MA2741: Spring Term – Exercise sheet 1

Exercises involving the Divergence theorem

1. A closed region Ω is bounded by a simple surface S. Use the Divergence theorem to prove that

$$\int_S \underline{r} \cdot \mathrm{d}\underline{s} = 3V$$

where $\underline{r}$ is the position vector of a point on the surface and V is the volume of the region Ω.

2. Use the Divergence theorem to evaluate

$$\int_S \underline{F} \cdot \mathrm{d}\underline{s},$$

where

$$\underline{F} = (z^2 - 1)(xy^2\underline{i} + xy\underline{j} + y^2\underline{k})$$

and S is the closed surface of the cube centred at the origin and with sides of length 2 units with each side parallel to one of the planes $x = 0$, $y = 0$ and $z = 0$. Check you answer by doing the surface integrals.

3. Show that

$$\int_S \underline{q} \cdot \mathrm{d}\underline{s} = \frac{\pi}{6}$$

where $\underline{q} = z^2\underline{k}$ and S is the whole of the surface of the cone $x^2 + y^2 = (1 - z)^2$, $0 \le z \le 1$, including the base $x^2 + y^2 = 1$, $z = 0$. Use direct evaluation and the Divergence theorem.

4. A closed region Ω is bounded by a simple surface S. Use the Divergence theorem to prove that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \;\mathrm{d}v = \int_S \phi\frac{\partial\psi}{\partial n}\;\mathrm{d}s - \int_\Omega \phi\nabla^2\psi\;\mathrm{d}v$$

where ϕ and ψ are scalar fields. Hence, prove Green's second identity which is

$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}v = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\,\mathrm{d}s.$$

Exercises involving Stokes' theorem

1. Given that S is the hemisphere of unit radius described by

$$\underline{r}(u,v) = \sin v \cos u \underline{i} + \sin v \sin u \underline{j} + \cos v \underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le \pi/2$$

   and C is the closed curve that bounds the hemisphere in the $xy-$plane, evaluate

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r} \qquad \text{and} \qquad \int_S (\nabla \times \underline{q}) \cdot \mathrm{d}\underline{s}$$

   where

$$\text{i)} \quad \underline{q} = Uy\underline{i}, \ \ U \text{ constant}, \qquad \text{ii)} \quad \underline{q} = y^2\underline{i} + x\underline{j}.$$

   What do you notice about your answers?

2. Verify Stokes' theorem for the vector field $\underline{F} = x^2y\underline{i} + z\underline{j}$ and the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

3. Evaluate

$$\int_S (\nabla \times \underline{q}) \cdot \mathrm{d}\underline{s}$$

   where

$$\underline{q} = (x^2 + y - 4)\underline{i} + 3xy\underline{j} + (2xz + z^2)\underline{k}$$

   and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

Exercises involving Green's theorem in the plane

1. Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)\ \mathrm{d}x + x^2\ \mathrm{d}y$$

   where C is the closed curve bounded by $y = x$ and $y = x^2$, $0 \le x \le 1$.

2. Use Green's theorem in the plane to evaluate

$$\oint_C (x^2 - 2xy)\ \mathrm{d}x + (x^2y + 3)\ \mathrm{d}y$$

   where C is the boundary of the region enclosed by $y = 8x^2$, $x = 2$ and $y = 0$. Check your answer by direct integration.