i} - y\underline{j} = u\underline{i} + v\underline{j}, \quad u = \frac{x}{1+t}, \quad v = -y.$$

For the terms needed to obtain the acceleration we have

$$\text{local acceleration} = \frac{\partial \underline{q}_E}{\partial t} = -\frac{x}{(1+t)^2}\underline{i}.$$

The gradients of u and v are

$$\nabla u = \frac{1}{1+t}\underline{i}, \quad \nabla v = -\underline{j}.$$

$$\begin{aligned}\text{convective acceleration} &= (\underline{q}\cdot\nabla)\underline{q} = (\underline{q}\cdot\nabla u)\underline{i} + (\underline{q}\cdot\nabla v)\underline{j} \\ &= \frac{x}{(1+t)^2}\underline{i} + y\underline{j}.\end{aligned}$$

Combining the local acceleration and convective acceleration gives

$$\frac{\partial \underline{q}_E}{\partial t} + (\underline{q}\cdot\nabla)\underline{q} = y\underline{j}.$$

This agrees with (2.4.11) as $y = y_0 \mathrm{e}^{-t}$.

2.5 Fluid flows: some basic definitions

We now collect together some of the quantities introduced to describe fluids and define a few more terms.

In the context of fluids we have so far introduced the density ρ, the pressure p and the velocity $\underline{q}$ and in an Eulerian description of the fluid the velocity $\underline{q}$ at some point $\underline{r}$ and time t is in components

$$\underline{q} = \underline{q}(\underline{r},t) = u(\underline{r},t)\underline{i} + v(\underline{r},t)\underline{j} + w(\underline{r},t)\underline{k}. \tag{2.5.1}$$

For the flows that we will consider we have the following terms.

Two-dimensional flow: The flow is two-dimensional and parallel to the x, y plane if $w = 0$ and u and v plus all other physical quantities (such as density and pressure) are independent of z, i.e.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial \rho}{\partial z} = 0.$$

Steady flow: Here all quantities such as velocity $\underline{q}$ and density ρ are independent of the time t and thus

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \underline{q}}{\partial t} = 0.$$

When we have steady flow the material derivatives become

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = \frac{\partial \rho}{\partial t} + (\underline{q}\cdot\nabla)\rho = (\underline{q}\cdot\nabla)\rho$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t}\underline{q} = \frac{\partial \underline{q}}{\partial t} + (\underline{q}\cdot\nabla)\underline{q} = (\underline{q}\cdot\nabla)\underline{q}. \tag{2.5.2}$$

There may still be inertia terms to consider as steady flow does not mean that the convective acceleration term is zero.

Stagnation Point: This is a point at which $\underline{q} = \underline{0}$ for all time.

Particle Paths: These are the paths that each fluid particle takes during the flow which we have discussed in the previous section. To determine these now from the velocity $\underline{q}$ described in Eulerian terms involves solving the differential equation

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q}(\underline{r},t), \quad \text{with } \underline{r}(0) = \underline{r}_0. \tag{2.5.3}$$

This leads to a path parametrised by the time t which starts at $\underline{r}_0$.

Streamlines: The streamlines of a flow at time t are curves that are everywhere tangential to the velocity field $\underline{q}(\underline{r}, t)$ at that time t. The streamlines provide a pictorial representation of the flow at a fixed time t. To determine the streamlines let

$$x(s)\underline{i} + y(s)\underline{j} + z(s)\underline{k}$$

denote a parameterisation of a such a curve. For this to be a streamline we need the tangent to be parallel to $\underline{q}$ at each point, i.e.

$$x'(s)\underline{i} + y'(s)\underline{j} + z'(s)\underline{k} = (\text{const.})(u\underline{i} + v\underline{j} + w\underline{k}).$$

This can be written as

$$\frac{x'(s)}{u} = \frac{y'(s)}{v} = \frac{z'(s)}{w}$$

which is sometimes abbreviated to

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}.$$

In the Eulerian description we have u, v and w expressed in terms of x, y and z and these give differential equations we need to solve to get the streamlines.

In the most general case the velocity field $\underline{q}(\underline{r}, t)$ changes as t changes and thus the streamlines change with t. However, in the case of steady flow we just have $\underline{q} = \underline{q}(\underline{r})$ and the streamlines are fixed in time which in turn implies from (2.5.3) that the streamlines coincide with the particle paths.

One consequence of how streamlines are defined is that two different streamlines cannot intersect except at stagnation points, i.e. where $\underline{q} = \underline{0}$. This is because if two streamlines were to cross each other at a point then this would imply that the velocity had two different directions at the intersection point which is not possible. Also, two different streamlines cannot just touch at an isolated point as this would then imply that in a steady state situation the particle path of a particle at this point has more than one possibility which conflicts with there being a unique solution to the initial value problem (2.5.3). Particle paths can however intersect in that a particle at position $\underline{r}_0$ and time $t = 0$ may reach the same position again at a later time.

Examples of determining particle paths and streamlines

1. Find the particle paths and streamlines for the two-dimensional flow defined by the velocity field

$$\underline{q}(\underline{r}, t) = \underline{q}_E(\underline{r}, t) = xt\,\underline{i} - y\underline{j}.$$

<u>**Solution**</u>

Observe first that $\underline{q}(\underline{r}, t)$ does depend on t and hence we can expect the particle paths and streamlines to be different and we consider next how to determine these.

(a) Particle Paths.

With

$$\underline{r} = x\,\underline{i} + y\,\underline{j} \quad \text{and} \quad \underline{q} = u\,\underline{i} + v\,\underline{j}$$

the system of ordinary differential equations which determine the path are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u = xt \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v = -y$$

and we can write

$$\frac{\mathrm{d}x}{x} = t \quad \text{and} \quad \frac{\mathrm{d}y}{y} = -1 \tag{2.5.4}$$

and integrating gives

$$\ln(x) = \frac{t^2}{2} + c_0 \quad \text{and} \quad \ln(y) = -t + c_1$$

where c_0 and c_1 are constants. For a particular path which is at (x_0, y_0) at time $t = 0$ we have

$$\ln(x_0) = c_0 \quad \text{and} \quad \ln(y_0) = c_1$$

Hence

$$\ln\left(\frac{x}{x_0}\right) = \ln(x) - \ln(x_0) = \frac{t^2}{2}$$

and

$$\ln\left(\frac{y}{y_0}\right) = -t.$$

The above describes the particle path in parametric way. In this case we can also explicitly express in the path in a form which does not involve t by just substituting

$$t = -\ln\left(\frac{y}{y_0}\right) = \ln\left(\frac{y_0}{y}\right)$$

into the expression for x to give

$$\ln\left(\frac{x}{x_0}\right) = \frac{1}{2}\left(\ln\left(\frac{y_0}{y}\right)\right)^2$$

which can be written as

$$\left(\ln\left(\frac{x^2}{x_0^2}\right)\right)^{1/2} = \ln\left(\frac{y_0}{y}\right)$$

and by taking the exponential we get

$$\frac{y_0}{y} = \exp\left(\left(\ln\left(\frac{x^2}{x_0^2}\right)\right)^{1/2}\right),$$

i.e.

$$y = y_0 \exp\left(-\left(\ln\left(\frac{x^2}{x_0^2}\right)\right)^{1/2}\right). \tag{2.5.5}$$

To get a rough idea of the paths that this gives observe that if $y_0 > 0$ then $y > 0$ throughout and $y \to 0$ as $|x| \to \infty$. Similarly, if $y_0 < 0$ then $y < 0$ throughout and $y \to 0$ as $|x| \to \infty$. If $y_0 = 0$ then $y = 0$ is the path and we also get this if we let $x_0 \to 0$.

(b) Streamlines.

Streamlines are determined from the velocity field at a specific time and thus in the computations t is constant. Thus, corresponding to (2.5.4) we now have

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v}$$

which is

$$\frac{\mathrm{d}}{xt} = \frac{\mathrm{d}y}{-y}$$

and, with t being constant, integration of this gives

$$\frac{1}{t}\ln x = -\ln y + \ln C,$$

where C is a constant. Hence

$$x^{1/t}\, y = C \tag{2.5.6}$$

with different values of C giving different streamlines. Note that the streamlines are different at different times and also that the paths given by (2.5.6) are not the same as those of the particle paths given in (2.5.5).

2. Consider the unsteady, two-dimensional flow defined by the velocity field

$$\underline{q} = U\,\underline{i} + \frac{x}{(1+t)}\,\underline{j}, \quad t \ge 0,$$

where $U > 0$ is constant. Show that the streamlines are parabolic and that the particle paths are given by

$$y - y_0 = x - x_0 + (x_0 - U)\ln\left(1 + \frac{x - x_0}{U}\right),$$

where (x_0, y_0) is the initial position of the particle.

Solution

Observe first that $\underline{q}(\underline{r}, t)$ does depend on t, as in the previous example, and hence we can expect the particle paths and streamlines to be different and we consider next how to determine these.

(a) Particle Paths.

The differential equations which determine the particle paths are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u = U, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{x}{1+t}. \end{aligned}$$

The equation for x with the initial condition $x(0) = x_0$ can be solved immediately as we just have

$$x = x_0 + Ut.$$

If we substitute this in the equation for y then we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{Ut + x_0}{1+t} = \frac{U(1+t) - U + x_0}{1+t} = U + \frac{x_0 - U}{1+t}.$$

In this form we can integrate with respect to t to give

$$y = Ut + (x_0 - U)\ln(1+t) + C, \quad \text{where } C \text{ is a constant.}$$

The initial condition $y(0) = y_0$ gives $C = y_0$. The particle path in parametric form is

$$x = x_0 + Ut, \quad y = y_0 + Ut + (x_0 - U)\ln(1+t).$$

To express this instead as a relation in x and y is not too hard here as

$$Ut = x - x_0, \quad t = \frac{x - x_0}{U}$$

and substituting in the expression for y gives

$$y = y_0 + (x - x_0) + (x_0 - U)\ln\left(1 + \frac{x - x_0}{U}\right)$$

as required.

(b) Streamlines.

In the case of the streamlines the time is fixed and the differential equations to consider are

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v}$$

which in this case are

$$\frac{\mathrm{d}x}{U} = \frac{\mathrm{d}y}{\left(\dfrac{x}{1+t}\right)}.$$

Rearranging we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{U(1+t)}.$$

As U is constant and t is constant in this context we can integrate to give

$$y = \frac{x^2}{2U(1+t)} + C$$

where C denotes a constant. This is a standard equation for a parabola and we get different parabolas for different values of C.

2.6 The equation of mass conservation

In this section and the next section we consider some fundamental laws of physics governing the flow of a fluid and in this section we consider conservation of mass.

In an Eulerian description let Ω be an arbitrary fixed region with surface S and we assume that Ω contains no sources or sinks by which fluid can leave the region or enter the region. Let, as before, $\rho = \rho(\underline{r}, t)$ denote density and we consider next how this might change as fluid flows through the region.

Now the mass of the fluid contained in Ω is given by

$$\int_\Omega \rho \, \mathrm{d}v$$

and the rate as which this decreases in given by

$$-\frac{\partial}{\partial t}\int_\Omega \rho \, \mathrm{d}v = -\int_\Omega \frac{\partial \rho}{\partial t} \, \mathrm{d}v. \tag{2.6.1}$$

The principle of mass conservation requires that the only way the fluid in Ω can change is by flow across the surface S and we consider next now to express this.

Consider a surface element $\underline{n}\,\mathrm{d}s$ with $\underline{n}$ denoting the unit vector and with $\mathrm{d}s$ denoting the infinitesimal area. In an infinitesimal time increment $\mathrm{d}t$ points on the surface move by an amount $\underline{q}\,\mathrm{d}t$ where $\underline{q}$ is the velocity. The interest here concerns the component of this displacement in the direction of $\underline{n}$ as this indicates the part which is leaving the region and this component is given by $(\underline{q} \cdot \underline{n})\mathrm{d}t$. This distance in the $\underline{n}$ direction times the area $\mathrm{d}s$ of the surface element gives the volume of fluid that leaves the region in the time increment by passing across this part of the surface. Hence the mass of fluid which leaves by passing through this part is

$$\rho(\underline{q} \cdot \underline{n}) \, \mathrm{d}s \, \mathrm{d}t$$

and the rate at which it is leaving is

$$\rho(\underline{q} \cdot \underline{n}) \, \mathrm{d}s.$$

Now if we consider the entire surface S then the rate at which fluid is passing through S is

$$\int_S \rho(\underline{q} \cdot \underline{n}) \, \mathrm{d}s = \int_\Omega \nabla \cdot (\rho \underline{q}) \, \mathrm{d}v. \tag{2.6.2}$$

where the right hand side expression follows by the divergence theorem.

The principle of mass conservation requires that (2.6.1) and (2.6.2) must be the same, i.e.

$$\int_\Omega \nabla \cdot (\rho \underline{q}) \, \mathrm{d}v = -\int_\Omega \frac{\partial \rho}{\partial t} \, \mathrm{d}v$$

which we write as

$$\int_\Omega \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) \right) \mathrm{d}v.$$

This must be true for all possible regions Ω and this in turn implies that the integrand must be 0 everywhere, i.e.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0.} \tag{2.6.3}$$

This is known as the **equation of mass conservation** and it also known as the **equation of continuity**. By using vector identities involving ∇ there are other ways of expressing this condition. We have

$$\nabla \cdot (\rho \underline{q}) = \underline{q} \cdot \nabla \rho + \rho \nabla \cdot \underline{q}$$

and hence we get

$$\left(\frac{\partial \rho}{\partial t} + \underline{q} \cdot \nabla \rho\right) + \rho \nabla \cdot \underline{q} = 0. \qquad (2.6.4)$$

The term in brackets is the material time derivative of ρ where the material time derivative is defined in (2.4.4), and hence we can write

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \underline{q} = 0.} \qquad (2.6.5)$$

If a material is incompressible then the density of a particle does not change and thus the material time derivative of ρ is 0 and by (2.6.5) this is the case if and only if the velocity satisfies

$$\boxed{\nabla \cdot \underline{q} = 0.} \qquad (2.6.6)$$

The incompressibility assumption does not imply that $\rho(\underline{r}, t)$ cannot vary with position $\underline{r}$ and time t but just implies that any variation must be such that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \underline{q} \cdot \nabla \rho = 0.$$

In many practical cases we do have however that ρ is constant, i.e. there is no dependence on $\underline{r}$ and t. To a good approximation water can be considered as incompressible when flows are modelled. Water is not however exactly incompressible and this fact is needed when waves are considered in the last part of the module.

2.7 Euler's equations of motion

In previous modules in mechanics you would have considered Newton's second law of motion in the case of a rigid particle and when the mass m is constant the motion $\underline{r}(t)$ of the particle satisfies

$$m \frac{\mathrm{d}^2 \underline{r}}{\mathrm{d}t^2} = \underline{F}$$

where $\underline{F}$ is the net force on the particle. In the case of a continuum this is assumed to be satisfied for each particle which forms the body being considered and the additional thing that needs to be taken into account is the force on the particle due to the neighbouring particles. This was described in sections 2.2 and 2.3 when the equilibrium of a fluid was considered. If the fluid is inviscid then the state of stress is always of the form

$$\boldsymbol{\sigma} = -p\, I$$

and the force on a region Ω with surface S due to the neighbouring fluid is

$$-\int_S p\underline{n}\, \mathrm{d}s = -\int_\Omega \nabla p\, \mathrm{d}v$$

as in section 2.3. Again we assume that the only other force on Ω is due to gravity and this can be expressed as

$$-\int_\Omega \rho g \underline{k}\, \mathrm{d}v = -\int_\Omega \nabla(\rho g z)\, \mathrm{d}v.$$

With an Eulerian description and with $\underline{q} = \underline{q}(\underline{r}, t)$ being the velocity the Lagrangean acceleration is

$$\frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = \frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \nabla)\underline{q}$$

and hence Newton's second law for the continuum is

$$\int_\Omega \rho \frac{\mathrm{D}\underline{q}}{\mathrm{D}t} \,\mathrm{d}v = -\int_\Omega \nabla(p + \rho g z) \,\mathrm{d}v.$$

As this must hold for all regions Ω we get

$$\boxed{\rho \frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = -\nabla(p + \rho g z).} \tag{2.7.1}$$

This is known as Euler's equations and this will be used again when waves are considered.

Further consideration of Euler's equations for fluid flow goes beyond what will be examined in this module although some brief comments will next be given to further justify why streamlines are worth considering for steady flow and the simplifications that arise when the flow is irrotational.

If we assume that the density is constant then we have incompressible flow, i.e. $\nabla \cdot \underline{q} = 0$ and when the flow is steady we have

$$\frac{\partial \underline{q}}{\partial t} = \underline{0}$$

and the material time derivative of the velocity reduces to

$$\frac{\mathrm{D}\underline{q}}{\mathrm{D}t} = (\underline{q} \cdot \nabla)\underline{q}. \tag{2.7.2}$$

Now recall that in chapter 1 we had the identity

$$(\underline{q} \cdot \nabla)\underline{q} = \nabla\left(\frac{1}{2}|\underline{q}|^2\right) + (\nabla \times \underline{q}) \times \underline{q}$$

and if we substitute this in (2.7.1) and collect together the terms involving the gradient then we obtain

$$\underline{\omega} \times \underline{q} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{q}|^2 + gz\right), \quad \text{where } \underline{\omega} = \nabla \times \underline{q} \text{ is the vorticity.} \tag{2.7.3}$$

It is convenient here to let

$$H = \frac{p}{\rho} + \frac{1}{2}|\underline{q}|^2 + gz.$$

As $\underline{\omega} \times \underline{q}$ is orthogonal to $\underline{q}$ and streamlines are curves those tangent is always in the direction of $\underline{q}$ it follows that if we take the dot product of (2.7.3) with the unit vector in the direction of $\underline{q}$ then the directional derivative of H is zero along a streamline, i.e. we have on a streamline that

$$H = \frac{p}{\rho} + \frac{1}{2}|\underline{q}|^2 + gz = \text{const.} \tag{2.7.4}$$

Hence if we know the velocity $\underline{q}$ and we have determined a streamline then this enables the variation of the pressure p along the streamline to be determined.

If the flow described above is such that $\underline{\omega} = \underline{0}$ throughout the region of the flow then we have that H is a constant not just along streamlines but on the entire region. As already mentioned a flow with the property that $\underline{\omega} = \underline{0}$ is called an irrotational flow and we give a few more comments to justify why these should be considered further. With a bit more use of vector calculus, which will not be done here, if we take the curl of (2.7.2) an equation for $\underline{\omega}$ can be obtained which can be expressed in the form

$$\frac{\mathrm{D}\underline{\omega}}{\mathrm{D}t} = (\underline{\omega} \cdot \nabla)\underline{q}$$

which is known as the vorticity equation. Now in the case of a two-dimensional flow of the form

$$\underline{q} = u(x, y, t)\underline{i} + v(x, y, t)\underline{j}$$

we get

$$\underline{\omega} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = \omega \underline{k}, \quad \text{with } \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

and hence

$$\underline{\omega} \cdot \nabla = \omega \frac{\partial}{\partial z}$$

and

$$\frac{\mathrm{D}\underline{\omega}}{\mathrm{D}t} = \underline{0}.$$

The vorticity $\underline{\omega}$ of a particle in such a fluid does not change with time and hence if it is $\underline{0}$ at some time then it is $\underline{0}$ for all time. Thus, for example, if the flow is uniform somewhere then the vorticity is $\underline{0}$ and this never changes during the flow. We consider the case of two-dimensional irrotational flows later in this module.

As a final comment, when the fluid flows past an obstacle the relation (2.7.4) is used to determine how the pressure varies around the obstacle and from this the force on the obstacle is determined. Classical aerofoil theory makes use of this to determine the lift on a wing shaped object which is moving through a fluid.

2.8 A discussion of the limitations of the inviscid fluid assumption

In this chapter the fluid has been assumed to be inviscid which means that it has no viscosity and the term frictionless flow is also used to describe how such a fluid behaves. The term frictionless flow means that adjacent fluid particles move freely past each other. The mathematical outcome of this regarding the derivation of the equations is that the stress is only ever a hydrostatic pressure, i.e.

$$\boldsymbol{\sigma} = -pI. \tag{2.8.1}$$

There is no fluid for which this is exactly true but it can be a good approximation for water in many situations. When it is necessary to take account of viscous effects and we have what is known as an incompressible Newtonian fluid we need to replace (2.8.1) by

$$\boldsymbol{\sigma} = -pI + \mu \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} \tag{2.8.2}$$

where μ is known as the viscosity which is generally assumed to be constant. In the case of a flow for which the velocity is of the form

$$u = \alpha y, \quad v = 0, \quad w = 0,$$

which is known as a shear flow, things simplify and if we just consider the $2, 1$ off diagonal term then we have

$$\sigma_{21} = \mu \frac{\partial u}{\partial y} = \mu\alpha.$$

This tells us that the viscosity μ is the ratio of a shear stress (σ_{21} in this case) and a velocity gradient term. When μ is very small then for moderately sized velocity gradients we only get small shear stresses whilst if μ is much larger then the shear stresses are much larger for the same velocity gradients. There is, for example, a much higher value for μ for a fluid such as syrup compared with the value for water.

When we need to replace (2.8.1) by (2.8.2) the equations of motion becomes more complicated with Euler's equations being replaced by the Navier-Stokes' equations which will not be described here. When we have viscous flow the type of flow that results depends on what is known as the Reynold's number, which we do not present here other than to say that it is related to the ratio

$$\frac{|\text{inertia terms in equation of motion}|}{|\text{viscous terms in equation of motion}|}.$$

The inviscid fluid case corresponds to a Reynold's number of ∞ when there are no viscous terms.

In many situations modelling an actual fluid by assuming that it is inviscid fluid leads to a reasonable approximation to what actualy happens over most of the region being considered but there are some important situations where a sequence of solutions corresponding to a sequence of smaller and smaller values of μ converging to 0 do not tend to the solution when $\mu = 0$. The difficulty that may occur when $\mu > 0$ is for the part of the flow close to a rigid boundary where the velocity gradients may be very large although such velocity gradients are much smaller away from such boundaries. The rapid change in the velocity $\underline{q}$ very close to the boundary is due to the fluid not slipping on the boundary and the small region close to a boundary where this happens is known as a boundary layer. In contrast when $\mu = 0$ the fluid flows without resistance along such boundaries. To show the difference between the case $\mu = 0$ and $\mu > 0$ figures 2.2–2.5 show streamlines at one specific time for different values of the Reynold's number which was briefly introduced above. (In all the cases shown when $\mu > 0$ a steady state has not been reached although in the case of figure 2.3 it is close to a steady state.)

In each case the flow is basically from left to right. For information, the results in these figures were obtained by running a Matlab program obtained from downloading the zip file `http://www.cfmbyexample.com/resources/Cylinder_06October2011.zip` and adjusting the parameters in some of the Matlab files and each run took a few minutes. For each case when $\mu > 0$ the Navier Stokes' equations are approximately solved using a numerical scheme.

The effect of a non-zero viscosity and a small boundary layer causes the flow to separate as it flows around the cylindrical obstacle and this separation effect increases as the Reynold's number increases, compare for example figures 2.4 and 2.5. As the Reynold's number gets larger and larger it becomes more and more difficult for the numerical scheme to determine a reliable solution. This is an example where the actual flow is quite complicated and the inviscid model does not describe the situation very well when the Reynold's number is high.

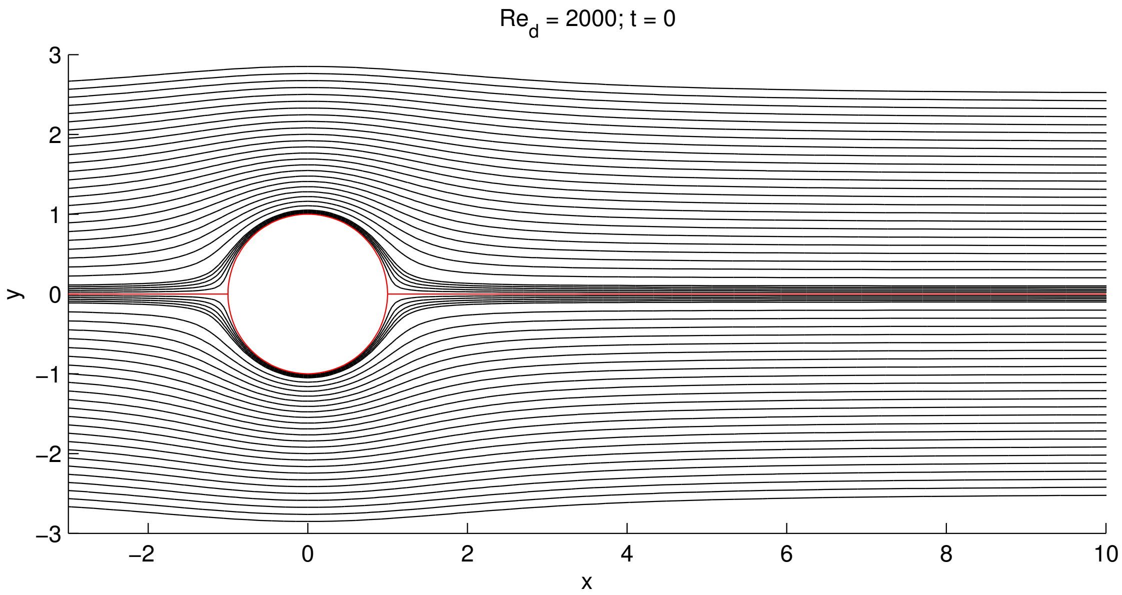


Figure 2.2: The streamlines when the viscosity $\mu = 0$ for the flow around a rigid cylinder with the flow being from left to right. This is actually the initial condition in every case when it necessary to approximately solve the Navier Stokes' equations.

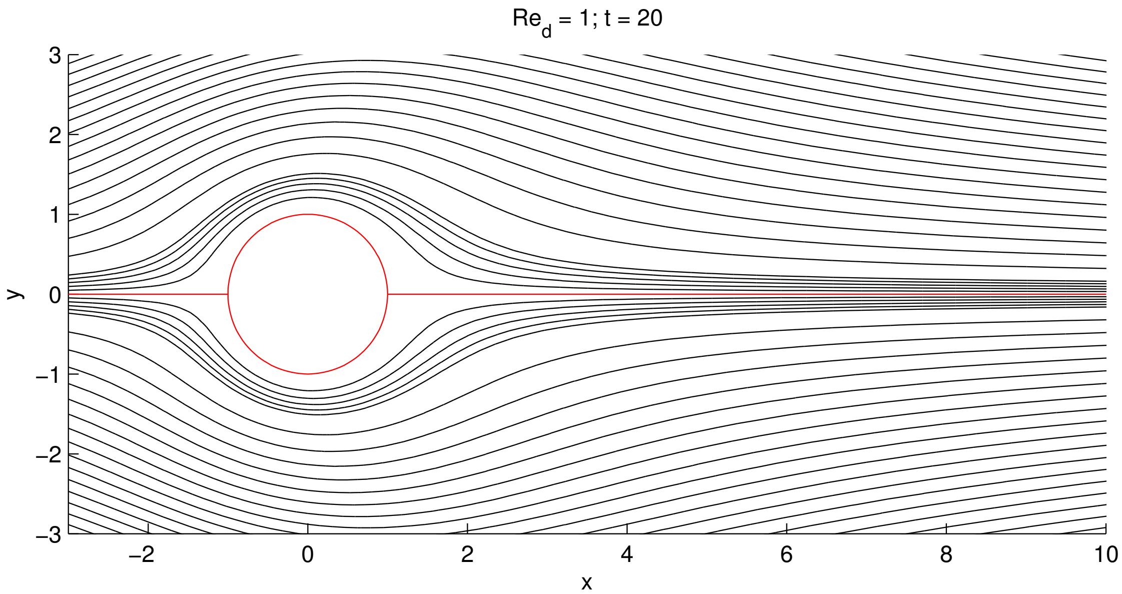


Figure 2.3: The streamlines when the Reynold's number Re = 1 at a certain time for the flow around a rigid cylinder with the flow being from left to right.

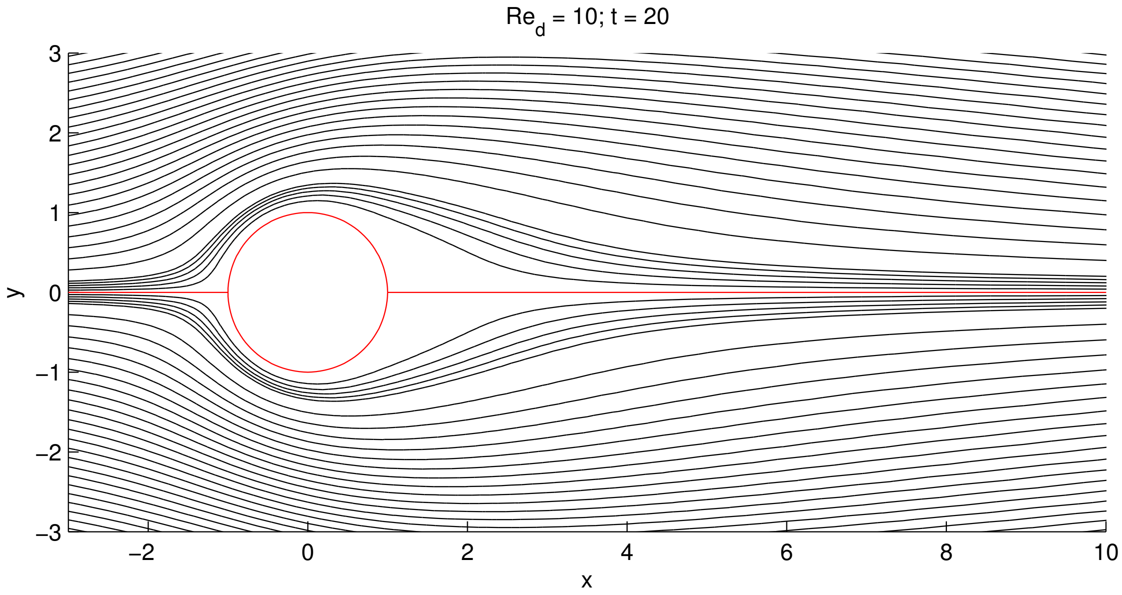


Figure 2.4: The streamlines when the Reynold's number Re = 10 at the same time as in figure 2.3 for the flow around a rigid cylinder with the flow being from left to right.

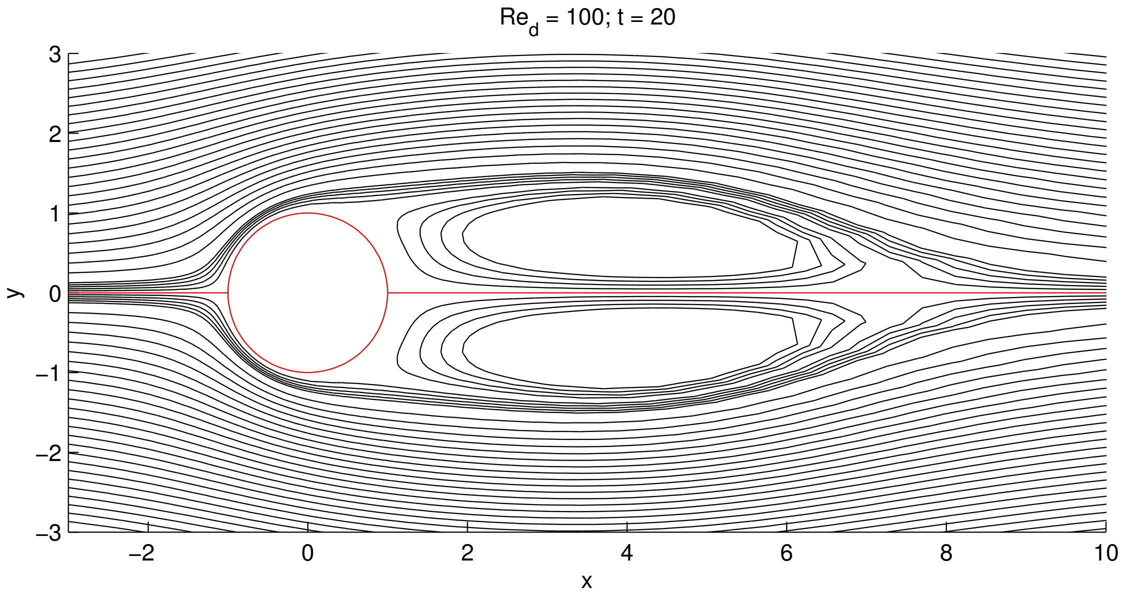


Figure 2.5: The streamlines when the Reynold's number Re = 100 at the same time as in figure 2.3 for the flow around a rigid cylinder with the flow being from left to right. There is quite a large region to the right of the cylinder where the flow has separated and the flow in this part is more complicated than elsewhere.

Chapter 3

Steady two dimensional incompressible flows, vorticity and irrotational flows

3.1 The two dimensional simplification and the stream function ψ

In this chapter we restrict throughout to fluids which are incompressible and consider flows which are steady and are such that the velocity $\underline{q}$ only has components in the $\underline{i}$ and $\underline{j}$ directions with all physical quantities only depending on x, y. Thus in a Eulerian description the velocity $\underline{q}$ is of the form

$$\underline{q} = u(x,y)\underline{i} + v(x,y)\underline{j} \tag{3.1.1}$$

and the incompressible assumption implies that

$$\nabla \cdot \underline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.1.2}$$

In chapter 1 we noted that for any vector field $\underline{A}$

$$\nabla \cdot (\nabla \times \underline{A}) = 0$$

and we also indicated in that chapter that if $\nabla \cdot \underline{q} = 0$ then it can be shown that there exists a vector field $\underline{A}$ such that $\underline{q} = \nabla \times \underline{A}$. We use this now for our two dimensional case and first note that if $\underline{A} = \underline{A}(x,y)$ (i.e. no dependence on z) then

$$\nabla \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \underline{i}\frac{\partial A_3}{\partial y} - \underline{j}\frac{\partial A_3}{\partial x} + \underline{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right).$$

If $A_1 = A_2 = 0$ then $\nabla \times \underline{A}$ has no $\underline{k}$ component and if we take $\underline{A} = \psi\underline{k}$ then

$$\underline{q} = \nabla \times (\psi\underline{k}) = u\underline{i} + v\underline{j} = \frac{\partial \psi}{\partial y}\underline{i} - \frac{\partial \psi}{\partial x}\underline{j}. \tag{3.1.3}$$

Now observe that

$$(\nabla\psi) \times \underline{k} = \frac{\partial\psi}{\partial x}(\underline{i} \times \underline{k}) + \frac{\partial\psi}{\partial y}(\underline{j} \times \underline{k}) = -\frac{\partial\psi}{\partial x}\underline{j} + \frac{\partial\psi}{\partial y}\underline{i}$$

which is the same as above and thus we have

$$\boxed{\underline{q} = u\,\underline{i} + v\,\underline{j} = (\nabla\psi) \times \underline{k} = \frac{\partial\psi}{\partial y}\,\underline{i} - \frac{\partial\psi}{\partial x}\,\underline{j}.} \tag{3.1.4}$$

Thus to summarise, the velocity $\underline{q}$ in the case of two-dimensional steady incompressible flow can be described using just one scalar valued function $\psi = \psi(x, y)$, which is known as the **stream function**, and we consider examples of stream functions later in the chapter. First however to explain how this relates to streamlines defined in section 2.5 observe that

$$\underline{q} \cdot \nabla\psi = \left(\frac{\partial\psi}{\partial y}\underline{i} - \frac{\partial\psi}{\partial x}\underline{j}\right) \cdot \left(\frac{\partial\psi}{\partial x}\underline{i} + \frac{\partial\psi}{\partial y}\underline{j}\right) = 0.$$

Now recall that the gradient $\nabla\psi$ is a vector which has a direction which is orthogonal to curves of the form $\psi(x, y) = \text{const.}$ and this hence implies that these curves must always have a tangent parallel to $\underline{q}(x, y)$ at each point being considered. But recall that streamlines are defined such that the tangent at a point (x, y) is in the direction of $\underline{q}(x, y)$ and thus the curves of the form $\psi(x, y) = \text{const.}$ are the streamlines of the flow.

The relation $\underline{q} = (\nabla\psi) \times \underline{k}$ in polars

As a final point we consider the expression for $\underline{q}$ when we replace $\psi(x, y)$ by $\psi(r, \theta)$ where r and θ are polar coordinates as later several of the examples will involve polar coordinates.

With $\underline{e}_r$ and $\underline{e}_\theta$ being the base vectors in polars and with no dependence on z we have

$$\nabla\psi = \frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta \tag{3.1.5}$$

and this gives

$$\underline{q} = (\nabla\psi) \times \underline{k} = \frac{\partial\psi}{\partial r}(\underline{e}_r \times \underline{k}) + \frac{1}{r}\frac{\partial\psi}{\partial\theta}(\underline{e}_\theta \times \underline{k}),$$

i.e.

$$\boxed{\underline{q} = (\nabla\psi) \times \underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\underline{e}_r - \frac{\partial\psi}{\partial r}\,\underline{e}_\theta.} \tag{3.1.6}$$

3.2 Vorticity $\underline{\omega}$, irrotational flow and the velocity potential ϕ

Before some examples of flows are considered in the next section we consider now what is known as the vorticity of a flow which has been briefly mentioned a few times. Specifically it was mentioned in connection with a vector identity on page 1-6 in chapter 1 and it was mentioned again on page 2-16 in chapter 2 in connection with one of the terms in the expression involving the Lagrangean acceleration. The **vorticity** is defined as

$$\underline{\omega} = \nabla \times \underline{q} \tag{3.2.1}$$

and for our two dimensional flow in the x, y plane this reduces to

$$\underline{\omega} = \nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u(x,y) & v(x,y) & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{k}. \tag{3.2.2}$$

With a two-dimensional incompressible flow we have a stream function ψ and the velocity components are given by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

and thus

$$\underline{\omega} = \left(-\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) \right) \underline{k} = -\nabla^2 \psi \underline{k}. \tag{3.2.3}$$

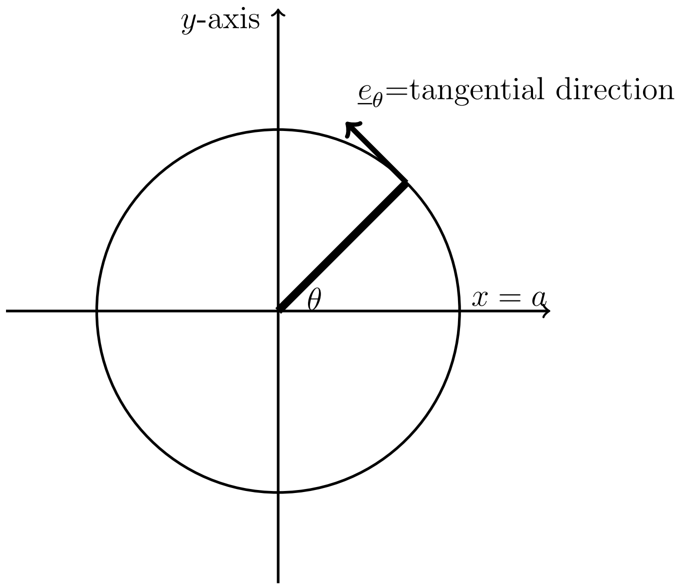
When fluid particles move with velocity $\underline{q} = \underline{q}(x, y)$ the vorticity gives a measure of the local rotation or spin of each particle. One way to think of this is to consider how the earth orbits the sun with a complete orbit taking one year and with the spin part of the earth about an axis taking one day (one year and one day being defined so that this is the case). The spin of the earth about its axis is related to the vorticity of the motion. With this as a rough idea of what vorticity is we consider now more precisely the terms in (3.2.2) by considering a circle of small radius a centered at 0 and consider how the velocity changes in the vicinity of $\underline{0}$ which contains this circle. When $|x|$ and $|y|$ are small the first few terms in the multi-variable Taylor expansion are

$$\underline{q}(x, y) \approx \underline{q}(\underline{0}) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \underline{i} + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \right) \underline{j}$$

where the partial derivatives are evaluated at $\underline{0}$. Now on the circle we have $x = a \cos \theta$ and $y = a \sin \theta$ and thus

$$\underline{q}(a \cos \theta, a \sin \theta) \approx \underline{q}(\underline{0}) + a \left(\cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y} \right) \underline{i} + a \left(\cos \theta \frac{\partial v}{\partial x} + \sin \theta \frac{\partial v}{\partial y} \right) \underline{j}.$$

The unit vector tangent to the circle in the anti-clockwise sense is $\underline{e}_\theta = -\sin \theta \underline{i} + \cos \theta \underline{j}$. We illustrate the geometry part of this set-up below.



To shorten the notation let $c = \cos\,\theta$ and $s = \sin\,\theta$ and consider the component of the velocity in the direction tangent to the circle which is

$$\underline{q}(a\cos\,\theta, a\sin\,\theta)\cdot\underline{e}_\theta \approx \underline{q}(\underline{0})\cdot\underline{e}_\theta + a\left(sc\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)+c^2\frac{\partial v}{\partial x}-s^2\frac{\partial u}{\partial y}\right).$$

As specific cases, when $\theta = 0$ we have

$$\underline{q}(a,0)\cdot\underline{j} \approx \underline{q}(\underline{0})\cdot\underline{j} + a\frac{\partial v}{\partial x}$$

and when $\theta = \pi/2$ we have

$$\underline{q}(0,a)\cdot(-\underline{i}) \approx \underline{q}(\underline{0})\cdot(-\underline{i}) - a\frac{\partial u}{\partial y}.$$

If we integrate over a 2π range and note that

$$\int_0^{2\pi}\cos\,\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\,\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\,\theta\cos\,\theta\,\mathrm{d}\theta = 0$$

and

$$\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \pi$$

then we get an average over the length of the circle of

$$\frac{1}{2\pi a}\int_0^{2\pi}\underline{q}(a\cos\,\theta, a\sin\,\theta)\cdot\underline{e}_\theta\,\mathrm{d}\theta \approx \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

which is half the component in the vorticity. We get equality in the above if we consider the limit as $a \to 0$. Observe that the left hand side here is exactly the same as

$$\frac{1}{2(\text{area of the disk})}\oint_C \underline{q}\cdot\mathrm{d}\underline{r}, \quad \text{as } \mathrm{d}\underline{r} = a\underline{e}_\theta\mathrm{d}\theta$$

and from the definition of curl as a limit we have

$$\underline{k}\cdot(\nabla\times\underline{q}) = \lim_{a\to 0}\frac{1}{\pi a^2}\oint_C \underline{q}\cdot\mathrm{d}\underline{r}.$$

It is in this sense that vorticity measures how particles are rotating about an axis with the direction of $\underline{\omega}$ giving this axis and this direction is $\underline{k}$ in the two-dimensional cases considered in this chapter.

When the vorticity $\underline{\omega} = \nabla\times\underline{q} = \underline{0}$ the flow is said to be **irrotational** and this is the case in all the examples considered in this chapter except the case of simple shear flow. From (3.2.3) the flow is irrotational when

$$\nabla^2\psi = 0 \tag{3.2.4}$$

which means that the stream function satisfies Laplace's equation. Now recall from an identity in chapter 1 we had that for any scalar function ϕ we have $\nabla\times\nabla\phi = 0$ and it was also stated that if $\nabla\times\underline{q} = \underline{0}$ then there exists a function ϕ such that

$$\underline{q} = \nabla\phi. \tag{3.2.5}$$

This function ϕ is known as the **velocity potential**. Thus when we have flow which is steady, two dimensional, incompressible and irrotational we have a stream function ψ and a velocity gradient ϕ which are related such that in cartesian coordinates the velocity is given by

$$\underline{q} = \frac{\partial \psi}{\partial y}\,\underline{i} - \frac{\partial \psi}{\partial x}\,\underline{j} = \frac{\partial \phi}{\partial x}\,\underline{i} + \frac{\partial \phi}{\partial y}\,\underline{j}. \tag{3.2.6}$$

As $\nabla \cdot \underline{q} = 0$ the function ϕ is also such that

$$\nabla^2 \phi = 0, \tag{3.2.7}$$

i.e. ψ and ϕ both satisfy Laplace's equation. By equating the coefficients of $\underline{i}$ and $\underline{j}$ we get the relations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{3.2.8}$$

which are known as the **Cauchy Riemann equations**. These appear when functions of a complex variable are considered which are differentiable in a complex sense and this is done in a level 3 module.

3.3 Examples of basic stream functions

In the following examples we give the velocity $\underline{q}$, we verify that $\nabla\cdot\underline{q} = 0$ for incompressible flow, we investigate whether or not the flow is irrotational and we derive the stream function ψ. To attempt to visualise the flow we also show the streamlines.

3.3.1 Uniform flow

In a uniform flow the velocity $\underline{q}(x, y)$ is a constant vector, i.e. the velocity is the same at every point (x, y). Let

$$\underline{q} = a_1\,\underline{i} + a_2\,\underline{j} \tag{3.3.1}$$

denote the velocity with a_1 and a_2 being constant. All partial derivatives are zero and thus trivially $\nabla \cdot \underline{q} = 0$ confirming that the flow is incompressible and also $\underline{\omega} = \nabla \times \underline{q} = \underline{0}$ which confirms that the flow is irrotational. To determine the stream function ψ we need to satisfy

$$\frac{\partial \psi}{\partial y} = a_1 \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -a_2$$

giving

$$\psi(x, y) = -a_2 x + a_1 y + c$$

for any constant c. The streamline pattern does not depend on c and as the velocity components and the vorticity just involve various partial derivatives of ψ we can take any convenient value for c and in this case we choose to take $c = 0$. A stream function $\psi(x, y)$ for a uniform flow is thus of the form

$$\psi(x, y) = -a_2 x + a_1 y \tag{3.3.2}$$

when we have the constant velocity given in (3.3.1). The streamlines are such that $\psi(x, y)$ = const. and these are parallel straight lines. The cases where the direction of the flow is $\underline{i}$ and $\underline{i} + \underline{j}$ respectively are shown in figure 3.1.

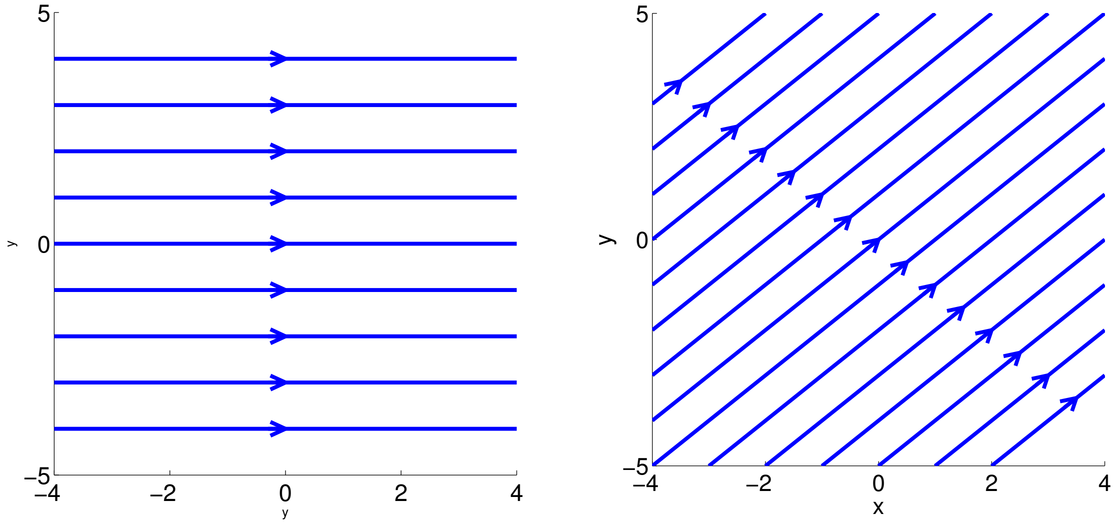


Figure 3.1: Streamlines for uniform flow in the $\underline{i}$ direction (left hand side plot) and $\underline{i}+\underline{j}$ direction (right hand side plot).

3.3.2 A simple shear flow

A simple shear flow in the direction of $\underline{i}$ is described by

$$\underline{q} = \beta y\, \underline{i}, \tag{3.3.3}$$

where β is a constant and in the following discussion we assume that $\beta > 0$. For $y > 0$ the direction of the flow is in the positive y direction with the speed increasing as y increases. If you think of layers of fluid corresponding to y being fixed then the layer corresponding to $y = \alpha + \epsilon$ is moving a bit faster than the layer corresponding to $y = \alpha > 0$ when $\epsilon > 0$. With viscous fluids there is a resistance to this type of motion and we have frictional forces and shear stresses but when a fluid is assumed to be inviscid the layers can move freely move past each other without any resistance.

In terms of components we have $\underline{q} = u\,\underline{i} + v\,\underline{j}$ with

$$u = \beta y, \quad v = 0$$

and as a check

$$\nabla \cdot \underline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

confirming that the flow is incompressible. To determine the stream function, which is unique up to an arbitrary additive constant, we have

$$\frac{\partial \psi}{\partial y} = u = \beta y, \quad \frac{\partial \psi}{\partial x} = -v = 0$$

and we can take

$$\psi = \frac{1}{2}\beta y^2. \tag{3.3.4}$$

The streamlines are the lines corresponding to $\psi(x, y) = C$ where C is a constant and these are shown in figure 3.2 corresponding to equally spaced values for the constant C.

We have straight lines of the form $y = \text{const.}$ as in the case of the uniform flow considered in the previous section but remember now that the speed is different on different lines. The vorticity $\underline{\omega}$ in this case is given by

$$\underline{\omega} = -\nabla^2\psi\,\underline{k} = -\beta\,\underline{k} \tag{3.3.5}$$

and is thus not zero. This tells us that particles are moving in a straight line and they are also rotating.

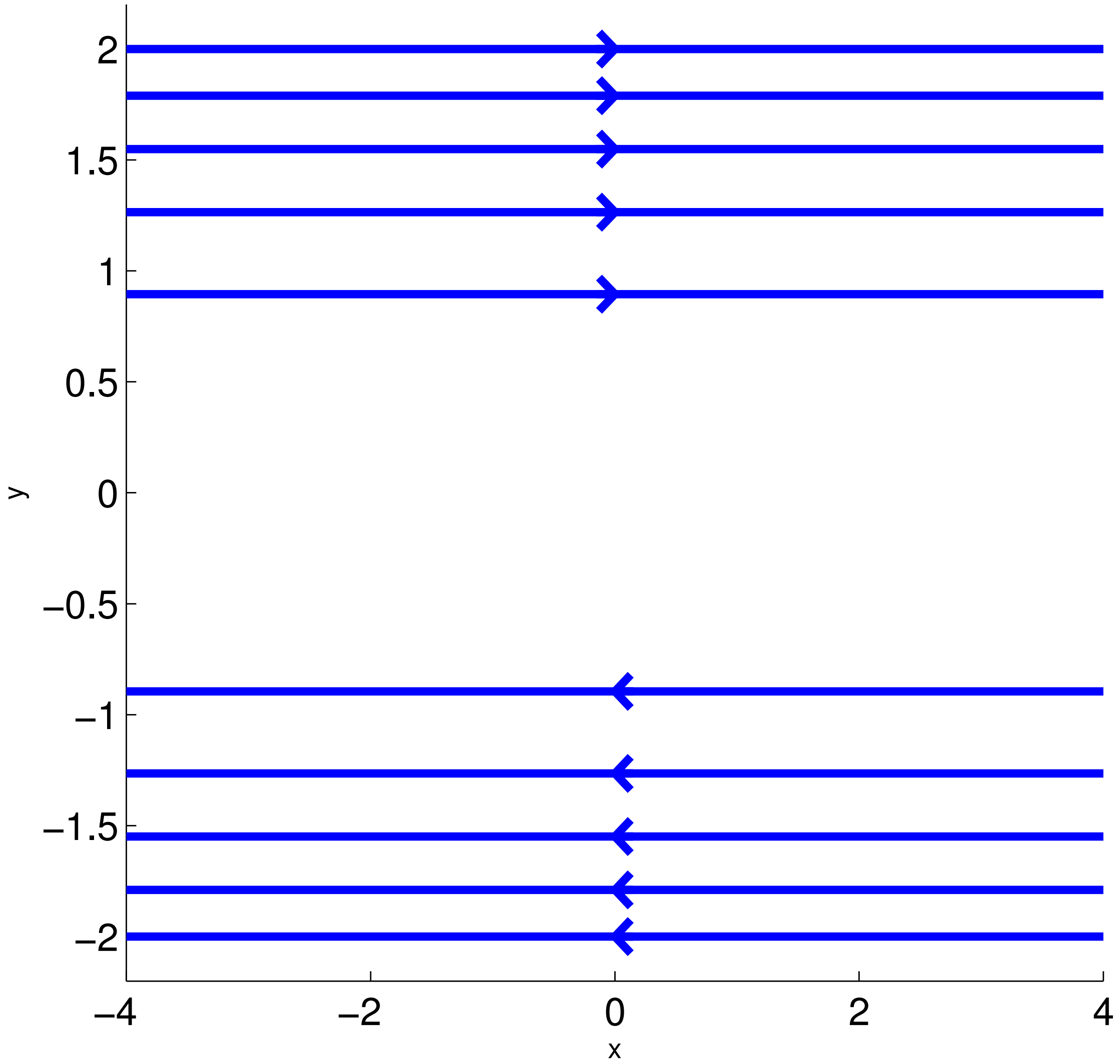


Figure 3.2: Streamlines for a simple shear flow corresponding to $\underline{q} = y\,\underline{i}$ which has stream function $\psi = y^2/2$. The lines correspond to equal spacing of the ψ value. The fluid is stationary when $y = 0$.

3.3.3 A line source and a line sink

If a long hose with multiple small holes is put in a deep water tank then the water from the hose will spray out radially with roughly the same speed in all directions with the speed depending on the pressure of the water in the hose. If the hose is located on the z-axis then in much of the tank we approximately have two-dimensional flow described by the velocity $\underline{q} = \underline{q}(r, \theta)$ of the form

$$\underline{q}(r, \theta) = f(r)\underline{e}_r(\theta) \tag{3.3.6}$$

for some function $f(r)$ which we determine next. If a stream function $\psi = \psi(r, \theta)$ exists in this case then it satisfies

$$\underline{q} = (\nabla\psi) \times \underline{k} = \left(\frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta\right) \times \underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta. \tag{3.3.7}$$

For this to be of the form given in (3.3.6) requires that

$$\frac{\partial\psi}{\partial r} = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = f(r).$$

Now

$$\frac{\partial\psi}{\partial r} = 0 \quad \text{implies that } \psi = \psi(\theta).$$

If we substitute this into the second relation and rearrange slightly then we have

$$rf(r) = \frac{\partial\psi}{\partial\theta}.$$

As the left hand side is a function of r only and the right hand side is a function of θ only this implies both must be constant, i.e.

$$\frac{\partial\psi}{\partial\theta} = rf(r) = A$$

where A is a constant. Thus to summarize the form of the stream function ψ and the velocity $\underline{q}$ are as follows.

$$\psi = A\theta \quad \text{and} \quad \underline{q} = \frac{A}{r}\,\underline{e}_r. \tag{3.3.8}$$

As we have a stream function this confirms that the flow is incompressible.

As in the other examples we consider next if the flow is irrotational. Now we have already shown that

$$\underline{\omega} = \nabla \times \underline{q} = -\nabla^2\psi\,\underline{k}.$$

In this case

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\underline{e}_\theta = \frac{A}{r}\,\underline{e}_\theta$$

and thus

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{A}{r}\right) = 0$$

indicating that the flow is irrotational.

When we are putting water into the tank the flow is away from the hose and this corresponds to $A > 0$ and the streamlines are of the form shown in the left hand side plot in figure 3.3. If we instead have a small tube in the tank at $z = 0$ which is at a lower pressure than the water in the tank then the flow is in the opposite direction and this corresponds to $A < 0$ and is shown in the plot on the right hand side of figure 3.3. In both cases the streamlines are radial lines with the case $A > 0$ being called **a line source** and the case $A < 0$ being called **a line sink**.

When the equation of mass conservation on page 2-13 was derived it was assumed that the region being considered contained no sources or sinks and this condition is violated here at $r = 0$ but is true for all regions which exclude $r = 0$. This is reflected in the expressions for ψ and $\underline{q}$ with $|\underline{q}| \to \infty$ as $r \to 0$ and with $\nabla \cdot \underline{q} = 0$ being 0 when $r > 0$ and it is not defined at $r = 0$. The magnitude of the constant A gives us information about the amount of fluid entering or leaving the region but to do this a bit more precisely we consider the rate at which fluid flows across a circle centered at 0 which involves a flux integral in the two-dimensional set-up. If the circle is $C = \{(r, \theta) : \; r = a, \; 0 \le \theta < 2\pi\}$ then

$$\int_C \underline{q} \cdot \underline{n} \, \mathrm{d}s = \int_0^{2\pi} \underline{q} \cdot \underline{e}_r \, a \mathrm{d}\theta = 2\pi A. \tag{3.3.9}$$

The constant $2\pi A$ is known as **the strength of the source**.

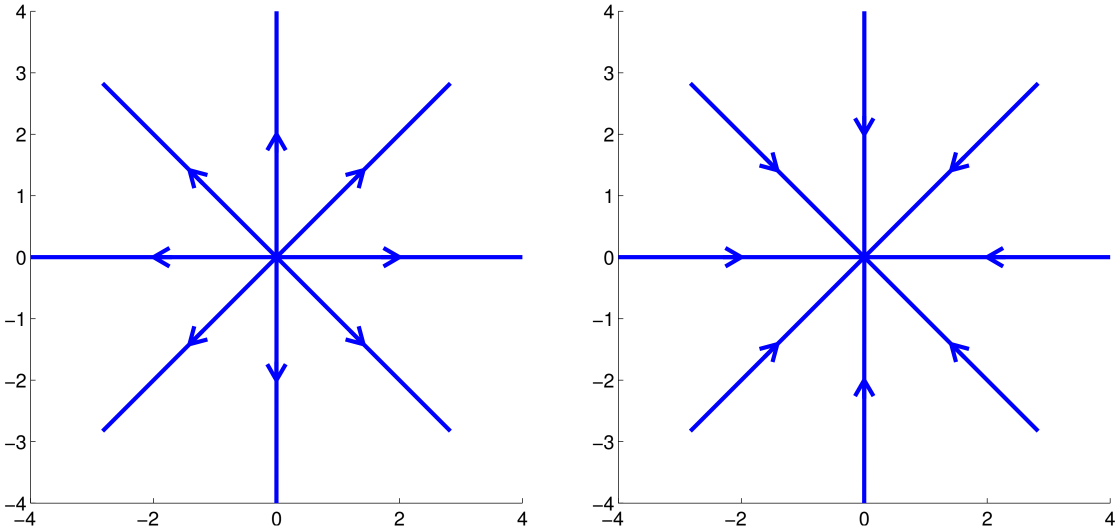


Figure 3.3: Streamlines for a source at $(0, 0)$ (left hand side plot) and a sink at $(0, 0)$ (right hand side plot). The fluid particles move away from the source along radial lines and they move towards the sink along radial lines.

3.3.4 A line vortex

In the previous example we considered flow which is radially towards or radially away from a point. We now consider a flow which circulates about a point involving a velocity of the form

$$\underline{q} = f(r) \, \underline{e}_\theta \tag{3.3.10}$$

where $f(r)$ is some function which we consider below. Now if a stream function ψ exists in this case then

$$\underline{q} = (\nabla \psi) \times \underline{k} = \left(\frac{\partial \psi}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \underline{e}_\theta \right) \times \underline{k} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \underline{e}_r - \frac{\partial \psi}{\partial r} \underline{e}_\theta = f(r) \, \underline{e}_\theta. \tag{3.3.11}$$

This gives

$$\frac{\partial \psi}{\partial \theta} = 0 \quad \text{and} \quad f(r) = -\frac{\partial \psi}{\partial r}.$$

Thus $\psi = \psi(r)$ and $f(r)$ depends on the derivative of this function but there are no other constraints. To motivate a specific choice for $f(r)$ consider the circle

$$C = \{(r, \theta) : \ r = a, \ 0 \le \theta < 2\pi\},$$

as we did in the case of a line source/sink, but consider now the line integral involving $\underline{q}$ around this curve. The value obtained is known as **the circulation** and is given by

$$\Gamma = \oint_C \underline{q} \cdot \mathrm{d}\underline{r} = \int_0^{2\pi} f(a) a \, \mathrm{d}\theta. \tag{3.3.12}$$

The value obtained is independent of a when $f(r)$ is such that

$$f(r) = \frac{B}{r} \quad \text{giving } \Gamma = 2\pi B \tag{3.3.13}$$

and this corresponds to

$$\psi(r) = -B \ln r + \text{const}.$$

As we are free to choose the constant it is convenient to take $B \ln a$ so that

$$\psi(r) = -B \ln\left(\frac{r}{a}\right) = -\frac{\Gamma}{2\pi} \ln\left(\frac{r}{a}\right). \tag{3.3.14}$$

To summarise, one specific line vortex involves a stream function ψ and velocity $\underline{q}$ given by

$$\psi(r) = -\left(\frac{\Gamma}{2\pi}\right) \ln\left(\frac{r}{a}\right) \quad \text{and} \quad \underline{q} = \left(\frac{\Gamma}{2\pi}\right) \frac{1}{r} \underline{e}_\theta. \tag{3.3.15}$$

As we have a stream function this confirms that the flow is incompressible. To check whether of not it is irrotational we need to consider

$$\underline{\omega} = \nabla \times \underline{q} = -\nabla^2 \psi \, \underline{k}.$$

Now let

$$\underline{g} = g_1 \, \underline{e}_r = \nabla(\ln r) = \frac{1}{r} \underline{e}_r$$

and

$$\nabla \cdot \underline{g} = \frac{1}{r} \frac{\partial}{\partial r}(r \, g_1(r)) = 0.$$

Hence

$$\underline{\omega} = \underline{0}$$

and we do have irrotational flow. Thus although the streamlines shown in figure 3.4 are circles the particles themselves are not spinning and remember that vorticity describes the local behaviour of each particle whilst the streamlines shows the global pattern of the flow.

Streamlines of the form shown in figure 3.4 correspond approximately to how air moves when there is a hurricane although a three dimensional model is needed to model this accurately.

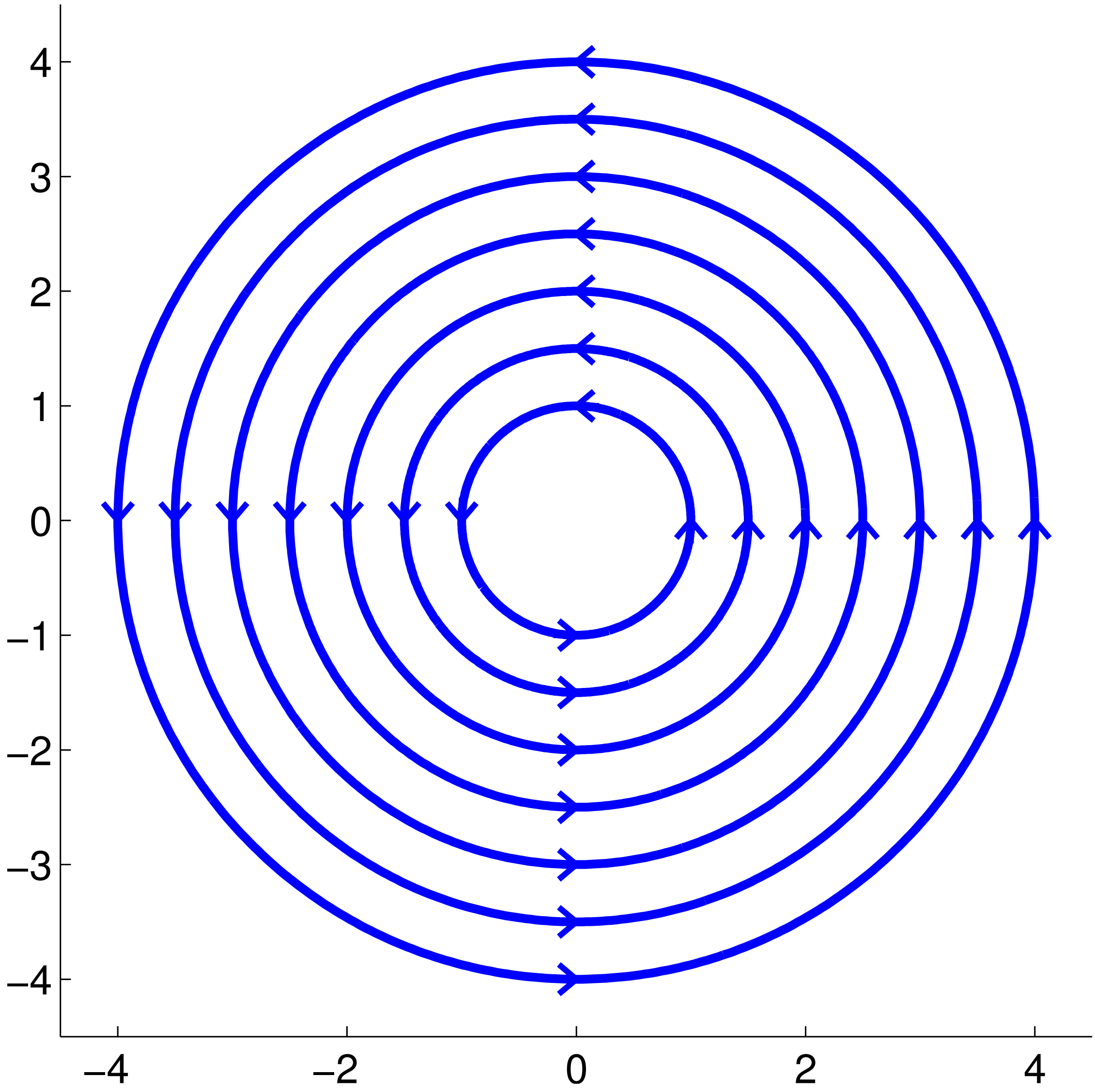


Figure 3.4: Streamlines for a line vortex at $(0, 0)$.

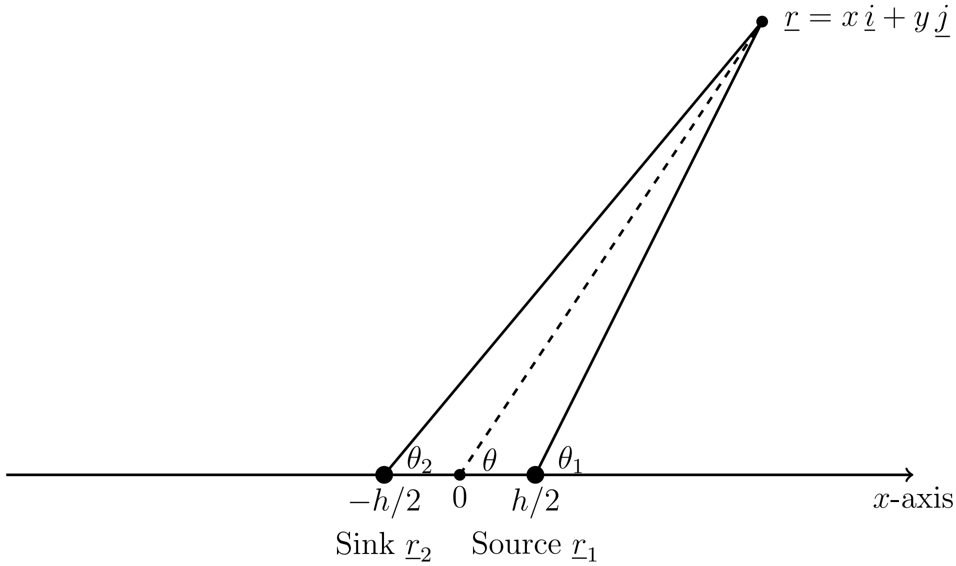
3.4 Examples of combining basic stream functions

3.4.1 The dipole

Suppose that we have a source at $\underline{r}_1 = (h/2)\underline{i}$ and a sink at $\underline{r}_2 = -(h/2)\underline{i}$, $h > 0$ with both having equal strength. From what was done in section 3.3.3 it follows that the stream function is of the form

$$\psi(\underline{r}) = A(\theta_1 - \theta_2) \tag{3.4.1}$$

where θ_1 is the angle associated with $\underline{r} - \underline{r}_1$ and where θ_2 is the angle associated with $\underline{r} - \underline{r}_2$ as shown in the following diagram.



In the following a limit is going to be considered which involves first writing

$$\psi = Ah\left(\frac{\theta_1-\theta_2}{h}\right) = \mu\left(\frac{\theta_1-\theta_2}{h}\right) \quad \text{with } \mu = Ah.$$

The dipole corresponds to the case $h \to 0$ with $\mu = Ah$ remaining constant, i.e. it is the limiting case when the points $\underline{r}_1$ and $\underline{r}_2$ move together with the strengths tending to ∞ in such a way that $\mu = Ah$ is constant. Now to express θ_1 and θ_2 in a form which involves $\underline{r}$ and h we have

$$\sin^2\theta_1 = \frac{y^2}{(x-h/2)^2+y^2} \quad \text{and} \quad \sin^2\theta_2 = \frac{y^2}{(x+h/2)^2+y^2}.$$

These two values are the same when $x=0$ and when $y=0$ and if we exclude these cases for the moment then we can write

$$\frac{\theta_1-\theta_2}{h} = \left(\frac{\theta_1-\theta_2}{\sin^2\theta_1-\sin^2\theta_2}\right)\left(\frac{\sin^2\theta_1-\sin^2\theta_2}{h}\right)$$

and we consider the limit of each bracketed terms separately. Firstly as $h \to 0$ we have $\theta_1 \to \theta$ and $\theta_2 \to \theta$ and from the defintion of the derivative

$$\lim_{h\to 0}\left(\frac{\sin^2\theta_1-\sin^2\theta_2}{\theta_1-\theta_2}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^2\theta\right) = 2\sin\theta\cos\theta. \tag{3.4.2}$$

For the other term

$$\begin{aligned}\frac{\sin^2\theta_1-\sin^2\theta_2}{h} &= \frac{y^2(2hx)}{h((x-h/2)^2+y^2)((x+h/2)^2+y^2)} \\ &\to \frac{y^2 2x}{(x^2+y^2)^2} = \frac{2\sin^2\theta\,\cos\theta}{r} \quad \text{as } h\to 0. \end{aligned} \tag{3.4.3}$$

By combining (3.4.2) and (3.4.3) we get

$$\frac{\theta_1-\theta_2}{h} \to \frac{\sin\theta}{r} \quad \text{as } h \to 0. \tag{3.4.4}$$

Note that this expression makes sense when $\theta \to 0$ and $\theta \to \pi$ corresponding to $x = 0$ and it also makes sense as $\theta \to \pm\pi/2$ corresponding to $y = 0$. The stream function for the dipole is thus

$$\psi = \mu \frac{\sin\,\theta}{r}. \tag{3.4.5}$$

The streamlines are curves corresponding to $\psi(r, \theta) = C$ where C is a constant. When $C = 0$ this corresponds to $\sin\,\theta = 0$ which gives the x-axis. In all cases with $C \neq 0$ the curves start and end at $r = 0$ and are bounded. When $C > 0$ this corresponds to $0 < \theta < \pi$ with the curve, in polar form, being

$$r = \frac{\mu}{C} \sin\,\theta$$

which has a maximum at $\theta = \pi/2$. When $C < 0$ this corresponds to $-\pi < \theta < 0$ with the curve being

$$r = \frac{\mu}{C} \sin\,\theta$$

with a minimum at $-\pi/2$. A plot of streamlines for this function ψ are shown in figure 3.5.

The dipole given above was derived for a source at $(h/2)\underline{i}$ and a sink at $-(h/2)\underline{i}$ and is said to have a direction of $\underline{i}$. If the source and sink are reversed then the direction is $-\underline{i}$ and the stream function is

$$\psi = -\mu \frac{\sin\,\theta}{r}. \tag{3.4.6}$$

You can think of this as in the direction of π and in the general case when the direction of the line segment from the sink to the source is θ_d it can be shown that

$$\psi = \mu \frac{\sin(\theta - \theta_d)}{r}. \tag{3.4.7}$$

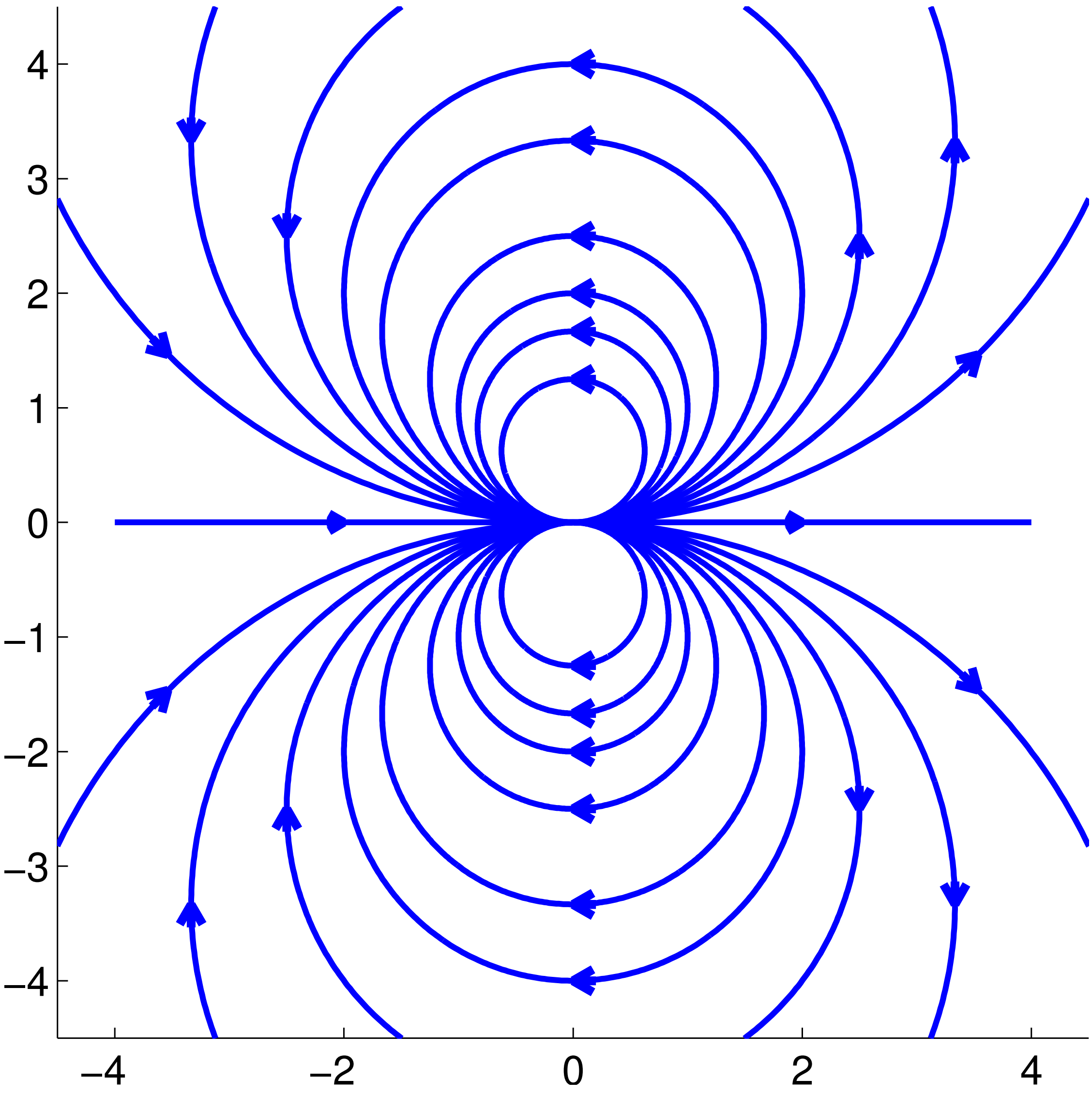


Figure 3.5: Streamlines for a dipole at $(0,0)$ in the $\underline{i}$ direction. The fluid particles move away from the source and towards the sink. The curves shown correspond to curves with polar descriptions of the form $r = K\sin(\theta)$ for different values of the constant K.

3.4.2 Flow round a cylinder

In section 3.3.1 we considered uniform flow in the direction of $\underline{i}$ and this involved

$$\underline{q} = \underline{q} = u\,\underline{i} + v\,\underline{j} = \frac{\partial \psi_1}{\partial y}\,\underline{i} - \frac{\partial \psi_1}{\partial x}\,\underline{j} = U\,\underline{i}, \quad \text{with } \psi_1 = Uy = Ur\sin\theta.$$

We now consider a dipole corresponding to (3.4.6) and label the stream function here as

$$\psi_2 = -\mu \frac{\sin\theta}{r}$$

and consider combining ψ_1 and ψ_2 to give

$$\psi = \psi_1 + \psi_2 = \left(Ur - \frac{\mu}{r}\right)\sin\theta. \tag{3.4.8}$$

In figure 3.6 we show the two streamlines side-by-side to attempt to see how they partly cancel or combine close to the dipole.

If we consider the streamline corresponding to $\psi = 0$ then this has different parts corresponding to

$$\sin\theta = 0 \quad \text{and} \quad r^2 = \frac{\mu}{U}.$$

The part $\sin\theta = 0$ corresponds to the x-axis whilst the other part corresponds to a circle of radius a satisfying

$$a^2 = \frac{\mu}{U} \quad \text{so that } \mu = Ua^2.$$

The velocity field is

$$\begin{aligned}
\underline{q} &= \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\underline{e}_r - \frac{\partial \psi}{\partial r}\,\underline{e}_\theta \\
&= \frac{1}{r}\left(Ur - \frac{\mu}{r}\right)\cos\theta\,\underline{e}_r - \left(U + \frac{\mu}{r^2}\right)\sin\theta\,\underline{e}_\theta \\
&= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\,\underline{e}_r - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\,\underline{e}_\theta.
\end{aligned}$$

When $r = a$ we have

$$\underline{q} = -2U\sin\theta\,\underline{e}_\theta$$

and the velocity is tangential to the circle with no fluid moving across the circle. What we have stumbled upon is the flow of an inviscid fluid past a rigid cylinder of radius a placed in a uniform flow as this corresponds to what we get with instead having a dipole in the direction of $-\underline{i}$ with strength $\mu = Ua^2$. The stream function for this flow is given by

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{3.4.9}$$

The stagnation points of this flow are points where $\underline{q} = \underline{0}$ and these occur when both of the following are satisfied

$$\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0 \quad \text{and} \quad \left(1 + \frac{a^2}{r^2}\right)\sin\theta = 0.$$

Now $\cos\theta$ and $\sin\theta$ are not both zero at the same θ and thus the only stagnation points are when $r = a$ and $\sin\theta = 0$ which gives the points on the circle with $\theta = 0$ and $\theta = \pi$. The streamlines for flow round a cylinder are given in figure 3.7 and this corresponds to the first figure on page 2-20 when the limitations of the inviscid model were discussed.

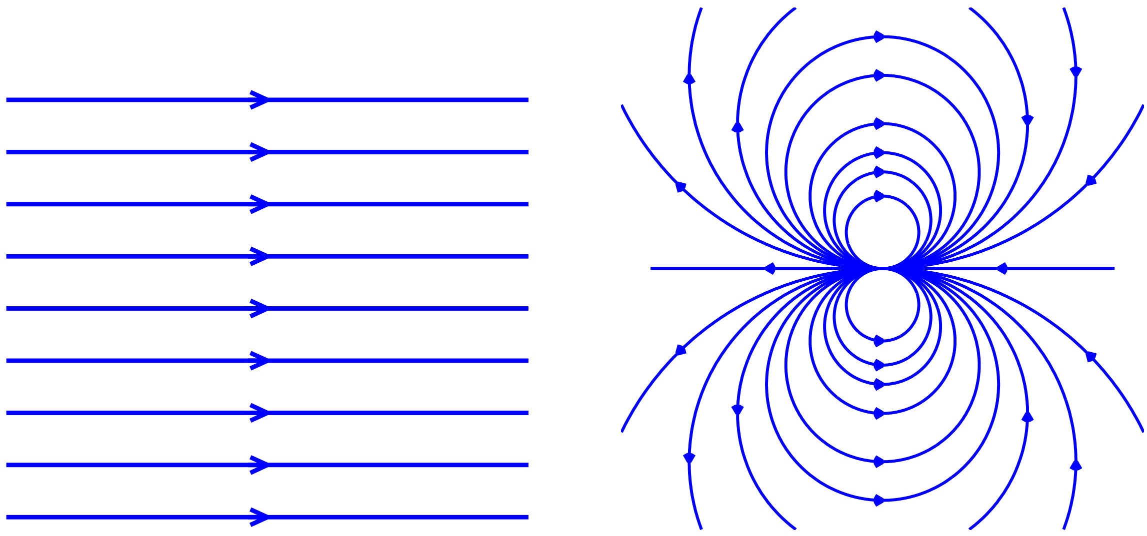


Figure 3.6: Streamlines for uniform flow in the direction of $\underline{i}$ (left plot for ψ_1) and a dipole in the direction of $-\underline{i}$ (right plot for ψ_2).

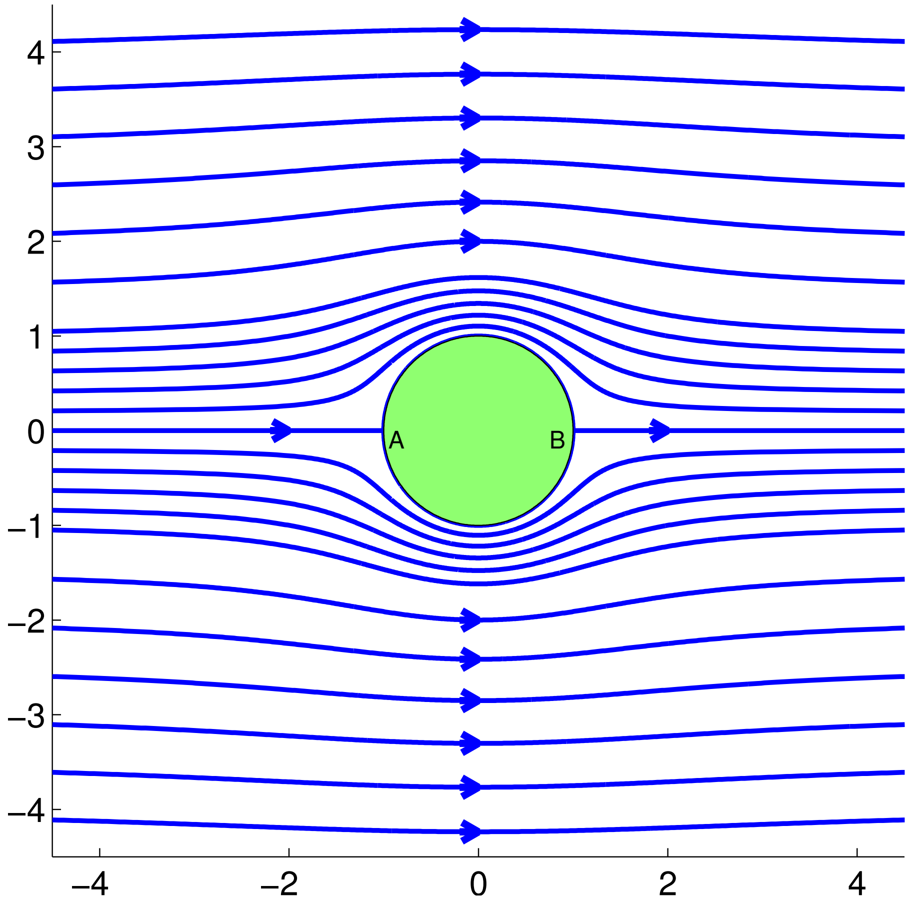


Figure 3.7: Streamlines for a cylinder of radius 1 placed in a uniform flow in the direction of $\underline{i}$. The points $A = (-1, 0)$ and $B = (1, 0)$ are the stagnation points.