MA2741: Spring Term – Exercise sheet 1 with answers

Exercises involving the Divergence theorem

1. A closed region Ω is bounded by a simple surface S. Use the Divergence theorem to prove that
$$\int_S \underline{r} \cdot \mathrm{d}\underline{s} = 3V$$
where $\underline{r}$ is the position vector of a point on the surface and V is the volume of the region Ω.

Answer

In the expression for the surface integral we have
$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}.$$
The divergence of this vector is
$$\nabla \cdot \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$
Then by the divergence theorem
$$\int_S \underline{r} \cdot \mathrm{d}\underline{s} = \int_\Omega \nabla \cdot \underline{r}\,\mathrm{d}v = \int_\Omega 3\,\mathrm{d}v = 3 \times (\text{volume of } \Omega).$$

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2. Use the Divergence theorem to evaluate
$$\int_S \underline{F} \cdot \mathrm{d}\underline{s},$$
where
$$\underline{F} = (z^2 - 1)(xy^2\underline{i} + xy\underline{j} + y^2\underline{k})$$
and S is the closed surface of the cube centred at the origin and with sides of length 2 units with each side parallel to one of the planes $x = 0$, $y = 0$ and $z = 0$. Check you answer by doing the surface integrals.

Answer

In components $\underline{F} = F_1\,\underline{i} + F_2\,\underline{j} + F_3\,\underline{k}$ with
$$F_1 = (z^2 - 1)xy^2, \quad F_2 = (z^2 - 1)xy, \quad F_3 = (z^2 - 1)y^2.$$
For the partial derivatives in the divergence expression we have
$$\frac{\partial F_1}{\partial x} = (z^2 - 1)y^2, \quad \frac{\partial F_2}{\partial y} = (z^2 - 1)x, \quad \frac{\partial F_3}{\partial z} = 2zy^2,$$

giving

$$\nabla \cdot \underline{F} = (z^2-1)(y^2+x) + 2zy^2.$$

The region Ω is a cube and is described by

$$\Omega = \{(x,y,z):\ -1 \le x,y,z \le 1\}.$$

For the volume integral we have

$$I = \int_\Omega \nabla \cdot \underline{F}\,\mathrm{d}v = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (z^2-1)(y^2+x) + 2zy^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We consider the integral in parts. First note that

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (z^2-1)x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

because x is an odd function and the range on x is $(-1,1)$. Similarly

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 2zy^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

because z is an odd function and the range on z is $(-1,1)$. Hence

$$\begin{aligned} I &= \int_\Omega \nabla \cdot \underline{F}\,\mathrm{d}v = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (z^2-1)y^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{-1}^1 (z^2-1)\,\mathrm{d}z \int_{-1}^1 y^2\,\mathrm{d}y \int_{-1}^1 \mathrm{d}x \\ &= \left(\frac{2}{3}-2\right)\left(\frac{2}{3}\right)2 = -\frac{16}{9}. \end{aligned}$$

For the surface integral note that the cube has 6 faces.

Two of the faces correspond to $z^2=1$ and $\underline{F} = \underline{0}$ on these faces.

On the face $y=1$ the outward normal is $\underline{n} = \underline{j}$ and

$$\underline{F} \cdot \underline{j} = (z^2-1)x.$$

This is an odd function of x and we get 0 when we integrate over $-1 < x < 1$. We similarly get 0 when we consider the face $y=-1$.

On the face $x=1$ the outward normal is $\underline{n} = \underline{i}$ and

$$\underline{F} \cdot \underline{i} = (z^2-1)y^2.$$

Similarly on the face $x=-1$ the outward normal is $\underline{n} = -\underline{i}$ and

$$\underline{F} \cdot (-\underline{i}) = (z^2-1)y^2.$$

The combined contribution to the surface integral from these two faces is thus

$$2\int_{-1}^{1}\int_{-1}^{1}(z^2-1)y^2\,\mathrm{d}y\mathrm{d}z = 2\left(\frac{2}{3}-2\right)\left(\frac{2}{3}\right) = -\frac{16}{9}.$$

This confirms the value as $-16/9$.

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3. Show that

$$\int_S \underline{q}\cdot \mathrm{d}\underline{s} = \frac{\pi}{6}$$

where $\underline{q} = z^2\underline{k}$ and S is the whole of the surface of the cone $x^2+y^2=(1-z)^2$, $0\le z\le 1$, including the base $x^2+y^2=1$, $z=0$. Use direct evaluation and the Divergence theorem.

Answer

The cone is most easily described using cylindrical polar coordinates (r,θ,z) with points on the surface corresponding to $r=1-z$ and thus the position vector of a point on the surface is given by

$$\underline{r}(\theta,z) = (1-z)\underline{e}_r(\theta) + z\,\underline{k}.$$

If we partially differentiate with respect to θ and z we get vectors tangential to the cone and we get a vector normal to the cone if we takes the cross product of such vectors. In this case

$$\frac{\partial \underline{r}}{\partial\theta} = (1-z)\underline{e}_\theta, \qquad \frac{\partial \underline{r}}{\partial z} = -\underline{e}_r + \underline{k}.$$

and

$$\frac{\partial \underline{r}}{\partial\theta}\times\frac{\partial \underline{r}}{\partial z} = (1-z)(-\underline{e}_\theta\times\underline{r} + \underline{e}_\theta\times\underline{k}) = (1-z)(\underline{k}+\underline{e}_r).$$

If we let S_1 denote the cone then

$$\begin{aligned}\int_{S_1}\underline{q}\cdot\mathrm{d}\underline{s} &= \int_{\theta=-\pi}^{\pi}\int_{z=0}^{1}\underline{q}\cdot\left(\frac{\partial \underline{r}}{\partial\theta}\times\frac{\partial \underline{r}}{\partial z}\right)\mathrm{d}z\mathrm{d}\theta.\\ &= \int_{\theta=-\pi}^{\pi}\int_{z=0}^{1}(1-z)z^2\,\mathrm{d}z\mathrm{d}\theta\\ &= 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6}.\end{aligned}$$

The surface S is the closed surface which consists of S_1 and the base of the cone and on the base of the cone $z=0$ and thus $\underline{q}=\underline{0}$. Thus

$$\int_S \underline{q}\cdot\mathrm{d}\underline{s} = \frac{\pi}{6}.$$

To evaluate using the divergence theorem and a volume integral involves using

$$\int_S \underline{q}\cdot\mathrm{d}\underline{s} = \int_\Omega \nabla\cdot\underline{q}\,\mathrm{d}v$$

with

$$\nabla \cdot \underline{q} = \frac{\partial (z^2)}{\partial z} = 2z.$$

Let Ω denote the region interior to S which is described by

$$\Omega = \left\{ (r, \theta, z) : \ 0 \le r < 1 - z, \ 0 < z < 1, \ -\pi < \theta \le \pi \right\}.$$

With cylindrical polars the volume element is $r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ and the volume integral to consider is

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \int_{z=0}^{1} \int_{r=0}^{1-z} 2zr\mathrm{d}r\mathrm{d}z\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \int_{z=0}^{1} \left[r^2\right]_0^{1-z} z\mathrm{d}z\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \int_{z=0}^{1} (1-z)^2 z\mathrm{d}z\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \int_{z=0}^{1} (z - 2z^2 + z^3)\,\mathrm{d}z\mathrm{d}\theta \\ &= 2\pi \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{\pi}{6}. \end{aligned}$$

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4. A closed region Ω is bounded by a simple surface S. Use the Divergence theorem to prove that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \,\mathrm{d}v = \int_S \phi \frac{\partial \psi}{\partial n}\,\mathrm{d}s - \int_\Omega \phi \nabla^2 \psi \,\mathrm{d}v$$

where ϕ and ψ are scalar fields. Hence, prove Green's second identity which is

$$\int_\Omega \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right)\,\mathrm{d}v = \int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right)\,\mathrm{d}s.$$

Answer

Note first the vector identity

$$\nabla \cdot (\phi \nabla \psi) = \nabla\phi \cdot \nabla\psi + \phi \nabla^2 \psi.$$

Since

$$(\nabla \psi) \cdot \underline{n} = \frac{\partial \psi}{\partial n}$$

the divergence theorem gives

$$\int_\Omega \nabla \cdot (\phi \nabla \psi)\,\mathrm{d}v = \int_S \phi \frac{\partial \psi}{\partial n}\,\mathrm{d}s,$$

i.e.

$$\int_\Omega \left(\nabla\phi \cdot \nabla\psi + \phi \nabla^2 \psi \right)\,\mathrm{d}v = \int_S \phi \frac{\partial \psi}{\partial n}\,\mathrm{d}s.$$

If we swap ϕ and ψ then we get

$$\int_{\Omega} \left(\nabla\psi \cdot \nabla\phi + \psi\nabla^2\phi \right) \, \mathrm{d}v = \int_S \psi \frac{\partial\phi}{\partial n} \, \mathrm{d}s.$$

Green's second identity follows by subtracting this relation from the previous relation.

Exercises involving Stokes' theorem

1. Given that S is the hemisphere of unit radius described by

$$\underline{r}(u,v) = \sin v \cos u \underline{i} + \sin v \sin u \underline{j} + \cos v \underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le \pi/2$$

and C is the closed curve that bounds the hemisphere in the $xy-$plane, evaluate

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r} \qquad \text{and} \qquad \int_S (\nabla \times \underline{q}) \cdot \mathrm{d}\underline{s}$$

where

i) $\underline{q} = Uy\underline{i}$, U constant, ii) $\underline{q} = y^2\underline{i} + x\underline{j}$.

What do you notice about your answers?

Answer

When $\underline{q}$ is given as in (i) we have

$$\nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Uy & 0 & 0 \end{vmatrix} = -U\underline{k}.$$

When $\underline{q}$ is given as in (ii) we have

$$\nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x & 0 \end{vmatrix} = (1-2y)\underline{k}.$$

Points on the unit circle in the x, y plane are described using Cartesian base vectors as

$$\underline{r} = \cos\, u\, \underline{i} + \sin\, u\, \underline{j}, \quad -\pi < u \le \pi,$$

giving

$$\mathrm{d}\underline{r} = (-\sin\, u\underline{i} + \cos\, u\underline{j})\mathrm{d}u.$$

In the case of (i) we have $y = \sin\, u$ and we have

$$\underline{q} \cdot \mathrm{d}\underline{r} = -U \sin^2 u \,\mathrm{d}u.$$

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r} = \int_0^{2\pi} (-U \sin^2 u)\,\mathrm{d}u. = -\pi\, U.$$

In the case of (ii) we have $x = \cos\, u$ and $y = \sin\, u$ and we have

$$\underline{q} \cdot \mathrm{d}\underline{r} = (-y^2 \sin\, u + x \cos\, u)\,\mathrm{d}u. = (-\sin^3 u + \cos^2 u)\,\mathrm{d}u.$$

As $\sin^3 u$ is an odd function we have

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r} = \int_{-\pi}^{\pi} \cos^2 u \, \mathrm{d}u = \pi.$$

To evaluate the surface integrals we need to first determine

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial u} &= -\sin v \sin u \, \underline{i} + \sin v \cos u \underline{j}, \\
\frac{\partial \underline{r}}{\partial v} &= \cos v \cos u \, \underline{i} + \cos v \sin u \underline{j} - \sin v \underline{k}.
\end{aligned}$$

The surface integral is then

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin v \sin u & \sin v \cos u & 0 \\ \cos v \cos u & \cos v \sin u & -\sin v \end{vmatrix} \\
&= (-\sin^2 v \cos u)\underline{i} - (\sin^2 v \sin u)\underline{j} - (\sin v \cos v)\underline{k} \\
&= (-\sin v)\underline{r}.
\end{aligned}$$

This is in the direction of the inward normal and for the outward normal we need

$$\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} = (\sin v)\underline{r}.$$

In the case of (i) we have

$$(\nabla \times \underline{q}) \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} \right) = -U(\sin v)\underline{r} \cdot \underline{k} = -U(\sin v)\cos v = -\frac{U}{2} \sin 2v.$$

As

$$\int_0^{\pi/2} \sin 2v \, \mathrm{d}v = 1$$

it follows that

$$\int_{u=0}^{2\pi} \int_{v=0}^{\pi/2} (\nabla \times \underline{q}) \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} \right) \mathrm{d}v\mathrm{d}u = -\pi U$$

which agrees with what was obtained by using the line integral.
In the case of (ii) we have

$$(\nabla \times \underline{q}) \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} \right) = (1-2y)(\sin v)\underline{r} \cdot \underline{k} = (1 - 2\sin v \sin u)(\sin v)(\cos v).$$

The last part involves $-2\sin^2 v \cos v \sin u$ and when we integrate with respect to u on the range $0 \le u < 2\pi$ this gives 0. Hence

$$\int_{u=0}^{2\pi} \int_{v=0}^{\pi/2} (\nabla \times \underline{q}) \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} \right) \mathrm{d}v\mathrm{d}u = \frac{2\pi}{2} \int_{v=0}^{\pi/2} \sin(2v) \, \mathrm{d}v = \pi.$$

which agrees with that obtained using the line integral.

2. Verify Stokes' theorem for the vector field $\underline{F} = x^2y\underline{i} + z\underline{j}$ and the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

Answer

We need to verify that

$$\int_S (\nabla \times \underline{F}) \cdot \underline{n}\, \mathrm{d}s = \oint_C \underline{F} \cdot \mathrm{d}\underline{r}.$$

To start we need a parametric description for S and C and in the case of the surface S we can take

$$\underline{r}(s,t) = a\left(\cos\, s(\cos\, t\,\underline{i} + \sin\, t\,\underline{j}) + \sin\, s\,\underline{k}\right), \quad 0 \le s \le \frac{\pi}{2}, \quad -\pi < t \le \pi.$$

The perimeter corresponds to $s = 0$ and is the circle

$$\underline{r}(0,t) = a\left(\cos\, t\,\underline{i} + \sin\, t\,\underline{j}\right), \quad -\pi < t \le \pi$$

and as t increases this corresponds to moving round the circle in the anti-clockwise direction. For the surface integral we need the curl which is

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & z & 0 \end{vmatrix} = \underline{i} - \underline{j}(0) + \underline{k}(-x^2) = \underline{i} - x^2\,\underline{k}.$$

For the line integral we need

$$\frac{\mathrm{d}\underline{r}(0,t)}{\mathrm{d}t} = -a\sin\, t\,\underline{i} + a\cos\, t\,\underline{j}.$$

Also, for points on the circle

$$\underline{F}(a\cos\, t, a\sin\, t, 0) = a^3\cos^2 t\,\sin\, t\,\underline{i}$$

and

$$\underline{F}(a\cos\, t, a\sin\, t, 0) \cdot \frac{\mathrm{d}\underline{r}(0,t)}{\mathrm{d}t} = -a^4\cos^2 t\,\sin^2 t = \frac{-a^4\sin^2(2t)}{4}.$$

Thus

$$\oint_C \underline{F} \cdot \mathrm{d}\underline{r} = \int_{-\pi}^{\pi} \left(\frac{-a^4\sin^2(2t)}{4}\right) \mathrm{d}t = -\frac{a^4\pi}{4}.$$

If you want to consider a simpler surface integral which also has C as the perimeter then you could take $\{(x,y,0):\ x^2 + y^2 < a^2\}$. In this case the normal is $\underline{n} = \underline{k}$ and with $x = r\cos\, t$

$$(\nabla \times \underline{F}) \cdot \underline{n} = -x^2 = -r^2\cos^2 t.$$

The flat surface in this case is

$$r(\cos\, t\,\underline{i} + \sin\, t\,\underline{j}), \quad 0 \le r < a, \quad -\pi < t \le \pi,$$

i.e. with a polar description, and $\mathrm{d}s = r\mathrm{d}r\mathrm{d}t$. The surface integral in this case is

$$\int_S (\nabla \times \underline{F}) \cdot \underline{n}\,\mathrm{d}s = -\int_{-\pi}^{\pi}\int_0^a r^3 \cos^2 t\,\mathrm{d}r\mathrm{d}t = -\frac{a^4\pi}{4}.$$

To actually do the surface integral for the hemisphere in the question we need to determine

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t} &= a^2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin s \cos t & -\sin s \sin t & \cos s \\ -\cos s \sin t & \cos s \cos t & 0 \end{vmatrix} \\
&= a^2\left((-\cos^2 s \cos t)\underline{i} - (\cos^2 s \sin t)\underline{j} + (-\cos s \sin s)\underline{k}\right) \\
&= -a \cos s\, \underline{r}.
\end{aligned}$$

As $0 < \cos s < 1$ for $0 < s < \pi/2$ this vector is pointing towards the centre of the sphere and thus for the outward normal direction we need instead

$$\frac{\partial \underline{r}}{\partial t} \times \frac{\partial \underline{r}}{\partial s} = a \cos s\, \underline{r}.$$

For the integrand in the surface integral to consider we have

$$\begin{aligned}
(\nabla \times \underline{F}) \cdot \left(\frac{\partial \underline{r}}{\partial t} \times \frac{\partial \underline{r}}{\partial s}\right) &= (\underline{i} - x^2\,\underline{k}) \cdot (a \cos s\, \underline{r}) \\
&= a(a\cos s)(\underline{i} - a^2 \cos^2 s \cos^2 t\,\underline{k}) \cdot (\cos s(\cos t\,\underline{i} + \sin t\,\underline{j}) + \sin s\,\underline{k}) \\
&= a^2 \cos^2 s \cos t - a^4 \cos^3 s \sin s \cos^2 t.
\end{aligned}$$

Now

$$\int_{-\pi}^{\pi} \cos t\,\mathrm{d}t = 0, \quad \int_{-\pi}^{\pi} \cos^2 t\,\mathrm{d}t = \pi, \quad -\int_0^{\pi/2} \cos^3 s \sin s\,\mathrm{d}s = \left[\frac{\cos^4 s}{4}\right]_0^{\pi/2} = -\frac{1}{4}.$$

Thus

$$\int_{t=-\pi}^{\pi}\int_{s=0}^{\pi/2} (\nabla \times \underline{F}) \cdot \left(\frac{\partial \underline{r}}{\partial t} \times \frac{\partial \underline{r}}{\partial s}\right) \mathrm{d}s\mathrm{d}t = -\frac{a^4\pi}{4}.$$

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3. Evaluate

$$\int_S (\nabla \times \underline{q}) \cdot \mathrm{d}\underline{s}$$

where

$$\underline{q} = (x^2 + y - 4)\underline{i} + 3xy\underline{j} + (2xz + z^2)\underline{k}$$

and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

Answer

By Stokes' theorem we have

$$\int_S (\nabla \times \underline{q}) \cdot \mathrm{d}\underline{s} = \oint_C \underline{q} \cdot \mathrm{d}\underline{r}$$

where C is the perimeter of the surface which is the circle $x^2 + y^2 = 4$ in the plane $z = 0$. A parametric description of this circle is

$$C = \left\{ \underline{r}(t) = 2(\cos\, t\, \underline{i} + \sin\, t\, \underline{j}) : \ -\pi < t \le \pi \right\}$$

and on this circle

$$\underline{q} = (x^2 + y - 4)\underline{i} + 3xy\underline{j} = (4\cos^2 t + 2\sin\, t - 4)\underline{i} + 12\cos\, t \sin\, t\underline{j}.$$

Now

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = 2(-\sin\, t\, \underline{i} + \cos\, t\, \underline{j})$$

and

$$\underline{q} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = (-8\cos^2 t \sin\, t - 4\sin^2 t + 8\sin\, t) + 12\cos^2 t \sin\, t.$$

Only one of the terms is not an odd function of t and thus

$$\int_{-\pi}^{\pi} \underline{q} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\, \mathrm{d}t = -4 \int_{-\pi}^{\pi} \sin^2 t\, \mathrm{d}t = -4\pi.$$

Exercises involving Green's theorem in the plane

1. Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)\,\mathrm{d}x + x^2\,\mathrm{d}y$$

where C is the closed curve bounded by $y = x$ and $y = x^2$, $0 \le x \le 1$.

Answer

Green's theorem in the plane is a special case of Stokes' theorem and the integrand in the area integral involves

$$\underline{k} \cdot (\nabla \times \underline{F}) = \underline{k} \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Green's theorem is

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y.$$

For this question

$$F_1 = xy + y^2, \quad F_2 = x^2,$$

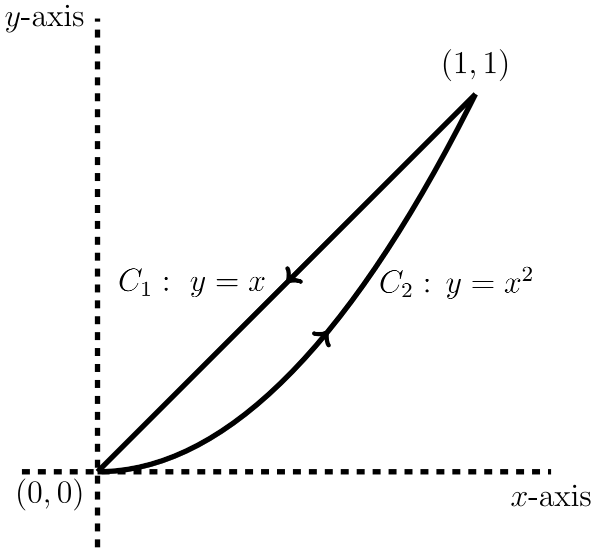
for the partial derivatives

$$\frac{\partial F_1}{\partial y} = x + 2y, \quad \frac{\partial F_2}{\partial x} = 2x \quad \text{giving} \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x - 2y.$$

and

$$\underline{F} \cdot \mathrm{d}\underline{r} = F_1 \mathrm{d}x + F_2 \mathrm{d}y = (xy + y^2)\mathrm{d}x + x^2 \mathrm{d}y.$$

The curve C has 2 parts corresponding to $y = x$ and to $y = x^2$ and as $0 \le x \le 1$ the part corresponding to $y = x^2$ is the lower of the two curves in $0 < x < 1$ with the 2 curves meeting at $x = 0$ and $x = 1$. Let C_1 denote the straight line segment and let C_2 denote the quadratic and note the direction of the integration along each part of C as shown in the diagram below.



On C_1 we have $y = x$ and hence

$$\int_{C_1} \underline{F}\cdot \mathrm{d}\underline{r} = \int_{x=1}^{0} (x^2+x^2)\,\mathrm{d}x + x^2\,\mathrm{d}x = -\int_0^1 3x^2\,\mathrm{d}x = -1.$$

On C_2 we have $y = x^2$, $\mathrm{d}y = 2x\mathrm{d}x$ and hence

$$\int_{C_2} \underline{F}\cdot \mathrm{d}\underline{r} = \int_{x=0}^{1} (x^3+x^4)\,\mathrm{d}x + 2x^3\,\mathrm{d}x = \int_0^1 (3x^3+x^4)\,\mathrm{d}x = \frac{3}{4}+\frac{1}{5} = \frac{19}{20}.$$

Combining these two results gives

$$\int_{C} \underline{F}\cdot \mathrm{d}\underline{r} = -\frac{1}{20}.$$

To compute instead the area integral we have

$$\begin{aligned}\int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}s &= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^{1} \left[xy-y^2\right]_{x^2}^{x}\,\mathrm{d}x \\ &= \int_{x=0}^{1} (x^2-x^2)-(x^3-x^4)\,\mathrm{d}x \\ &= \int_{x=0}^{1} (-x^3+x^4)\,\mathrm{d}x \\ &= -\frac{1}{4}+\frac{1}{5} = -\frac{1}{20}.\end{aligned}$$

2. Use Green's theorem in the plane to evaluate

$$\oint_C (x^2 - 2xy)\,\mathrm{d}x + (x^2 y + 3)\,\mathrm{d}y$$

where C is the boundary of the region enclosed by $y = 8x^2$, $x = 2$ and $y = 0$. Check your answer by direct integration.

Answer

In this question

$$F_1 = x^2 - 2xy, \quad F_2 = x^2 y + 3$$

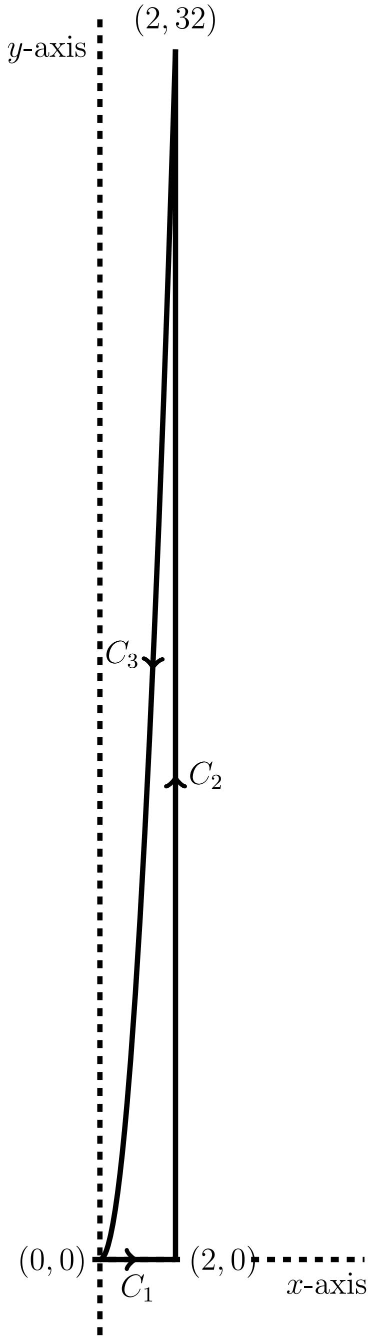
giving

$$\frac{\partial F_1}{\partial y} = -2x, \quad \frac{\partial F_2}{\partial x} = 2xy \quad \text{so that} \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x(y+1).$$

and we need to verify that

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y$$

for the curve C specified. The curve C has 3 parts as shown in the following diagram corresponding to the x-axis from 0 to 2 (the part C_1), the line $x = 2$ from $y = 0$ to $y = 32$ (the part C_2) and the curve $y = 8x^2$ from $x = 2$ to $x = 0$ (the part C_3).


(2, 32)
y-axis
C3
C2
(0, 0)
(2, 0)
C1
x-axis

For the area integral we have

$$\begin{aligned}\int_{x=0}^{2}\int_{y=0}^{8x^2} 2x(1+y)\,\mathrm{d}y\mathrm{d}x &= \int_{x=0}^{2} 2x\left[y+\frac{y^2}{2}\right]_0^{8x^2}\,\mathrm{d}x \\ &= \int_0^2 16x^3+64x^5\,\mathrm{d}x \\ &= 16\left(\frac{2^4}{4}\right)+64\left(\frac{2^6}{6}\right) \\ &= 64\left(1+\frac{32}{3}\right)=64\left(\frac{35}{3}\right).\end{aligned}$$

To calculate the line integral we consider each part separately as follows. On C_1, $y=0$, $\mathrm{d}y=0$ and $F_1=x^2$. Thus

$$\int_{C_1}\underline{F}\cdot\mathrm{d}\underline{r}=\int_0^2 F_1\,\mathrm{d}x=\int_0^2 x^2\,\mathrm{d}x=\frac{8}{3}.$$

On C_2, $x=2$, $\mathrm{d}x=0$ and $F_2=4y+3$. Thus

$$\int_{C_2}\underline{F}\cdot\mathrm{d}\underline{r}=\int_0^{32} F_2\,\mathrm{d}y=\int_0^{32}(4y+3)\,\mathrm{d}y=2(32)^2+3(32).$$

On C_3, $y=8x^2$, $\mathrm{d}y=16x$ and to express F_1 and F_2 in terms of x we have

$$F_1=x^2-2xy=x^2-16x^3,\quad F_2=x^2y+3=8x^4+3.$$

For the direction of the integration it is from $x=2$ to $x=0$ and thus

$$\begin{aligned}\int_{C_2}\underline{F}\cdot\mathrm{d}\underline{r} &= \int_2^0 F_1\,\mathrm{d}x+F_2\,\mathrm{d}y \\ &= -\int_0^2 x^2-16x^3+(8x^3+3)(16x)\,\mathrm{d}x \\ &= \int_0^2(-48x-x^2+16x^3-128x^5)\,\mathrm{d}x \\ &= -48\left(\frac{4}{2}\right)-\left(\frac{8}{3}\right)+16\left(\frac{16}{4}\right)-128\left(\frac{64}{6}\right) \\ &= -32-\left(\frac{8}{3}\right)-128\left(\frac{32}{3}\right).\end{aligned}$$

Combining the contributions from C_1, C_2 and C_3 gives

$$\begin{aligned}&\frac{8}{3}+(2(32)^2+3(32))+\left(-32-\left(\frac{8}{3}\right)-128\left(\frac{32}{3}\right)\right) \\ =\ & 64+64(32)-64\left(\frac{64}{3}\right)=64\left(1+32-\left(\frac{64}{3}\right)\right)=64\left(\frac{35}{3}\right).\end{aligned}$$
