MA2741: Spring Term – Exercise sheet 2 relating to chapter 2: Particle paths, streamlines, material time derivative etc. with answers

Note: In one of the question circulation about a closed curve C is mentioned and this is

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r}$$

and will be mentioned in chapter 3. Incompressibility will be considered in the lectures from week 19 and examples in the lectures involving streamlines will probably not be done until week 19.

1. Suppose that a particle starting at $\underline{r}_0$ at time $t = 0$ is at position $\underline{r}(\underline{r}_0, t)$ at time t and suppose that we have a function $f_L(\underline{r}_0, t)$. If we define $f(\underline{r}, t)$ by the relation

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t)$$

then show that

$$\frac{\partial}{\partial t} f_L(\underline{r}_0, t) = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f$$

where $\underline{q}$ denotes the velocity of the particle being considered.

Answer

In terms of x, y, z we have

$$f(x, y, z, t) = f_L(x_0, y_0, z_0, t)$$

with x, y and z depending on t. By the chain rule we have

$$\frac{\partial}{\partial t} f_L(x_0, y_0, z_0, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f$$

where

$$\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} \quad \text{and} \quad \underline{q} = \frac{\mathrm{d}x}{\mathrm{d}t}\underline{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\underline{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\underline{k}.$$

$\underline{q}$ is the velocity.

2. The Lagrangian description of a flow is

$$x = x_0 e^{\alpha t}, \quad y = y_0 e^{-\alpha t}, \quad z = z_0$$

where α is a positive constant and (x_0, y_0, z_0) is the initial position of a given particle.

i) Write down the position vector of a fluid particle. Determine the Lagrangian velocity and write this in Eulerian form.

Answer

The position vector is

$$\underline{r} = x_0 e^{\alpha t}\, \underline{i} + y_0 e^{-\alpha t}\, \underline{j} + z_0\, \underline{k}.$$

We get the velocity described in a Lagrangian way by differentiating with respect to t, i.e.

$$\underline{q}_L = x_0 \alpha \mathrm{e}^{\alpha t}\, \underline{i} - y_0 \alpha \mathrm{e}^{-\alpha t}\, \underline{j}.$$

To describe the same velocity in an Eulerian form we need to express x_0 and y_0 in terms of x, y and t. In this case we immediately have

$$x = x_0 \mathrm{e}^{\alpha t}, \quad y = y_0 \mathrm{e}^{-\alpha t}$$

giving

$$\underline{q}_E = \alpha(x\, \underline{i} - y\, \underline{j}).$$

---

ii) Show that $\nabla \cdot \underline{q}_E = 0$ where $\underline{q}_E$ denotes the Eulerian velocity.

Answer

$$\nabla \cdot \underline{q}_E = \alpha \left(\frac{\partial x}{\partial x} + \frac{\partial(-y)}{\partial y} \right) = 0.$$

---

iii) Determine the Lagrangian acceleration, the local accelerations and the convective acceleration and use the material time derivative relation to show that the two results are consistent.

Answer

The Lagrangian acceleration is

$$\frac{\partial \underline{q}_L}{\partial t} = x_0 \alpha^2 \mathrm{e}^{\alpha t}\, \underline{i} + y_0 \alpha^2 \mathrm{e}^{-\alpha t}\, \underline{j}.$$

The local acceleration is obtained from the Eulerian description of the velocity and is

$$\frac{\partial \underline{q}_E}{\partial t} = \underline{0}.$$

The convective acceleration is

$$(\underline{q}_E \cdot \nabla)\underline{q}_E.$$

In this case with $\underline{q}_E = u\,\underline{i} + v\,\underline{j}$ we have

$$\nabla u = \alpha\,\underline{i}, \quad \nabla v = -\alpha\,\underline{j}$$

giving

$$\underline{q}_E \cdot (\nabla u) = \alpha^2 x, \quad \underline{q}_E \cdot (\nabla v) = \alpha^2 y.$$

Thus

$$(\underline{q}_E \cdot \nabla)\underline{q}_E = \alpha^2(x\,\underline{i} + y\,\underline{j}).$$

In this case

$$\frac{\mathrm{D}\underline{q}_E}{\mathrm{D}t} = (\underline{q}_E \cdot \nabla)\underline{q}_E = \alpha^2(x\,\underline{i} + y\,\underline{j}).$$

This agrees with the expression for the Lagrangian acceleration as $x = x_0\mathrm{e}^{\alpha t}$ and $y = y_0\mathrm{e}^{-\alpha t}$.

---

iv) Determine the streamlines for the flow. Do they differ from the particle paths?

Answer

As

$$\frac{\partial \underline{q}_E}{\partial t} = \underline{0}$$

the flow is steady and the streamlines coincide with the particle paths.

The streamline at (x, y, z) has a tangent in the direction of the velocity at this point and thus it is in the direction of

$$x\,\underline{i} - y\,\underline{j}.$$

One way to express this property is to write

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{-y}.$$

Integrating gives

$$\ln x = -\ln y + \ln C$$

where C is a constant. Thus the streamlines are of the form

$$xy = C$$

with the streamline passing through (x_0, y_0, z_0) being

$$xy = x_0 y_0.$$

---

3. An incident at a chemical plant causes the release of a pollutant into a river which flows with velocity
$$\underline{q} = \alpha x \underline{i} - \alpha y \underline{j}, \quad y > 0.$$
It is known that the concentration of the pollutant at position (x, y) and time t is described by
$$c(x, y, t) = \beta x^2 y e^{-\alpha t}$$
where β is constant. Determine whether or not the pollutant concentration for a typical particle changes with time?

Answer

The concentration c is described in a Eulerian way and to get the time derivative relating to a particular particle we need to determine the material time derivative of c which is
$$\frac{\mathrm{D}}{\mathrm{D}t} c(x, y, t) = \frac{\partial c}{\partial t} + \underline{q} \cdot \nabla c.$$
For the derivatives
$$\frac{\partial c}{\partial t} = -\alpha c, \quad \nabla c = \beta \mathrm{e}^{-\alpha t}(2xy\, \underline{i} + x^2\, \underline{j}).$$
Thus
$$\underline{q} \cdot \nabla c = \alpha \beta \mathrm{e}^{-\alpha t}(2x^2 y - x^2 y) = \alpha \beta \mathrm{e}^{-\alpha t} x^2 y = \alpha c$$
and
$$\frac{\mathrm{D}}{\mathrm{D}t} c(x, y, t) = \alpha c - \alpha c = 0.$$
The concentration for a typical particle does not change with time.

---

4. Consider the unsteady, 2-D flow described by
$$\underline{q} = U \underline{i} + \alpha t \underline{j}, \quad t > 0$$
where U and α are positive constants. Show that the streamlines are straight lines and that any fluid particle follows a parabolic path as time proceeds.

Answer

As the flow is unsteady the particle paths differ from the streamlines.

Let $\underline{r} = x\, \underline{i} + y\, \underline{j}$ denote a position at time t of a particle. To get the particle paths we need to consider
$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q},$$
i.e.
$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= U, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \alpha t. \end{aligned}$$

For the particle which is at (x_0, y_0) at time $t = 0$ we have

$$x = x_0 + Ut, \quad y = y_0 + \alpha \frac{t^2}{2}.$$

To express the particle path in a form which does not involve t is straightforward here as the expression for $x(t)$ is not complicated.

$$t = \frac{x - x_0}{U}, \quad y = y_0 + \frac{\alpha}{2} \left(\frac{x - x_0}{U} \right)^2.$$

We have the equation of a parabola.

The streamlines at a specific time t have a tangent which is parallel to the velocity and this condition can be expressed in the form

$$\frac{\mathrm{d}x}{U} = \frac{\mathrm{d}y}{\alpha t}.$$

As t is constant here integration gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha t}{U}, \quad y = \left(\frac{\alpha t}{U} \right) x + C,$$

where C is a constant, and this is the equation of a straight line.

---

5. *Apart from notational changes, this was the most of question 3 of the* MA2841 *paper in 2012. The other part of the question was on the previous exercise sheet.*

   A factory worker is filling barrels of beer using a funnel, the surface of which is described by the Cartesian position vector

   $$\underline{r} = \mathrm{e}^{-v} \cos u \underline{i} + \mathrm{e}^{-v} \sin u \underline{j} - v \underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

   Given that the fluid velocity within the funnel is

   $$\underline{q} = (x + 2y)\underline{i} - y\underline{j} - 3\underline{k},$$

   determine whether or not the flow is incompressible.

Answer

To determine if the flow is incompressible we just need to check if $\nabla \cdot \underline{q} = 0$.

If $\underline{q} = u \, \underline{i} + v \, \underline{j} + w \, \underline{k}$ then

$$\nabla \cdot \underline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 1 - 1 + 0 = 0$$

and hence the flow is incompressible.

---

6. *Apart from notational changes and re-wording this was question 4 of the* MA2841 *paper in 2012.*

   A scientist releases some dye into a river and notices that the dye colours an ellipsoidal shaped volume of water and the surface of the ellipsoidal is observed to vary with time t according to

$$F(x,y,t)=1 \quad \text{where } F(x,y,t)=\frac{x^2}{1+t}+y^2\mathrm{e}^{-2t}, \quad 0\le t<T.$$

   (i) Show that if the velocity $\underline{q}$ of the river is given by

$$\underline{q}=\frac{x}{2(1+t)}\underline{i}+y\underline{j}, \quad 0\le t<T$$

   then a point on the surface at time $t=0$ remains on the surface for all t, $0\le t<T$.

Answer
With the surface given by
$$F(x,y,t)=1$$
this corresponds to the same particles throughout if
$$\frac{\mathrm{D}F}{\mathrm{D}t}=\frac{\partial F}{\partial t}+\underline{q}\cdot\nabla F=0.$$
For the function F in the question we have
$$\frac{\partial F}{\partial t}=-\frac{x}{(1+t)^2}-2y^2\mathrm{e}^{-2t}, \quad \frac{\partial F}{\partial x}=\frac{2x}{1+t}, \quad \frac{\partial F}{\partial y}=2y\mathrm{e}^{-2t}.$$
Hence
$$\underline{q}\cdot\nabla F=\frac{x^2}{(1+t)^2}+2y^2\mathrm{e}^{-2t}$$
and we get
$$\frac{\mathrm{D}F}{\mathrm{D}t}=0.$$

   (ii) Show that the particle paths are given by

$$y=y_0\exp\left(\frac{x^2}{x_0^2}-1\right)$$

   where (x_0,y_0) is the initial position of a particle.

Answer
The particle paths satisfy
$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t}=\underline{q},$$

i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{2(1+t)},$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y.$$

With $y(0) = y_0$ we have

$$y = y_0 \mathrm{e}^t.$$

In the case of x we have

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{2(1+t)}.$$

Integrating gives

$$\ln x = \frac{1}{2}\ln(1+t) + \ln A$$

where A is a constant. If $x(0) = x_0$ then $A = x_0$. Using the properties of logarithms enables us to write this as

$$x = x_0(1+t)^{1/2}.$$

From this relation we get

$$t = \left(\frac{x}{x_0}\right)^2 - 1 \quad \text{and} \quad y = y_0 \exp\left(\frac{x^2}{x_0^2} - 1\right).$$

(iii) Determine the streamlines of the flow.

Answer

At a fixed time t the streamlines are parallel to the velocity $\underline{q} = u\,\underline{i} + v\,\underline{j}$ and can be expressed in the form

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v}$$

giving

$$\frac{\mathrm{d}x}{\left(\dfrac{x}{2(1+t)}\right)} = \frac{\mathrm{d}y}{y}.$$

To state again, t is fixed here and thus integrating gives

$$2(1+t)\ln x = \ln y + \ln A$$

where A is a constant. Hence the streamlines are of the form

$$x^{2(1+t)} = Ay.$$

(iv) Suppose that at time $t = T$, $T > 0$, it starts to rain and this alters the fluid velocity such that

$$\underline{q} = \frac{y}{2}\underline{i} + x\underline{j}, \quad t \ge T.$$

Determine, for this velocity field, the circulation about the curve C described by $x^2 + y^2 = 1$.

Answer

The circulation around C is defined by

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r}.$$

To determine this we need a parametric description of the curve and for this we can take

$$x = \cos\,\theta, \quad y = \sin\,\theta, \quad 0 \le \theta < 2\pi.$$

With $\underline{r} = x\,\underline{i} + y\,\underline{j}$ this gives

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = -\sin\,\theta\,\underline{i} + \cos\,\theta\,\underline{j}$$

and

$$\underline{q}(\cos\,\theta, \sin\,\theta) \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta} = -\frac{\sin^2\theta}{2} + \cos^2\theta.$$

Hence

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r} = \int_0^{2\pi} \underline{q} \cdot \frac{\mathrm{d}\underline{r}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \pi\left(-\left(\frac{1}{2}\right) + 1\right) = \frac{\pi}{2}.$$

---

7. *Apart from notational changes, this was most of question 3 of the* `MA2841` *paper in 2011. The remaining parts of the question (parts (ii) and (iii)) may appear on the next exercise sheet.*

   The surface of a bowl is mapped by the position vector

   $$\underline{r} = v \cos u \, \underline{i} + v \sin u \, \underline{j} + v^2 \, \underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

   A fluid within the bowl moves such that its velocity is given by

   $$\underline{q} = \frac{y^2}{x} \underline{i} - y \underline{j}, \quad x > 0.$$

   (i) Determine whether or not the flow is incompressible.

[3 marks]

Answer

The flow is incompressible if $\nabla \cdot \underline{q} = 0$. With $\underline{q} = u \, \underline{i} + v \, \underline{j} + w \, \underline{k}$ we have

$$\nabla \cdot \underline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{y^2}{x}\right) + \frac{\partial}{\partial y}(-y) = -\left(\frac{y^2}{x^2} + 1\right).$$

In the funnel region $x^2 + y^2 < 1$ and thus the flow is not incompressible at any point.

(iv) A fluid particle is initially at the position (x_0, y_0, z_0) within the bowl, with $x_0 > 0$. Show that its location for subsequent time is described by the position vector

$$\underline{r} = \sqrt{x_0^2 + y_0^2 - y_0^2 \mathrm{e}^{-2t}} \, \underline{i} + y_0 \mathrm{e}^{-t} \, \underline{j} + z_0 \underline{k}, \quad t \ge 0,$$

where, as usual, the square root notation means that the positive square root is taken.

[8 marks]

Answer

With $\underline{r} = x \, \underline{i} + y \, \underline{j}$ the particle paths are described by

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q},$$

i.e.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{y^2}{x}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 0. \end{aligned}$$

The equations for y and z are immediately solved to give

$$y(t) = y_0 \mathrm{e}^{-t}, \quad z(t) = z_0.$$

Substituting in the equation for x gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{y_0^2 \mathrm{e}^{-2t}}{x}$$

and rearranging we have

$$x \, \mathrm{d}x = y_0^2 \mathrm{e}^{-2t}.$$

Integrating and using the initial condition gives

$$\frac{x^2 - x_0^2}{2} = y_0^2 \left(\frac{\mathrm{e}^{-2t} - 1}{-2} \right)$$

which we can re-write as

$$x^2 = x_0^2 + y_0^2 (1 - \mathrm{e}^{-2t}).$$

By taking the square root we get the expression in the question.

(v) Show that the particle moves in a circle in the plane $z = z_0$.

[2 marks]

Answer

By using $y(t) = y_0 \mathrm{e}^{-t}$ the expression giving $x(t)$ can be written as

$$x^2 = x_0^2 + y_0^2 (1 - \mathrm{e}^{-2t}) = x_0^2 + y_0^2 - y_0^2 \mathrm{e}^{-2t} = x_0^2 + y_0^2 - y^2$$

and we get

$$x^2 + y^2 = x_0^2 + y_0^2.$$

This is the equation of a circle of radius $\sqrt{x_0^2 + y_0^2}$ centred at the origin.

---

8. *Apart from notational changes, this was most of question 3 of the* `MA2941` *paper in 2010. The remaining parts of the question (part (ii)) may appear on the next exercise sheet.*

A fluid flows such that its velocity is given by

$$\underline{q} = 3\underline{i} + 2tx\,\underline{j} - z\,\underline{k}, \quad t \ge 0$$

where t is time.

(i) Determine whether or not the flow is incompressible.

[3 marks]

Answer

With $\underline{q} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$ we have incompressible flow if

$$\nabla \cdot \underline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

In this case

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = -1.$$

Hence $\nabla \cdot \underline{q} = -1$ and the flow is not incompressible.

(iii) A fluid particle is initially at the position (x_0, y_0, z_0). Show that its location for subsequent time is described by the position vector

$$\underline{r} = (3t + x_0)\underline{i} + (2t^3 + x_0t^2 + y_0)\underline{j} + z_0\mathrm{e}^{-t}\underline{k}, \quad t \ge 0.$$

[6 marks]

Answer

With $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ the particle paths satisfy

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{q},$$

i.e.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 3,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2tx,\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= -z.\end{aligned}$$

Integrating and using the initial condition gives

$$x = x_0 + 3t, \quad z = z_0\mathrm{e}^{-t}.$$

Substituting in the equation for y gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t(x_0 + 3t), \quad y = y_0 + x_0 t^2 + 2t^3.$$

(iv) Show that particles initially in the plane $z = 0$ remain in this plane and find the corresponding particle paths in parametric form.

[4 marks]

Answer

If $z_0 = 0$ then

$$\underline{r}(t) = (3t + x_0)\,\underline{i} + (2t^3 + x_0 t^2 + y_0)\,\underline{j}$$

is a parametric description of a path. As there is no component in the $\underline{k}$ direction the path is always in the plane $z = 0$.

(v) Determine the streamlines for particles that move in the plane $z = 0$ and sketch them for $t = 1$.

[4 marks]

Answer

Let $\underline{q} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$ denote the velocity. The streamlines at time t are curves with a tangent parallel to the velocity at each point.

At time $t = 1$ we have

$$u = 3, \quad v = 2x, \quad w = -z.$$

The parallel condition can be written as

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}$$

and in this case we have

$$\frac{\mathrm{d}x}{3} = \frac{\mathrm{d}y}{2x} = \frac{\mathrm{d}z}{-z}.$$

The equation for z gives

$$z(t) = z_0 \mathrm{e}^{-t}$$

and if $z_0 = 0$ then $z(t) = 0$. The other part gives

$$2x\mathrm{d}x = 3\mathrm{d}y$$

and integrating gives

$$x^2 - x_0^2 = 3(y - y_0)$$

where $x(0) = x_0$ and $y(0) = y_0$. Different values of (x_0, y_0) give different streamlines in the x, y plane.

---

9. *Apart from notational changes and a little re-wording this was question 4 of the* MA2941 *paper in 2009.*

   (a) A fluid particle moves in a two-dimensional (2-D) space with position vector

$$\underline{r} = 2\cos t\,\underline{i} + (1+\sin t)\underline{j}.$$

   (i) Show that the particle path is an ellipse.

[3 marks]

Answer
With $\underline{r} = x\,\underline{i} + y\,\underline{j}$ we have

$$x = 2\cos t, \quad y - 1 = \sin t.$$

Thus

$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{2}\right)^2 + (y-1)^2$$

which is the equation of an ellipse with centre at $(0,1)$ with semi-major axes of length 2 in the $\underline{i}$ direction and the semi-minor axes of length 1 in the $\underline{j}$ direction.

   (ii) Find the Lagrangian velocity $\underline{q}_L$, and show that this can be expressed in Eulerian form as

$$\underline{q}_E = -2(y-1)\underline{i} + \frac{x}{2}\,\underline{j}.$$

[3 marks]

Answer
The velocity in a Lagrangian description is given by

$$\underline{q}_L(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underline{r}(t) = -2\sin t\,\underline{i} + \cos t\,\underline{j}.$$

The same velocity in Eulerian form can be obtained by using the relations for x and y in terms of t to give

$$\underline{q}_E = \underline{q}_E(x,y) = -2(y-1)\,\underline{i} + \frac{x}{2}\,\underline{j}.$$

   (iii) Determine the Lagrangian and local accelerations.

[2 marks]

Answer
The Lagrangian acceleration is

$$\underline{a}_L = \frac{\mathrm{d}}{\mathrm{d}t}\underline{q}_L(t) = -2\cos t\,\underline{i} - \sin t\,\underline{j}.$$

The local acceleration is
$$\frac{\partial \underline{q}_E}{\partial t} = \underline{0}.$$

(iv) Use the material time derivative to show that your results for part (iii) are consistent.

[5 marks]

Answer

The material time derivative of $\underline{q}_E$ is the same as $\underline{a}_L$ and is given by

$$\frac{\mathrm{D}\underline{q}_E}{\mathrm{D}t} = \frac{\partial \underline{q}_E}{\partial t} + (\underline{q}_E \cdot \nabla)\underline{q}_E.$$

In this case

$$\nabla u = \nabla(-2(y-1)) = -2\underline{j}, \quad \nabla v = \nabla\left(\frac{x}{2}\right) = \left(\frac{1}{2}\right)\underline{i}$$

and
$$\underline{q}_E \cdot (\nabla u) = -x, \quad \underline{q}_E \cdot (\nabla v) = -(y-1).$$

The convective acceleration is hence

$$(\underline{q}_E \cdot \nabla)\underline{q}_E = -x\,\underline{i} - (y-1)\underline{j}.$$

As the local acceleration is $\underline{0}$ this is the same as $\underline{a}_L$ because $x = 2\cos t$, $y - 1 = \sin t$.

(b) An accident at a chemical plant causes the release of a toxic pollutant into a river. The concentration of the pollutant is given by

$$c(x, y, t) = \sinh\left(\frac{x}{t}\right)\mathrm{e}^{-ty}, \quad t > 0.$$

Given that the pollutant concentration in a given fluid element does not vary with time, determine a two-dimensional (2-D) velocity of the river consistent with this and determine the streamlines of this flow.

[7 marks]

Answer

The pollutant concentration in a given fluid element does not vary with time means that the material time derivative is 0, i.e.

$$\frac{\mathrm{D}c}{\mathrm{D}t} = \frac{\partial c}{\partial t} + \underline{q} \cdot \nabla c = 0.$$

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= \cosh\left(\frac{x}{t}\right)\left(\frac{-x}{t^2}\right)\mathrm{e}^{-ty} + \sinh\left(\frac{x}{t}\right)(-y)\mathrm{e}^{-ty},\\
\frac{\partial c}{\partial x} &= \cosh\left(\frac{x}{t}\right)\left(\frac{1}{t}\right)\mathrm{e}^{-ty},\\
\frac{\partial c}{\partial y} &= \sinh\left(\frac{x}{t}\right)(-t)\mathrm{e}^{-ty}.
\end{aligned}
$$

With $\underline{q} = u\,\underline{i} + v\,\underline{j}$ we have to consider

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = 0$$

and substituting in the above relations and cancelling the common factor of e^{-ty} gives

$$\left(\frac{-x}{t^2}\right)\cosh\left(\frac{x}{t}\right) + (-y)\sinh\left(\frac{x}{t}\right) + \frac{u}{t}\cosh\frac{x}{t} + (-tv)\sinh\left(\frac{x}{t}\right) = 0$$

and collecting terms we can write

$$\frac{1}{t^2}\cosh\left(\frac{x}{t}\right)(-x+tu) + \sinh\left(\frac{x}{t}\right)(-y-tv) = 0.$$

The question only asks for a velocity field which is consistent with this relation and we can get this by taking

$$-x + tu = 0 \quad \text{and} \quad -y - tv = 0.$$

Streamlines at a given time t are such that the tangent to the curves is in the direction of the velocity and this condition can be written in the form

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v},$$

i.e.

$$\frac{\mathrm{d}x}{\left(\frac{x}{t}\right)} = \frac{\mathrm{d}y}{\left(\frac{-y}{t}\right)}, \quad \text{which simplifies to} \quad \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y}{y}.$$

Integrating gives

$$\ln x = -\ln y + \ln A$$

where A is a constant. The streamlines are of the form

$$xy = A.$$

---

10. *Apart from notational changes and a little re-wording this was part of question 4 of the* `MA2941` *paper in 2008.*

For a particular fluid motion, the position vector of a typical particle is

$$\underline{r} = \left(x_0 - \cos\left(\omega t + \frac{\pi}{2}\right)\right)\underline{i} + y_0\underline{j}$$

where ω denotes a constant.

(i) Describe the motion of the particle.

[2 marks]

Answer
With $\underline{r} = x\,\underline{i} + y\,\underline{j}$ the value of y is fixed and the value of x oscillates between $x_0 - 1$ and $x_0 + 1$.

(ii) Find the Lagrangian velocity $\underline{q}_L$ and show that when $|t| \le \pi/(2\omega)$ this can be expressed in Eulerian form as

$$\underline{q}_E = \omega\sqrt{1-(x-x_0)^2}\,\underline{i}.$$

[3 marks]

Answer
The Lagrangian form of the velocity is

$$\underline{q}_L = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \omega \sin\left(\omega t + \frac{\pi}{2}\right)\underline{i}.$$

Now

$$x - x_0 = -\cos\left(\omega t + \frac{\pi}{2}\right)$$

and hence

$$\sin^2\left(\omega t + \frac{\pi}{2}\right) = 1 - (x-x_0)^2.$$

For the expression for the velocity in Eulerian form we have one of the following.

$$\underline{q}_E = \omega\sqrt{1-(x-x_0)^2}\,\underline{i} \quad \text{or} \quad \underline{q}_E = -\omega\sqrt{1-(x-x_0)^2}\,\underline{i}.$$

The Lagrangian velocity and Eulerian velocity refer to the same quantity and as the x-component of the velocity is positive when

$$0 < \omega t + \frac{\pi}{2} < \pi$$

we need the plus sign for time in this range.

(iii) Determine the Lagrangian and local accelerations.

[2 marks]

Answer

The Lagrangian acceleration is

$$\underline{a}_L = \frac{\mathrm{d}\underline{q}_L}{\mathrm{d}t} = \omega^2 \cos\left(\omega t + \frac{\pi}{2}\right) \underline{i}.$$

The local acceleration is

$$\frac{\partial \underline{q}_E}{\partial t} = \underline{0}.$$

(iv) Use the material time derivative to show that your results for part (iii) are consistent.

[3 marks]

Answer

The Lagrangian acceleration is the same as the material time derivative of $\underline{q}_E$ which is

$$\frac{\mathrm{D}\underline{q}_E}{\mathrm{D}t} = \frac{\partial \underline{q}_E}{\partial t} + \underline{q}_E \cdot \nabla \underline{q}_E = \underline{q}_E \cdot \nabla \underline{q}_E$$

in this case.

Let u be such that $\underline{q}_E = u\,\underline{i}$.

$$u = \omega\sqrt{1-(x-x_0)^2}.$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{2}\left(1-(x-x_0)^2\right)^{-1/2}(-2(x-x_0)) = -\frac{\underline{\omega}(x-x_0)}{\sqrt{1-(x-x_0)^2}}.$$

In this case

$$\underline{q}_E \cdot \nabla \underline{q}_E = u\frac{\partial u}{\partial x}\underline{i}$$

and

$$u\frac{\partial u}{\partial x} = -\omega^2(x-x_0) = \omega^2(x_0-x).$$

This is the same as the term in $\underline{a}_L$ as

$$x_0 - x = \omega^2 \cos\left(\omega t + \frac{\pi}{2}\right).$$

11. *Apart from notational changes and a slight re-wording, this was part of question 3 of the* MA2941 *paper in 2009. The other parts may appear in the next exercise sheet.*

A fluid flows inside a semi-infinite pipe whose surface is described by the Cartesian position vector

$$\underline{r} = \cos\, u\, \underline{i} + 2\sin\, u\, \underline{j} + v\, \underline{k}, \quad 0 \le v \le 2\pi, \quad -\infty < v < 0.$$

The fluid velocity is given by

$$\underline{q} = y\, \underline{i} - 4x\, \underline{j} + z\, \underline{k}.$$

(i) Determine whether or not the flow is incompressible.

[3 marks]

Answer

The flow is incompressible if

$$\nabla \cdot \underline{q} = 0.$$

In this case

$$\nabla \cdot \underline{q} = \frac{\partial y}{\partial x} + \frac{\partial(-4x)}{\partial y} + \frac{\partial z}{\partial z} = 1$$

and hence the flow is not incompressible.

---

12. *Apart from notational changes, this was part of question 3 of the* MA2941 *paper in 2008. The other parts may appear in the next exercise sheet.*

A motorist is pouring petrol into their car through a funnel, the surface of which is described by the Cartesian position vector

$$\underline{r} = \mathrm{e}^{-v} \cos\, u\underline{i} + \mathrm{e}^{-v} \sin\, u\underline{j} - v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

Suppose that the fluid velocity is given by

$$\underline{q} = -(x+y)\underline{i} + x\underline{j} - vk.$$

(i) Determine whether or not the flow is incompressible.

[3 marks]

Answer

The flow is incompressible if

$$\nabla \cdot \underline{q} = 0.$$

In this case

$$\nabla \cdot \underline{q} = \frac{\partial(-x-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(-1)}{\partial z} = -1.$$

Hence the flow is not incompressible.
