# MA2741: Spring Term – Exercise sheet 3 relating to chapter 3: 2D incompressible flows, stream function, vorticity, $\cdots$

In past exams formula sheets were available in case questions needed the use of polar coordinates and such sheets contained the following.

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In the following, underline symbols denote vector quantities.

In terms of cylindrical polar coordinates $(r, \theta, z)$ with unit vectors $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{k}$, the following definitions hold:

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\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{k}, \\
\nabla \cdot \underline{F} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}, \\
\nabla \times \underline{F} &= \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{k} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}, \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2},
\end{aligned}
$$

where $f(r, \theta, z)$ is any scalar function and

$$\underline{F} = F_r(r,\theta,z)\underline{e}_r + F_\theta(r,\theta,z)\underline{e}_\theta + F_z(r,\theta,z)\underline{k}.$$

The vector product of vectors $\underline{a}$ and $\underline{F}$, where

$$\underline{a}(r,\theta,z) = a_r(r,\theta,z)\underline{e}_r + a_\theta(r,\theta,z)\underline{e}_\theta + a_z(r,\theta,z)\underline{k},$$

is given by

$$\underline{a} \times \underline{F} = \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ a_r & a_\theta & a_z \\ F_r & F_\theta & F_z \end{vmatrix}.$$

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1. The stream function for a line vortex, with circulation $\Gamma$, in a uniform flow is

$$\psi = Ur\sin\theta - \frac{\Gamma}{2\pi}\ln r$$

where $U$ is constant and $(r, \theta)$ are plane polar coordinates. Obtain the velocity field and show that there is a stagnation point at

$$\left(\frac{\Gamma}{2\pi U}, \frac{\pi}{2}\right).$$

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2. The stream function

$$\psi = Ur\left(\theta\cos\theta + \frac{2}{\pi}\theta\sin\theta - \sin\theta\right), \quad 0 \le \theta \le \pi/2$$

where $U$ is constant and $(r, \theta)$ are plane polar coordinates, can be used to model 2-D, incompressible flow in the corner region $x > 0$, $y > 0$.

(a) Determine the velocity field and verify that the normal component vanishes at the rigid walls $\theta = 0$ and $\theta = \pi/2$.

(b) Calculate the vorticity, $\underline{\omega}$, of the flow where $\underline{\omega} = \nabla \times \underline{q}$.

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3. Consider the two-dimensional velocity field

$$\underline{q} = \left(\frac{\alpha x}{x^2 + y^2}\right)\underline{i} + \left(\frac{\alpha y}{x^2 + y^2}\right)\underline{j},$$

where $\alpha$ is constant. Show that this represents an incompressible flow and find a stream function and velocity potential for the flow.

[Hint: Determine the gradient of $\tan^{-1}(y/x)$.]

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4. A fluid flows through a section of pipe whose surface is described by the Cartesian position vector

$$\underline{r} = (1+v)\cos u\underline{i} + (1+v)\sin u\underline{j} + v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

The fluid velocity is given by

$$\underline{q} = \underline{q}(x, y) = y\underline{i} - x\underline{j}.$$

(a) Show that the flow is incompressible and determine the vorticity.

(b) Determine the normal, $\underline{n}$, to the pipe surface. Show that $\underline{q} \cdot \underline{n} = 0$ on the pipe surface and state the physical significance of this.

(c) Calculate the circulation round the circumference of the pipe in the plane $z = 0$ (i.e. $v = 0$).

(d) Calculate the surface area of this section of pipe.

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5. The stream function for a two-dimensional flow is given by:

$$\psi = \psi(x, y) = Uy^2 + 2Uxy.$$

Determine:

(a) the velocity field;
(b) the vorticity, $\underline{\omega}$;
(c) the circulation around $\Gamma$ where $\Gamma$ is the unit circle centred at the origin.

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6. A two-dimensional, incompressible flow comprises a source of strength $m \geq 0$ at the origin together with a uniform flow with speed $U > 0$ in the direction $\theta = \pi$ of a plane polar coordinate system $(r, \theta)$. The stream function is given by

$$\psi = -Ur\sin\theta + \frac{m\theta}{2\pi}.$$

Determine the fluid velocity and show that there is a stagnation point at

$$r = \frac{m}{2\pi U}, \quad \theta = 0.$$

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7. The stream function for an incompressible uniform flow past a particular multipole is

$$\psi = \psi(r, \theta) = Ur\sin\theta - \mu\frac{\sin 2\theta}{r^2}, \quad 0 \leq \theta \leq \pi$$

where $U$ and $\mu$ are constants and $(r, \theta)$ are plane polar coordinates.

(a) Obtain the polar equation of the curve corresponding to $\psi = 0$ for $0 \leq \theta \leq \pi$, and determine the normal vector, $\underline{n}$, to this curve.
(b) Show that $\underline{q} \cdot \underline{n} = 0$ on the curve found in i) above, where $\underline{q}$ is the fluid velocity.
(c) Determine the stagnation point and the vorticity of the flow.

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8. The stream function

$$\psi = Ur(\sin\theta - \theta\cos\theta), \quad 0 \leq \theta \leq \pi$$

where $U$ is constant and $(r, \theta)$ are plane polar coordinates, describes flow over a horizontal rigid surface which corresponds to $\theta = 0$ and $\theta = \pi$ (i.e. $y = 0$ in Cartesian form).

(a) Determine the velocity field.
(b) Determine the vorticity, $\underline{\omega}$, of the velocity field.
(c) Show that the circulation around the closed curve mapped by the polar position vector $\underline{r} = \sin(\theta)\underline{e}_r$, $0 \leq \theta \leq \pi$ is given by

$$\Gamma = U\int_0^\pi \left(\theta\sin(2\theta) - \sin^2\theta\right)\, d\theta.$$

Evaluate this integral.

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9. *Apart from notational changes, this was the most of question 3 of the* MA2841 *paper in 2012. The other part of the question was on the previous exercise sheet.*

   A factory worker is filling barrels of beer using a funnel, the surface of which is described by the Cartesian position vector

   $$\underline{r} = \mathrm{e}^{-v}\cos u\underline{i} + \mathrm{e}^{-v}\sin u\underline{j} - v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

   Suppose that the fluid velocity within the funnel is given by

   $$\underline{q} = (x + 2y)\underline{i} - y\underline{j} - 3\underline{k}.$$

   Determine the following.

   (ii) The vorticity $\underline{\omega}$ of the flow.

   **[3 marks]**

   (iii) The outward normal, $\underline{n}$, to the funnel surface.

   **[4 marks]**

   Show that $\underline{q}\cdot\underline{n} \ne 0$ on the funnel surface and state the physical significance of this.

   **[4 marks]**

   Hence, evaluate the flux integral for the fluid velocity across the funnel surface.

   **[6 marks]**

10. *Apart from notational changes, this was question 4a of the* MA2841 *paper in 2011 and was worth 12 marks (the complete question was worth 20 marks).*

A stream function is given, in plane polar coordinates $(r, \theta)$, by

$$\psi = r\theta \sin\theta, \quad 0 \le \theta \le 2\pi.$$

(i) Determine the velocity field, $\underline{q}$, for the flow.

**[3 marks]**

(ii) Calculate the vorticity, $\underline{\omega}$, of the velocity field.

**[3 marks]**

(iii) Show that the circulation, $\Gamma$, around the closed curve $r = \cos\theta$, $-\pi/2 \le \theta \le \pi/2$, is given by

$$\Gamma = -\int_{-\pi/2}^{\pi/2} (\sin^2\theta + \theta \sin 2\theta)\, \mathrm{d}\theta.$$

Evaluate this integral.

**[6 marks]**

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11. *Apart from notational changes, this was part of question 3 of the* MA2841 *paper in 2011. The other parts of the question were on the previous exercise sheet.*

The surface of a bowl is mapped by the position vector

$$\underline{r} = v\cos u\,\underline{i} + v\sin u\,\underline{j} + v^2\,\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

A fluid within the bowl moves such that its velocity is given by

$$\underline{q} = \frac{y^2}{x}\,\underline{i} - y\underline{j}, \quad x > 0.$$

(ii) Determine a normal vector $\underline{n}$ to the bowl surface.

**[4 marks]**

(iii) Show that $\underline{q} \cdot \underline{n} = 0$ on the bowl surface and state the physical significance of this.

**[3 marks]**

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12. *Apart from notational changes, this was question 4 of the* MA2941 *paper in 2010.*

Consider the stream function

$$\psi = r\theta\cos\theta - \frac{2\pi\cos\theta}{r}, \quad 0 \le \theta \le 2\pi,$$

where $(r,\theta)$ are plane polar coordinates.

(i) Show that one solution to $\psi = 0$ is $r^2\theta = 2\pi$ and sketch this curve for $0 < \theta < 2\pi$.

**[4 marks]**

(ii) Determine the velocity field $\underline{q}$ for the flow.

**[3 marks]**

(iii) Let $\varphi = r^2\theta - 2\pi$. Determine $\nabla\varphi$ and show that $\underline{q}\cdot\nabla\varphi = 0$ when $r^2\theta = 2\pi$. Explain this result.

**[7 marks]**

(iv) Show that the circulation around the closed curve $r = 1$, $0 \le \theta \le 2\pi$ is given by

$$-\int_0^{2\pi} (\theta\cos\theta + 2\pi\cos\theta)\,\mathrm{d}\theta$$

and evaluate this integral.

**[6 marks]**

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13. *Apart from notational changes, this was part of question 3 of the* MA2941 *paper in 2010. The other parts of the question appeared on the previous exercise sheet.*

A fluid flows such that its velocity is given by

$$\underline{q} = 3\underline{i} + 2tx\,\underline{j} - z\,\underline{k}, \quad t \ge 0$$

where $t$ is time.

(ii) Determine the vorticity of the flow.

**[3 marks]**

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14. *Apart from notational changes, this was question 3 of the* MA2941 *paper in 2009.*

A fluid flows inside a semi-infinite pipe whose surface is described by the Cartesian position vector

$$\underline{r} = \cos\, u\, \underline{i} + 2 \sin\, u\, \underline{j} + v\, \underline{k}, \quad 0 \le v \le 2\pi, \quad -\infty < v < 0.$$

The fluid velocity is given by

$$\underline{q} = y\, \underline{i} - 4x\, \underline{j} + z\, \underline{k}.$$

(ii) Determine the vorticity $\underline{\omega}$.

**[3 marks]**

(iii) Determine the normal, $\underline{n}$, to the pipe surface.

**[3 marks]**

(iv) Show that $\underline{q} \cdot \underline{n} = 0$ on the pipe surface and state the physical significance of this.

**[3 marks]**

(v) Calculate the circulation round the circumference of the pipe in the plane $z = 0$ (i.e. $v = 0$).

**[6 marks]**

(vi) Suppose now that a perforated, convex (i.e. the outward normal has a positive $\underline{k}$ component) disc is placed over the end of the pipe at $v = 0$. State Stokes' theorem and deduce the value of

$$\int_S \underline{\omega} \cdot \mathrm{d}\underline{s},$$

where $S$ is the surface of the disc.

**[3 marks]**