MA2741: Spring Term – Exercise sheet 2 relating to chapter 2: Particle paths, streamlines, material time derivative etc.

Note: In one of the question circulation about a closed curve C is mentioned and this is

$$\oint_C \underline{q} \cdot \mathrm{d}\underline{r}$$

and will be mentioned in chapter 3. Incompressibility will be considered in the lectures from week 19 and examples in the lectures involving streamlines will probably not be done until week 19.

1. Suppose that a particle starting at $\underline{r}_0$ at time $t = 0$ is at position $\underline{r}(\underline{r}_0, t)$ at time t and suppose that we have a function $f_L(\underline{r}_0, t)$. If we define $f(\underline{r}, t)$ by the relation

$$f(\underline{r}(\underline{r}_0, t), t) = f_L(\underline{r}_0, t)$$

then show that

$$\frac{\partial}{\partial t} f_L(\underline{r}_0, t) = \frac{\partial f}{\partial t} + \underline{q} \cdot \nabla f$$

where $\underline{q}$ denotes the velocity of the particle being considered.

2. The Lagrangian description of a flow is

$$x = x_0 e^{\alpha t}, \quad y = y_0 e^{-\alpha t}, \quad z = z_0$$

where α is a positive constant and (x_0, y_0, z_0) is the initial position of a given particle.

i) Write down the position vector of a fluid particle. Determine the Lagrangian velocity and write this in Eulerian form.

ii) Show that $\nabla \cdot \underline{q}_E = 0$ where $\underline{q}_E$ denotes the Eulerian velocity.

iii) Determine the Lagrangian acceleration, the local accelerations and the convective acceleration and use the material time derivative relation to show that the two results are consistent.

iv) Determine the streamlines for the flow. Do they differ from the particle paths?

3. An incident at a chemical plant causes the release of a pollutant into a river which flows with velocity

$$\underline{q} = \alpha x \underline{i} - \alpha y \underline{j}, \quad y > 0.$$

It is known that the concentration of the pollutant at position (x, y) and time t is described by

$$c(x, y, t) = \beta x^2 y e^{-\alpha t}$$

where β is constant. Determine whether or not the pollutant concentration for a typical particle changes with time?

4. Consider the unsteady, 2-D flow described by

$$\underline{q} = U\underline{i} + \alpha t\underline{j}, \quad t > 0$$

where U and α are positive constants. Show that the streamlines are straight lines and that any fluid particle follows a parabolic path as time proceeds.

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5. *Apart from notational changes, this was the most of question 3 of the* MA2841 *paper in 2012. The other part of the question was on the previous exercise sheet.*

A factory worker is filling barrels of beer using a funnel, the surface of which is described by the Cartesian position vector

$$\underline{r} = \mathrm{e}^{-v}\cos u\underline{i} + \mathrm{e}^{-v}\sin u\underline{j} - v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

Given that the fluid velocity within the funnel is

$$\underline{q} = (x + 2y)\underline{i} - y\underline{j} - 3\underline{k},$$

determine whether or not the flow is incompressible.

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6. *Apart from notational changes and re-wording this was question 4 of the* MA2841 *paper in 2012.*

A scientist releases some dye into a river and notices that the dye colours an ellipsoidal shaped volume of water and the surface of the ellipsoidal is observed to vary with time t according to

$$F(x, y, t) = 1 \quad \text{where } F(x, y, t) = \frac{x^2}{1+t} + y^2\mathrm{e}^{-t}, \quad 0 \le t < T.$$

(i) Show that if the velocity $\underline{q}$ of the river is given by

$$\underline{q} = \frac{x}{2(1+t)}\underline{i} + y\underline{j}, \quad 0 \le t < T$$

then a point on the surface at time $t = 0$ remains on the surface for all t, $0 \le t < T$.

(ii) Show that the particle paths are given by

$$y = y_0 \exp\left(\frac{x^2}{x_0^2} - 1\right)$$

where (x_0, y_0) is the initial position of a particle.

(iii) Determine the streamlines of the flow.

(iv) Suppose that at time $t = T$, $T > 0$, it starts to rain and this alters the fluid velocity such that

$$\underline{q} = \frac{y}{2}\underline{i} + x\underline{j}, \quad t \ge T.$$

Determine, for this velocity field, the circulation about the curve C described by $x^2 + y^2 = 1$.

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7. *Apart from notational changes, this was most of question 3 of the* MA2841 *paper in 2011. The remaining parts of the question (parts (ii) and (iii)) may appear on the next exercise sheet.*

   The surface of a bowl is mapped by the position vector

   $$\underline{r} = v \cos u \, \underline{i} + v \sin u \, \underline{j} + v^2 \, \underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

   A fluid within the bowl moves such that its velocity is given by

   $$\underline{q} = \frac{y^2}{x} \, \underline{i} - y \underline{j}, \quad x > 0.$$

   (i) Determine whether or not the flow is incompressible.

   [3 marks]

   (iv) A fluid particle is initially at the position (x_0, y_0, z_0) within the bowl, with $x_0 > 0$. Show that its location for subsequent time is described by the position vector

   $$\underline{r} = \sqrt{x_0^2 + y_0^2 - y_0^2 \mathrm{e}^{-2t}} \, \underline{i} + y_0 \mathrm{e}^{-t} \, \underline{j} + z_0 \underline{k}, \quad t \ge 0,$$

   where, as usual, the square root notation means that the positive square root is taken.

   [8 marks]

   (v) Show that the particle moves in a circle in the plane $z = z_0$.

   [2 marks]

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8. *Apart from notational changes, this was most of question 3 of the* MA2941 *paper in 2010. The remaining parts of the question (part (ii)) may appear on the next exercise sheet.*

A fluid flows such that its velocity is given by

$$\underline{q} = 3\underline{i} + 2tx\,\underline{j} - z\,\underline{k}, \quad t \geq 0$$

where t is time.

(i) Determine whether or not the flow is incompressible.

[3 marks]

(iii) A fluid particle is initially at the position (x_0, y_0, z_0). Show that its location for subsequent time is described by the position vector

$$\underline{r} = (3t + x_0)\underline{i} + (2t^3 + x_0t^2 + y_0)\underline{j} + z_0\mathrm{e}^{-t}\underline{k}, \quad t \geq 0.$$

[6 marks]

(iv) Show that particles initially in the plane $z = 0$ remain in this plane and find the corresponding particle paths in parametric form.

[4 marks]

(v) Determine the streamlines for particles that move in the plane $z = 0$ and sketch them for $t = 1$.

[4 marks]

9. *Apart from notational changes and a little re-wording this was question 4 of the* MA2941 *paper in 2009.*

   (a) A fluid particle moves in a two-dimensional (2-D) space with position vector

   $$\underline{r} = 2\cos t\,\underline{i} + (1 + \sin t)\underline{j}.$$

   (i) Show that the particle path is an ellipse.

   [3 marks]

   (ii) Find the Lagrangian velocity $\underline{q}_L$, and show that this can be expressed in Eulerian form as

   $$\underline{q}_E = -2(y-1)\underline{i} + \frac{x}{2}\,\underline{j}.$$

   [3 marks]

   (iii) Determine the Lagrangian and local accelerations.

   [2 marks]

   (iv) Use the material time derivative to show that your results for part (iii) are consistent.

   [5 marks]

   (b) An accident at a chemical plant causes the release of a toxic pollutant into a river. The concentration of the pollutant is given by

   $$c(x, y, t) = \sinh\left(\frac{x}{t}\right) \mathrm{e}^{-ty}, \quad t > 0.$$

   Given that the pollutant concentration in a given fluid element does not vary with time, determine a two-dimensional (2-D) velocity of the river consistent with this and determine the streamlines of this flow.

   [7 marks]

10. *Apart from notational changes and a little re-wording this was part of question 4 of the* MA2941 *paper in 2008.*

For a particular fluid motion, the position vector of a typical particle is

$$\underline{r} = \left(x_0 - \cos\left(\omega t + \frac{\pi}{2}\right)\right)\underline{i} + y_0\underline{j}$$

where ω denotes a constant.

(i) Describe the motion of the particle.

[2 marks]

(ii) Find the Lagrangian velocity $\underline{q}_L$ and show that when $|t| \le \pi/(2\omega)$ this can be expressed in Eulerian form as

$$\underline{q}_E = \omega\sqrt{1-(x-x_0)^2}\,\underline{i}.$$

[3 marks]

(iii) Determine the Lagrangian and local accelerations.

[2 marks]

(iv) Use the material time derivative to show that your results for part (iii) are consistent.

[3 marks]

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11. *Apart from notational changes and a slight re-wording, this was part of question 3 of the* MA2941 *paper in 2009. The other parts may appear in the next exercise sheet.*

A fluid flows inside a semi-infinite pipe whose surface is described by the Cartesian position vector

$$\underline{r} = \cos\,u\,\underline{i} + 2\sin\,u\,\underline{j} + v\,\underline{k}, \quad 0 \le v \le 2\pi, \quad -\infty < v < 0.$$

The fluid velocity is given by

$$\underline{q} = y\,\underline{i} - 4x\,\underline{j} + z\,\underline{k}.$$

(i) Determine whether or not the flow is incompressible.

[3 marks]

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12. *Apart from notational changes, this was part of question 3 of the* MA2941 *paper in 2008. The other parts may appear in the next exercise sheet.*

A motorist is pouring petrol into their car through a funnel, the surface of which is described by the Cartesian position vector

$$\underline{r} = \mathrm{e}^{-v}\cos\,u\underline{i} + \mathrm{e}^{-v}\sin\,u\underline{j} - v\underline{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

Suppose that the fluid velocity is given by

$$\underline{q} = -(x+y)\underline{i} + x\underline{j} - vk.$$

(i) Determine whether or not the flow is incompressible.

[3 marks]
