MA2741: Spring Term – Exercise sheet 3 relating to chapter 3: 2D incompressible flows, stream function, vorticity, $\cdots$ with answers

In past exams formula sheets were available in case questions needed the use of polar coordinates and such sheets contained the following.

In the following, underline symbols denote vector quantities.

In terms of cylindrical polar coordinates (r, θ, z) with unit vectors $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{k}$, the following definitions hold:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{k}, \\
\nabla \cdot \underline{F} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}, \\
\nabla \times \underline{F} &= \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}, \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2},
\end{aligned}$$

where $f(r, \theta, z)$ is any scalar function and

$$\underline{F} = F_r(r, \theta, z)\underline{e}_r + F_\theta(r, \theta, z)\underline{e}_\theta + F_z(r, \theta, z)\underline{k}.$$

The vector product of vectors $\underline{a}$ and $\underline{F}$, where

$$\underline{a}(r, \theta, z) = a_r(r, \theta, z)\underline{e}_r + a_\theta(r, \theta, z)\underline{e}_\theta + a_z(r, \theta, z)\underline{k},$$

is given by

$$\underline{a} \times \underline{F} = \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ a_r & a_\theta & a_z \\ F_r & F_\theta & F_z \end{vmatrix}.$$

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1. The stream function for a line vortex, with circulation Γ, in a uniform flow is

$$\psi = Ur\sin\theta - \frac{\Gamma}{2\pi}\ln r$$

where U is constant and (r, θ) are plane polar coordinates. Obtain the velocity field and show that there is a stagnation point at

$$\left(\frac{\Gamma}{2\pi U}, \frac{\pi}{2}\right).$$

Answer

In polars

$$\underline{q} = \nabla \times (\psi \underline{k}) = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_r - \frac{\partial \psi}{\partial r}\underline{e}_\theta.$$

At a stagnation point

$$\underline{q} = \underline{0}, \quad \text{i.e. } \frac{\partial \psi}{\partial \theta} = 0 \quad \text{and } \frac{\partial \psi}{\partial r} = 0.$$

In this case

$$\frac{\partial \psi}{\partial \theta} = Ur\cos\theta$$

and this is 0 when $\theta = \pm\pi/2$.

$$\frac{\partial \psi}{\partial r} = U\sin\theta - \frac{\Gamma}{2\pi r}.$$

When $\theta = -\pi/2$ we have $\sin\theta = -1$ and

$$\frac{\partial \psi}{\partial r} = -U - \frac{\Gamma}{2\pi r} < 0$$

and we have no solution.

When $\theta = \pi/2$ we have $\sin\theta = 1$ and

$$\frac{\partial \psi}{\partial r} = U - \frac{\Gamma}{2\pi r}$$

which is 0 when

$$r = \frac{\Gamma}{2\pi U}.$$

2. The stream function

$$\psi = Ur\left(\theta\cos\theta + \frac{2}{\pi}\theta\sin\theta - \sin\theta\right), \quad 0 \le \theta \le \pi/2$$

where U is constant and (r,θ) are plane polar coordinates, can be used to model 2-D, incompressible flow in the corner region $x > 0$, $y > 0$.

(a) Determine the velocity field and verify that the normal component vanishes at the rigid walls $\theta = 0$ and $\theta = \pi/2$.

Answer

$$\underline{q} = \nabla\times(\psi\underline{k}) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta.$$

In this case

$$\begin{aligned}\frac{\partial\psi}{\partial r} &= U\left(\theta\cos\theta + \frac{2}{\pi}\theta\sin\theta - \sin\theta\right),\\ \frac{1}{r}\frac{\partial\psi}{\partial\theta} &= U\left(-\theta\sin\theta + \cos\theta + \frac{2}{\pi}(\theta\cos\theta + \sin\theta) - \cos\theta\right)\\ &= U\left(-\theta\sin\theta + \frac{2}{\pi}(\theta\cos\theta + \sin\theta)\right).\end{aligned}$$

Each term has a factor of θ or $\sin\theta$ and hence $\underline{q} = \underline{0}$ when $\theta = 0$.
When $\theta = \pi/2$ we have $\cos\theta = 0$ and $\sin\theta = 1$ and

$$\underline{q} = U\left(-\frac{\pi}{2} + \frac{2}{\pi}\right)\underline{e}_r(\pi/2) = U\left(-\frac{\pi}{2} + \frac{2}{\pi}\right)\underline{j}.$$

When $\theta = \pi/2$ the normal to the surface has direction $\underline{n} = \underline{i}$ and thus $\underline{n}\cdot\underline{q} = 0$.

(b) Calculate the vorticity, $\underline{\omega}$, of the flow where $\underline{\omega} = \nabla\times\underline{q}$.

Answer

With $\underline{q} = q_r\underline{e}_r + q_\theta\underline{e}_\theta$ we have

$$\underline{\omega} = \nabla\times\underline{q} = \frac{1}{r}\begin{vmatrix}\underline{e}_r & r\underline{e}_\theta & \underline{k}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ q_r & rq_\theta & q_z\end{vmatrix} = \underline{k}\,\frac{1}{r}\left(\frac{\partial}{\partial r}(rq_\theta) - \frac{\partial}{\partial\theta}q_r\right).$$

Now

$$q_\theta = -\frac{\partial\psi}{\partial r} \quad \text{and} \quad \frac{\partial q_\theta}{\partial r} = 0.$$

Thus

$$\underline{\omega} = -\underline{k}\left(\frac{U}{r}\right)\left(-\sin\theta - \theta\cos\theta + \frac{2}{\pi}(2\cos\theta - \theta\sin\theta)\right).$$

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3. Consider the two-dimensional velocity field

$$\underline{q} = \left(\frac{\alpha x}{x^2+y^2}\right)\underline{i} + \left(\frac{\alpha y}{x^2+y^2}\right)\underline{j},$$

where α is constant. Show that this represents an incompressible flow and find a stream function and velocity potential for the flow.

[Hint: Determine the gradient of $\tan^{-1}(y/x)$.]

Answer

It helps to use polar coordinates to answer this question and note that $\theta = \tan^{-1}(y/x)$.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

and

$$\underline{q} = \frac{\alpha}{r}(\cos\theta\underline{i} + \sin\theta\underline{j}) = \frac{\alpha}{r}\underline{e}_r.$$

The divergence in polars is

$$\nabla\cdot\underline{q} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\alpha}{r}\right) = 0.$$

The flow is hence incompressible.

The flow is also irrotational as

$$\underline{\omega} = \nabla\times\underline{q} = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{\alpha}{r} & 0 & 0 \end{vmatrix} = \underline{0}.$$

Hence there is a stream function ψ and a velocity potential ϕ and

$$\underline{q} = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta.$$

To get the stream function we note first that

$$\frac{\partial\psi}{\partial r} = 0 \quad \text{giving } \psi = g(\theta)$$

for some function $g(\theta)$. If we now consider the $\underline{e}_r$ part then we get that

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}g'(\theta) = \frac{\alpha}{r}$$

which gives

$$g'(\theta) = \alpha, \quad g(\theta) = \alpha\theta + \text{const}.$$

The stream function can be taken as

$$\psi = \alpha\theta.$$

With similar workings for the velocity potential we have

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{giving } \psi = f(r)$$

for some function $f(r)$. If we now consider the $\underline{e}_r$ part then we get that

$$\frac{\partial \phi}{\partial r} = f'(r) = \frac{\alpha}{r} \quad \text{giving } f(r) = \alpha \ln r + \text{const}.$$

The velocity potential can be taken as

$$\phi = \alpha \ln r.$$

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4. A fluid flows through a section of pipe whose surface is described by the Cartesian position vector

$$\underline{r} = (1+v)\cos u\underline{i} + (1+v)\sin u\underline{j} + v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

The fluid velocity is given by

$$\underline{q} = \underline{q}(x,y) = y\underline{i} - x\underline{j}.$$

(a) Show that the flow is incompressible and determine the vorticity.

Answer

The flow is incompressible because

$$\nabla \cdot \underline{q} = \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} = 0 - 0 = 0.$$

The vorticity is

$$\underline{\omega} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \underline{i}(0) - \underline{j}(0) + \underline{k}(-1-1) = -2\underline{k}.$$

(b) Determine the normal, $\underline{n}$, to the pipe surface. Show that $\underline{q}\cdot\underline{n} = 0$ on the pipe surface and state the physical significance of this.

Answer

$$\frac{\partial \underline{r}}{\partial u} \quad \text{and} \quad \frac{\partial \underline{r}}{\partial v}$$

give vectors which are tangent to the surface and

$$\underline{n} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}$$

is a vector which is normal to the surface.

$$\begin{aligned} \underline{n} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -(1+v)\sin u & (1+v)\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} \\ &= \underline{i}(1+v)\cos u + \underline{j}(1+v)\sin u - \underline{k}(1+v). \end{aligned}$$

On the pipe surface $x = (1+v)\cos u$ and $y = (1+v)\sin u$ and

$$\underline{q}\cdot\underline{n} = (1+v)(y\cos u - x\sin u) = (1+v)^2(\cos u\sin u - \cos u\sin u) = 0.$$

The fluid does not flow through the surface of the pipe and hence this is a rigid boundary of the flow.

(c) Calculate the circulation round the circumference of the pipe in the plane $z = 0$ (i.e. $v = 0$).

Answer

When $v = 0$ we have the closed curve

$$\begin{aligned} \underline{r}(u, 0) &= \cos\, u\, \underline{i} + \sin\, u\, \underline{j}, \quad 0 \le \theta < 2\pi, \\ \frac{\partial \underline{r}}{\partial u} &= -\sin\, u\, \underline{i} + \cos\, u\, \underline{j}. \end{aligned}$$

Let C denote the closed curve. The circulation is defined by

$$\begin{aligned} \oint_C \underline{q} \cdot \mathrm{d}\underline{r} &= \int_0^{2\pi} \left(\sin\, u\underline{i} - \cos\, u\underline{j}\right) \cdot \frac{\partial \underline{r}}{\partial u}\, \mathrm{d}u \\ &= \int_0^{2\pi} \left(\sin\, u\underline{i} - \cos\, u\underline{j}\right) \cdot \left(-\sin\, u\underline{i} + \cos\, u\underline{j}\right)\, \mathrm{d}u \\ &= \int_0^{2\pi} \left(-\sin^2 u - \cos^2 u\right)\, \mathrm{d}u = -2\pi. \end{aligned}$$

(d) Calculate the surface area of this section of pipe.

Answer

The surface area is

$$\int_{v=0}^{1} \int_{u=0}^{2\pi} \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| \mathrm{d}u\mathrm{d}v = \int_{v=0}^{1} \int_{u=0}^{2\pi} |\underline{n}|\, \mathrm{d}u\mathrm{d}v.$$

Now

$$|\underline{n}|^2 = (1 + v)^2(1 + 1) = 2(1 + v)^2 \quad \text{and} \quad |\underline{n}| = \sqrt{2}(1 + v).$$

$$\int_0^1 (1 + v)\, \mathrm{d}v = 1 + \frac{1}{2} = \frac{3}{2}.$$

Hence the surface area is

$$3\sqrt{2}\pi.$$

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5. The stream function for a two-dimensional flow is given by:

$$\psi = \psi(x, y) = Uy^2 + 2Uxy.$$

Determine:

(a) the velocity field;

Answer
The velocity is

$$\underline{q} = (\nabla\psi) \times \underline{k} = \frac{\partial\psi}{\partial y}\underline{i} - \frac{\partial\psi}{\partial x}\underline{j} = 2U(y + x)\underline{i} - 2Uy\underline{j}.$$

(b) the vorticity, $\underline{\omega}$;

Answer
The vorticity is

$$\underline{\omega} = \nabla \times \underline{q} = 2U \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & -y & 0 \end{vmatrix} = -2U\underline{k}.$$

Recall that in the lectures we also showed that

$$\underline{\omega} = -\nabla^2\psi\,\underline{k}$$

and the result could have also been obtained in this way.

(c) the circulation around Γ where Γ is the unit circle centred at the origin.

Answer
The unit circle is described by

$$\underline{r}(\theta) = \cos\,\theta\,\underline{i} + \sin\,\theta\,\underline{j}, \quad 0 \le \theta \le 2\pi,$$

giving

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = -\sin\,\theta\,\underline{i} + \cos\,\theta\,\underline{j}.$$

Thus

$$\begin{aligned} \underline{q} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} &= 2U((\cos\,\theta + \sin\,\theta)(-\sin\,\theta) - \sin\,\theta\,\cos\,\theta) \\ &= -2U(2\sin\,\theta\,\cos\,\theta + \sin^2\theta) = -2U(\sin(2\theta) + \sin^2\theta). \end{aligned}$$

We have

$$\int_0^{2\pi} \sin(2\theta)\,\mathrm{d}\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \pi$$

and thus the circulation is

$$-2U\pi.$$

6. A two-dimensional, incompressible flow comprises a source of strength $m \geq 0$ at the origin together with a uniform flow with speed $U > 0$ in the direction $\theta = \pi$ of a plane polar coordinate system (r, θ). The stream function is given by

$$\psi = -Ur\sin\theta + \frac{m\theta}{2\pi}.$$

Determine the fluid velocity and show that there is a stagnation point at

$$r = \frac{m}{2\pi U}, \quad \theta = 0.$$

Answer

The velocity is

$$\begin{aligned} \underline{q} &= (\nabla\psi) \times \underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta \\ &= \left(-U\cos\theta + \frac{m}{2\pi r}\right)\underline{e}_r + U\sin\theta\,\underline{e}_\theta. \end{aligned}$$

At a stagnation point $\underline{q} = \underline{0}$.

The $\underline{e}_\theta$ component is 0 when $\theta = 0$ and when $\theta = \pi$.

When $\theta = 0$ we have $\cos\theta = 1$ and the $\underline{e}_r$ component is also 0 when

$$-U + \frac{m}{2\pi r} = 0 \quad \text{giving } r = \frac{m}{2\pi U}.$$

When $\theta = \pi$ we have $\cos\theta = -1$ and the $\underline{e}_r$ component is

$$U + \frac{m}{2\pi r} > 0.$$

There is no stagnation point when $\theta = \pi$.

7. The stream function for an incompressible uniform flow past a particular multipole is

$$\psi = \psi(r,\theta) = Ur\sin\theta - \mu\frac{\sin 2\theta}{r^2}, \quad 0 \le \theta \le \pi$$

where U and μ are constants and (r,θ) are plane polar coordinates.

(a) Obtain the polar equation of the curve corresponding to $\psi = 0$ for $0 \le \theta \le \pi$, and determine the normal vector, $\underline{n}$, to this curve.

Answer

As $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\sin\theta > 0$ when $0 < \theta < \pi$ the curve $\psi = 0$ is

$$Ur - 2\left(\frac{\mu\cos\theta}{r^2}\right) = 0.$$

The polar form of this curve hence corresponds to

$$r^3 = \frac{2\mu\cos\theta}{U}.$$

If we describe the curve in the form

$$g(r,\theta) = Ur^3 - 2\mu\cos\theta = 0$$

then a normal vector is given by

$$\underline{n} = \nabla g = \frac{\partial g}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial g}{\partial\theta} = 3r^2U\underline{e}_r + \left(\frac{2\mu\sin\theta}{r}\right)\underline{e}_\theta.$$

As we only need the direction we have

$$\left(\frac{r}{2}\right)\underline{n} = \left(\frac{3r^3U}{2}\right)\underline{e}_r + \mu\sin\theta\underline{e}_\theta = \mu(3\cos\theta\underline{e}_r + \sin\theta\underline{e}_\theta).$$

Hence we define

$$\tilde{\underline{n}} = 3\cos\theta\,\underline{e}_r + \sin\theta\,\underline{e}_\theta.$$

(b) Show that $\underline{q}\cdot\underline{n} = 0$ on the curve found in (a) above, where $\underline{q}$ is the fluid velocity.

Answer

The velocity is

$$\begin{aligned}\underline{q} &= (\nabla\psi)\times\underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r + \frac{\partial\psi}{\partial r}\underline{e}_\theta \\ &= \left(U\cos\theta - 2\mu\frac{\cos(2\theta)}{r^3}\right)\underline{e}_r + \left(U\sin\theta + 2\mu\frac{\sin(2\theta)}{r^3}\right)\underline{e}_\theta.\end{aligned}$$

On the curve $\psi = 0$ we have

$$\frac{2\mu}{r^3} = \frac{U}{\cos\theta}$$

giving

$$
\begin{aligned}
\underline{q} &= \left(U\cos\theta - U\frac{\cos(2\theta)}{\cos\theta}\right)\underline{e}_r - \left(U\sin\theta + U\frac{\sin(2\theta)}{\cos\theta}\right)\underline{e}_\theta \\
&= U\left(\cos\theta - \frac{\cos(2\theta)}{\cos\theta}\right)\underline{e}_r - 3U\sin\theta\,\underline{e}_\theta.
\end{aligned}
$$

Then

$$
\begin{aligned}
\underline{q}\cdot\tilde{n} &= 3U(\cos^2\theta - \cos(2\theta)) - 3U\sin^2\theta \\
&= 3U(\cos^2\theta - \sin^2\theta - \cos(2\theta)) = 0.
\end{aligned}
$$

(c) Determine the stagnation point and the vorticity of the flow.

Answer

At a stagnation point $\underline{q} = \underline{0}$. The $\underline{e}_\theta$ component is 0 when $\sin\theta = 0$ or

$$U + 4\mu\frac{\cos\theta}{r^3} = 0.$$

When $\theta = 0$ the $\underline{e}_r$ component is

$$U - \frac{2\mu}{r^3}$$

and this is 0 when

$$r^3 = \frac{2\mu}{U}.$$

We have found one stagnation point.
When $\theta = \pi$ the $\underline{e}_r$ component is

$$-U - \frac{2\mu}{r^3} < 0$$

and there is no value of r for which this is 0.
In this case there are actually other stagnation point resulting from the the condition

$$U + 4\mu\frac{\cos\theta}{r^3} = 0 \quad \text{i.e. when } \frac{\mu}{r^3} = -\frac{U}{4\cos\theta}$$

which when satisfied makes the coefficient of $\underline{e}_\theta$ equal to 0. If we substitute this in the expression for the coefficient of $\underline{e}_r$ then we get

$$
\begin{aligned}
U\cos\theta - 2\mu\frac{\cos(2\theta)}{r^3} &= U\cos\theta + \frac{U\cos(2\theta)}{2\cos\theta} \\
&= \left(\frac{U}{2\cos\theta}\right)\left(2\cos^2\theta + \cos(2\theta)\right) \\
&= \frac{U}{2\cos\theta}(4\cos^2\theta - 1).
\end{aligned}
$$

This is zero when $\cos^2\theta = 1/4$ and as r must be positive the only case to consider is when $\cos\,\theta = -1/2$ and

$$r^3 = \frac{2\mu}{U}.$$

When $\cos\,\theta = -1/2$ we have $\cos(2\theta) = -1$. The values of θ at this stagnation point are $\pm 2\pi/3$.

The vorticity is $\underline{\omega} = \nabla \times \underline{q}$.
Now the velocity has two parts which are

$$\underline{q} = \underline{q}_1 + \underline{q}_2$$

where $\underline{q}_1$ is associated with the uniform flow and $\underline{q}_2$ is associated with the other part.

$$\nabla \times \underline{q}_1 = \frac{U}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ \cos\,\theta & -r\sin\,\theta & 0 \end{vmatrix} = \underline{k}(-\sin\,\theta - (-\sin\,\theta)) = \underline{0}.$$

$$\begin{aligned} \nabla \times \underline{q}_2 &= \left(\frac{-2\mu}{r}\right)\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ \dfrac{\cos(2\theta)}{r^3} & -\dfrac{\sin(2\theta)}{r^2} & 0 \end{vmatrix} \\ &= = \underline{k}\left(\frac{-2\mu}{r}\right)\left(\frac{1}{r^3}(2\sin(2\theta) - 2\sin(2\theta))\right) = \underline{0}. \end{aligned}$$

Thus the vorticity $\underline{\omega} = \underline{0}$.

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8. The stream function

$$\psi = Ur(\sin\theta - \theta\cos\theta), \quad 0 \le \theta \le \pi$$

where U is constant and (r, θ) are plane polar coordinates, describes flow over a horizontal rigid surface which corresponds to $\theta = 0$ and $\theta = \pi$ (i.e. $y = 0$ in Cartesian form).

(a) Determine the velocity field.

Answer

$$\begin{aligned} \underline{q} &= (\nabla\psi) \times \underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta \\ &= U\left((\cos\,\theta - \cos\,\theta + \theta\sin\,\theta)\,\underline{e}_r - (\sin\,\theta - \theta\,\cos\theta)\,\underline{e}_\theta\right) \\ &= U\left((\theta\,\sin\,\theta)\,\underline{e}_r - (\sin\,\theta - \theta\,\cos\theta)\,\underline{e}_\theta\right). \end{aligned}$$

(b) Determine the vorticity, $\underline{\omega}$, of the velocity field.

Answer

Let q_r and q_θ denote the components of $\underline{q}$ for this two-dimensional flow. The vorticity is

$$\underline{\omega} = \nabla \times \underline{q} = \frac{1}{r}\left(\frac{\partial (r q_\theta)}{\partial r} - \frac{\partial q_r}{\partial \theta}\right)\underline{k}.$$

$$\begin{aligned} \frac{\partial q_r}{\partial \theta} &= U(\sin\theta + \theta\cos\theta), \\ r q_\theta &= -Ur(\sin\theta - \theta\cos\theta), \\ \frac{\partial (r q_\theta)}{\partial r} &= -U(\sin\theta - \theta\cos\theta). \end{aligned}$$

Thus

$$\underline{\omega} = -\left(\frac{2U\sin\theta}{r}\right)\underline{k}.$$

(c) Show that the circulation around the closed curve mapped by the polar position vector $\underline{r} = \sin(\theta)\underline{e}_r$, $0 \le \theta \le \pi$ is given by

$$\Gamma = U\int_0^\pi (\theta\sin(2\theta) - \sin^2\theta)\, d\theta.$$

Evaluate this integral.

Answer

Let C denote the closed curve. The circulation is defined by

$$\Gamma = \oint_C \underline{q}\cdot \mathrm{d}\underline{r} = \int_0^\pi \underline{q}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta}\,\mathrm{d}\theta.$$

As

$$\frac{\mathrm{d}\underline{e}_r}{\mathrm{d}\theta} = \underline{e}_\theta, \quad \text{we get} \quad \frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = \cos\theta\,\underline{e}_r + \sin\theta\,\underline{e}_\theta.$$

Hence

$$\begin{aligned} \left(\frac{1}{U}\right)\underline{q}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} &= \cos\theta(\theta\sin\theta) - \sin\theta(\sin\theta - \theta\cos\theta) \\ &= (2\theta\sin\theta\cos\theta - \sin^2\theta) = (\theta\sin(2\theta) - \sin^2\theta). \end{aligned}$$

Now

$$\int_0^\pi \sin^2\theta\,\mathrm{d}\theta = \frac{\pi}{2}.$$

For the other integral use integration by parts to give

$$\int_0^\pi \theta\sin(2\theta)\,\mathrm{d}\theta = \left[\frac{-\theta\cos(2\theta)}{2}\right]_0^\pi + \int_0^\pi \frac{\cos(2\theta)}{2}\,\mathrm{d}\theta = -\frac{\pi}{2}.$$

Thus

$$\Gamma = -U\pi.$$

9. *Apart from notational changes, this was the most of question 3 of the* MA2841 *paper in 2012. The other part of the question was on the previous exercise sheet.*

A factory worker is filling barrels of beer using a funnel, the surface of which is described by the Cartesian position vector

$$\underline{r} = \mathrm{e}^{-v}\cos u\underline{i} + \mathrm{e}^{-v}\sin u\underline{j} - v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

Suppose that the fluid velocity within the funnel is given by

$$\underline{q} = (x+2y)\underline{i} - y\underline{j} - 3\underline{k}.$$

Determine the following.

(ii) The vorticity $\underline{\omega}$ of the flow.

[3 marks]

Answer

The vorticity is

$$\begin{aligned}\underline{\omega} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y & -y & -3 \end{vmatrix} \\ &= \underline{i}\left(\frac{\partial(-3)}{\partial y} - \frac{\partial(-y)}{\partial z}\right) - \underline{j}\left(\frac{\partial(-3)}{\partial x} - \frac{\partial(x+2y)}{\partial z}\right) \\ &\quad + \underline{k}\left(\frac{\partial(-y)}{\partial x} - \frac{\partial(x+2y)}{\partial y}\right) = -2\underline{k}.\end{aligned}$$

(iii) The outward normal, $\underline{n}$, to the funnel surface.

[4 marks]

Answer

The vectors

$$\frac{\partial \underline{r}}{\partial u} \quad \text{and} \quad \frac{\partial \underline{r}}{\partial v}$$

are tangent to the surface of the funnel and the vector

$$\underline{n} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}$$

is in the direction normal to the surface.

$$\begin{aligned}\frac{\partial \underline{r}}{\partial u} &= \mathrm{e}^{-v}(-\sin u\,\underline{i} + \cos u\,\underline{j}), \\ \frac{\partial \underline{r}}{\partial v} &= -\mathrm{e}^{-v}(\cos u\,\underline{i} + \sin u\,\underline{j}) - \underline{k}.\end{aligned}$$

By inspection this can be expressed using the polar base vectors as

$$\begin{aligned}\frac{\partial \underline{r}}{\partial u} &= \mathrm{e}^{-v}\,\underline{e}_\theta,\\ \frac{\partial \underline{r}}{\partial v} &= -\mathrm{e}^{-v}\,\underline{e}_r - \underline{k}.\end{aligned}$$

and thus

$$\underline{n} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \mathrm{e}^{-2v}\underline{k} - \mathrm{e}^{-v}\,\underline{e}_r$$

As the radial component here is negative this vector has a component pointing towards the axis of the funnel and hence this vector corresponds to the inward normal.

To get the outward normal we re-define $\underline{n}$ to be

$$\underline{n} = \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial u} = -\mathrm{e}^{-2v}\underline{k} + \mathrm{e}^{-v}\,\underline{e}_r = -\mathrm{e}^{-2v}\underline{k} + \mathrm{e}^{-v}\,(\cos\,\theta\,\underline{i} + \sin\,\theta\,\underline{j}).$$

Show that $\underline{q}\cdot\underline{n} \neq 0$ on the funnel surface and state the physical significance of this.

[4 marks]

Hence, evaluate the flux integral for the fluid velocity across the funnel surface.

[6 marks]

Answer

On the surface of the funnel

$$x = \mathrm{e}^{-v}\cos\,u, \quad y = -\mathrm{e}^{-v}\sin\,u, \quad z = -v$$

and $\underline{q}$ is given by

$$\underline{q} = \mathrm{e}^{-v}(\cos\,u + 2\sin\,u)\underline{i} - \mathrm{e}^{-v}\,\sin\,u\,\underline{j} - 3\,\underline{k}.$$

$$\begin{aligned}\underline{q}\cdot\underline{n} &= \mathrm{e}^{-2v}\left((\cos^2 u + 2\sin\,u\,\cos\,u) - \sin^2 u + 3\right)\\ &= \mathrm{e}^{-2v}(\cos(2u) + \sin(2u) + 3).\end{aligned}$$

As

$$\cos(2u) + \sin(2u) + 3 \geq 1$$

this is not zero on the surface of the funnel and as a consequence fluid is flowing through the surface of the funnel.

The flux integral over the part of the surface specified is

$$\begin{aligned}\int_{v=0}^{1}\int_{u=0}^{2\pi} \underline{q}\cdot\left(\frac{\partial \underline{r}}{\partial v}\times\frac{\partial \underline{r}}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v &= \int_0^1\int_0^{2\pi} \mathrm{e}^{-2v}(\cos(2u)+\sin(2u)+3)\,\mathrm{d}u\,\mathrm{d}v\\ &= \left(\int_0^1 \mathrm{e}^{-2v}\,\mathrm{d}v\right)\int_0^{2\pi}(\cos(2u)+\sin(2u)+3)\,\mathrm{d}u\\ &= \left(\frac{1-\mathrm{e}^{-2}}{2}\right)(6\pi) = 3\pi(1-\mathrm{e}^{-2}).\end{aligned}$$

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10. *Apart from notational changes, this was question 4a of the* MA2841 *paper in 2011 and was worth 12 marks (the complete question was worth 20 marks).*

A stream function is given, in plane polar coordinates (r, θ), by

$$\psi = r\theta \sin\,\theta, \quad 0 \le \theta \le 2\pi.$$

(i) Determine the velocity field, $\underline{q}$, for the flow.

[3 marks]

Answer

The velocity $\underline{q}$ is given by

$$\begin{aligned} \underline{q} &= (\nabla\psi) \times \underline{k} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_r - \frac{\partial \psi}{\partial r}\underline{e}_\theta \\ &= (\theta\,\cos\,\theta + \sin\,\theta)\underline{e}_r - \theta\,\sin\,\theta\underline{e}_r. \end{aligned}$$

(ii) Calculate the vorticity, $\underline{\omega}$, of the velocity field.

[3 marks]

Answer

Let q_r and q_θ denote the components of the velocity. The vorticity is

$$\begin{aligned} \underline{\omega} = \nabla \times \underline{q} &= \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ q_r & rq_\theta & q_z \end{vmatrix} \\ &= \underline{k}\frac{1}{r}\left(\frac{\partial}{\partial r}(rq_\theta) - \frac{\partial}{\partial \theta}q_r\right) \\ &= \underline{k}\frac{1}{r}\left(-\theta\,\sin\,\theta - (-\theta\,\sin\,\theta + \cos\,\theta + \cos\,\theta)\right) \\ &= -\left(\frac{2\cos\,\theta}{r}\right)\underline{k}. \end{aligned}$$

(iii) Show that the circulation, Γ, around the closed curve $r = \cos\theta$, $-\pi/2 \le \theta \le \pi/2$, is given by

$$\Gamma = -\int_{-\pi/2}^{\pi/2} (\sin^2\theta + \theta\,\sin 2\theta)\,\mathrm{d}\theta.$$

Evaluate this integral.

[6 marks]

Answer

The parametric description of the curve is

$$\underline{r} = \cos\theta\,\underline{e}_r(\theta), \qquad \frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = -\sin\theta\,\underline{e}_r + \cos\theta\,\underline{e}_\theta.$$

Let C denote the closed curve. The circulation is

$$\oint_C \underline{q}\cdot\,\mathrm{d}\underline{r} = \int_{-\pi/2}^{\pi/2} \underline{q}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta}\,\mathrm{d}\theta.$$

In this case

$$\begin{aligned}\underline{q}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} &= -\sin\theta(\theta\,\cos\theta + \sin\theta) - \cos\theta(\theta\,\sin\theta)\\ &= -(\theta\,\sin(2\theta) + \sin^2\theta).\end{aligned}$$

For the integrals we have

$$\int_{-\pi/2}^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = \frac{\pi}{2}.$$

We use integration by parts for the other integral.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \theta\,\sin(2\theta)\,\mathrm{d}\theta &= 2\int_0^{\pi/2} \theta\,\sin(2\theta)\,\mathrm{d}\theta\\ &= [-\theta\,\cos(2\theta)]_0^{\pi/2} + \int_0^{\pi/2} \cos(2\theta)\,\mathrm{d}\theta = \frac{\pi}{2}.\end{aligned}$$

Thus

$$\Gamma = \oint_C \underline{q}\cdot\,\mathrm{d}\underline{r} = -\pi.$$

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11. *Apart from notational changes, this was part of question 3 of the* MA2841 *paper in 2011. The other parts of the question were on the previous exercise sheet.*

The surface of a bowl is mapped by the position vector

$$\underline{r} = v\cos u\,\underline{i} + v\sin u\,\underline{j} + v^2\,\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

A fluid within the bowl moves such that its velocity is given by

$$\underline{q} = \frac{y^2}{x}\,\underline{i} - y\underline{j}, \quad x > 0.$$

(ii) Determine a normal vector $\underline{n}$ to the bowl surface.

[4 marks]

Answer

Vectors tangential to the surface of the bowl are given by

$$\begin{aligned} \frac{\partial \underline{r}}{\partial u} &= -v\,\sin\,u\,\underline{i} + v\,\cos\,u\underline{j}, \\ \frac{\partial \underline{r}}{\partial v} &= \cos\,u\,\underline{i} + \sin\,u\,\underline{j} + 2v\,\underline{k}. \end{aligned}$$

By inspection this can be expressed using the polar base vectors with $u = \theta$ being the usual angle.

$$\begin{aligned} \frac{\partial \underline{r}}{\partial u} &= v\,\underline{e}_\theta, \\ \frac{\partial \underline{r}}{\partial v} &= \underline{e}_r + 2v\,\underline{k}. \end{aligned}$$

The cross-product gives a vector normal to the surface and this is

$$\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = -v\underline{k} + 2v\,\underline{e}_r.$$

As we only need the direction we can take

$$\underline{n} = -\underline{k} + 2\underline{e}_r.$$

(iii) Show that $\underline{q} \cdot \underline{n} = 0$ on the bowl surface and state the physical significance of this.

[3 marks]

Answer

On the surface of the bowl

$$x = v \cos u, \quad y = v \sin u.$$

$$\begin{aligned} \underline{q} &= v \left(\frac{\sin^2 u}{\cos u} \right) \underline{i} - v \sin u \, \underline{j} \\ &= -v \left(\frac{\sin u}{\cos u} \right) (-\sin u \, \underline{i} + \cos u \, \underline{j}) \\ &= -v \left(\frac{\sin u}{\cos u} \right) \underline{e_\theta}. \end{aligned}$$

It thus follows that $\underline{q} \cdot \underline{n} = 0$.

The physical significance of this is that fluid does not flow across the surface of the bowl.

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12. *Apart from notational changes, this was question 4 of the* MA2941 *paper in 2010.*

Consider the stream function

$$\psi = r\theta \cos \theta - \frac{2\pi \cos \theta}{r}, \quad 0 \le \theta \le 2\pi,$$

where (r, θ) are plane polar coordinates.

(i) Show that one solution to $\psi = 0$ is $r^2\theta = 2\pi$ and sketch this curve for $0 < \theta < 2\pi$.

[4 marks]

Answer

The curve $\psi = 0$ is

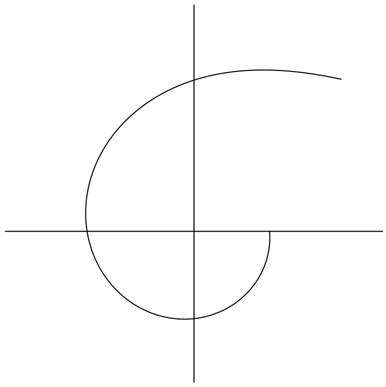
$$r\theta \cos \theta - \frac{2\pi \cos \theta}{r} = \cos \theta \left(r\theta - \frac{2\pi}{r} \right) = 0.$$

One solution is when the term in brackets is 0 which can be written as

$$r^2\theta = 2\pi.$$

$$r(\pi/2) = 2, \quad r(\pi) = \sqrt{2}, \quad r(2\pi) = 1.$$

As $\theta \to 0$, $r \to \infty$. The curve looks like the following.



(ii) Determine the velocity field $\underline{q}$ for the flow.

[3 marks]

Answer
The velocity $\underline{q}$ is given by

$$\underline{q} = (\nabla\psi) \times \underline{k} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_r - \frac{\partial\psi}{\partial r}\underline{e}_\theta.$$

$$\begin{aligned}\frac{\partial\psi}{\partial\theta} &= r(-\theta\,\sin\,\theta + \cos\,\theta) + \frac{2\pi\sin\,\theta}{r},\\ \frac{1}{r}\frac{\partial\psi}{\partial\theta} &= (-\theta\,\sin\,\theta + \cos\,\theta) + \frac{2\pi\sin\,\theta}{r^2},\\ \frac{\partial\psi}{\partial r} &= \theta\,\cos\,\theta + \frac{2\pi\cos\,\theta}{r^2}.\end{aligned}$$

Thus

$$\underline{q} = \left(-\theta\,\sin\,\theta + \cos\,\theta + \frac{2\pi\sin\,\theta}{r^2}\right)\underline{e}_r - \left(\theta\,\cos\,\theta + \frac{2\pi\cos\,\theta}{r^2}\right)\underline{e}_\theta.$$

(iii) Let $\varphi = r^2\theta - 2\pi$. Determine $\nabla\varphi$ and show that $\underline{q}\cdot\nabla\varphi = 0$ when $r^2\theta = 2\pi$. Explain this result.

[7 marks]

Answer

$$\begin{aligned}\nabla\varphi &= \frac{\partial\varphi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\underline{e}_\theta\\ &= 2r\theta\,\underline{e}_r + \frac{1}{r}(r^2)\underline{e}_\theta\\ &= 2r\theta\,\underline{e}_r + r\underline{e}_\theta = r(2\theta\,\underline{e}_r + \underline{e}_\theta).\end{aligned}$$

When $r^2\theta = 2\pi$ the velocity is given by

$$\begin{aligned} \underline{q} &= (-\theta \sin\theta + \cos\theta + \theta \sin\theta)\,\underline{e}_r - (\theta \cos\theta + \theta\cos\theta)\,\underline{e}_\theta \\ &= \cos\theta\,\underline{e}_r - 2\theta\cos\theta\,\underline{e}_\theta \\ &= \cos\theta\,(\underline{e}_r - 2\theta\,\underline{e}_\theta)\,. \end{aligned}$$

With $\nabla\varphi$ and $\underline{q}$ as just given it follows immediately that $\underline{q}\cdot\nabla\varphi = 0$.

This property indicates that the fluid does not cross the curve $r^2\theta = 2\pi$.

(iv) Show that the circulation around the closed curve $r = 1$, $0 \le \theta \le 2\pi$ is given by

$$-\int_0^{2\pi} (\theta\cos\theta + 2\pi\cos\theta)\,\mathrm{d}\theta$$

and evaluate this integral.

[6 marks]

Answer

The circle $r = 1$ is

$$\underline{r} = \underline{e}_r(\theta), \quad \text{with } \frac{\mathrm{d}\underline{e}_r}{\mathrm{d}\theta} = \underline{e}_\theta.$$

The circulation is

$$\begin{aligned} \oint_C \underline{q}\cdot \mathrm{d}\underline{r} &= \int_0^{2\pi} \underline{q}\cdot\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_0^{2\pi} -\frac{\partial\psi}{\partial r}(1,\theta)\,\mathrm{d}\theta \\ &= -\int_0^{2\pi} (\theta\cos\theta + 2\pi\cos\theta)\,\mathrm{d}\theta. \end{aligned}$$

We have

$$\int_0^{2\pi} \cos\theta\,\mathrm{d}\theta = 0.$$

By the periodic property

$$\int_0^{2\pi} \theta\cos\theta\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \theta\cos\theta\,\mathrm{d}\theta = 0$$

as the integrand is an odd function. Thus the circulation is 0.

13. *Apart from notational changes, this was part of question 3 of the* `MA2941` *paper in 2010. The other parts of the question appeared on the previous exercise sheet.*

A fluid flows such that its velocity is given by

$$\underline{q} = 3\underline{i} + 2tx\,\underline{j} - z\,\underline{k}, \quad t \geq 0$$

where t is time.

(ii) Determine the vorticity of the flow.

[3 marks]

Answer

The vorticity is

$$\underline{\omega} = \nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3 & 2tx & -z \end{vmatrix} = \underline{i}(0) - \underline{j}(0) + \underline{k}(2t) = 2t\,\underline{k}.$$

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14. *Apart from notational changes, this was question 3 of the* `MA2941` *paper in 2009.*

A fluid flows inside a semi-infinite pipe whose surface is described by the Cartesian position vector

$$\underline{r} = \cos\,u\,\underline{i} + 2\sin\,u\,\underline{j} + v\,\underline{k}, \quad 0 \leq v \leq 2\pi, \quad -\infty < v < 0.$$

The fluid velocity is given by

$$\underline{q} = y\,\underline{i} - 4x\,\underline{j} + z\,\underline{k}.$$

(ii) Determine the vorticity $\underline{\omega}$.

[3 marks]

Answer

The vorticity is

$$\underline{\omega} = \nabla \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -4x & z \end{vmatrix} = \underline{i}(0) - \underline{j}(0) + \underline{k}(-4-1) = -5\underline{k}.$$

(iii) Determine the normal, $\underline{n}$, to the pipe surface.

[3 marks]

Answer

The following vectors are tangent to the surface of the pipe.

$$\begin{aligned} \frac{\partial \underline{r}}{\partial u} &= -\sin u\,\underline{i} + 2\cos u\underline{j}, \\ \frac{\partial \underline{r}}{\partial v} &= \underline{k}. \end{aligned}$$

The cross product of these vectors gives a vector which is normal to the surface. Let

$$\begin{aligned} \underline{n} &= \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \underline{i}(2\cos u) - \underline{j}(-\sin u) + \underline{k}(0) = 2\cos u\,\underline{i} + \sin u\,\underline{j}. \end{aligned}$$

(iv) Show that $\underline{q} \cdot \underline{n} = 0$ on the pipe surface and state the physical significance of this.

[3 marks]

Answer

Using the vector $\underline{n}$ in part (iii) we have

$$\underline{q} \cdot \underline{n} = 2y\cos u - 4x\sin u.$$

On the surface of the pipe

$$x = \cos u \quad \text{and} \quad y = 2\sin u$$

and hence

$$\underline{q} \cdot \underline{n} = 0.$$

This property indicates that no fluid flows through the surface of the pipe.

(v) Calculate the circulation round the circumference of the pipe in the plane $z = 0$ (i.e. $v = 0$).

[6 marks]

Answer

Let C denote the circumference of the pipe in the plane $z = 0$. On C we have

$$\underline{r} = \cos u\,\underline{i} + 2\sin u\,\underline{j}, \quad \frac{\mathrm{d}\underline{r}}{\mathrm{d}u} = -\sin u\,\underline{i} + 2\cos u\,\underline{j}, \quad \underline{q} = 2\sin u\,\underline{i} - 4\cos u\,\underline{j}.$$

The circulation is

$$\oint_C \underline{q}\cdot \mathrm{d}\underline{r} = \int_0^{2\pi} \underline{q}\cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}u}\,\mathrm{d}u.$$

The integrand is

$$\underline{q}\cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}u} = -2\sin^2 u - 8\cos^2 u = -2 - 6\cos^2 u.$$

The circulation is

$$-4\pi - 6\pi = -10\pi.$$

(vi) Suppose now that a perforated, convex (i.e. the outward normal has a positive $\underline{k}$ component) disc is placed over the end of the pipe at $v = 0$. State Stokes' theorem and deduce the value of

$$\int_S \underline{\omega}\cdot \mathrm{d}\underline{s},$$

where S is the surface of the disc.

[3 marks]

Answer

Stokes' theorem for the surface S enclosed by C is that

$$\int_S \nabla\times\underline{q}\cdot \mathrm{d}\underline{s} = \oint_C \underline{q}\cdot \mathrm{d}\underline{r}.$$

As $\underline{\omega} = \nabla\times\underline{q}$ and the right hand side has been determined in part (iii) we have

$$\int_S \underline{\omega}\cdot \mathrm{d}\underline{s} = -10\pi.$$
