Answers to some MA2815 revision questions sent by email relating to MA2715

Contents

1 Chap 1, norms etc questions

1.1 Why is the vector $(1,-1,1)^T$ in question 9

The matrix is

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -5 & 1 \\ 4 & 1 & 1 \end{pmatrix}.$$

The largest sum of magnitudes occurs with row 2 and the value is 7. The second entry of $A\underline{x}$ is

$$(A\underline{x})_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = x_1 - 5x_2 + x_3.$$

When we restrict to vectors $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ we have

$$(A\underline{x})_2 = x_1 - 5x_2 + x_3 = 7 \quad \text{with} \quad x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$$

and we have

$$(A\underline{x})_2 = x_1 - 5x_2 + x_3 = -7 \quad \text{with} \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = -1.$$

The question says "give a vector $\underline{x}$" and this shows there are two different unit vectors in this norm such that $\|A\underline{x}\|_\infty = \|A\|_\infty$.

2 Chap 2, LU, columns of an inverse etc questions

2.1 Can we use any row operations to get a LU factorization?

No. You can use any row operations to reduce a general system to upper triangular form, as you did in year 1, but only a subset of these operations gives the upper triangular matrix U in a factorization $A = LU$ when the matrix has such a factorization.

The basic Gauss elimination process is one ways of doing things and there is no choice in the operations. When you are reducing in column 1 below the diagonal you are subtracting multiples of row 1 from the rows below. More generally, when you are reducing in column k below the diagonal you are subtracting multiples of row k from the rows $k+1 \ldots n$ when the matrix is $n \times n$. You should look at section 2.4 involving Gauss transformation matrices to understand why the entries in L below the diagonal are the multipliers used in the reduction process.

If you believe that you have matrices L and U for a given matrix A then you can always compute the matrix product LU and check if is the same as the matrix A.

2.2 How do you get the multipliers in the elimination process?

See the answer in section 2.1 for some of the explanation.

At the kth stage the current kth row is used to eliminate in rows $k+1, \ldots, n$. The multipliers are

$$m_{ik} = \frac{a_{ik}}{a_{kk}}, \quad i = k+1, \ldots, n.$$

When we subtract m_{ik} times row k from row i the next matrix has 0 in the i,k position.

2.3 What do you mean by a principal sub-matrix?

For a $n \times n$ matrix $A = (a_{ij})$ the $k \times k$ principal sub-matrix is

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$

The 1×1 and 2×2 cases are hence

$$(a_{11}) \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

2.4 What do you mean by a singular/non-singular matrix?

When A is a square matrix the term singular means that it is not invertible, i.e. $\det(A) = 0$, and the term non-singular means that it is invertible, i.e. $\det(A) \neq 0$. In the context of chapter 2 material we can uniquely solve a linear system of the form $A\underline{x} = \underline{b}$ when the matrix is non-singular.

2.5 Under what conditions does a matrix not have an LU factorisation?

A non-singular matrix A has a LU factorization if the basic Gauss elimination process runs to completion and this happens if and only if all the principal sub-matrices are non-singular. In this non-singular case and a $n \times n$ matrix A this requires that $a_{11} \neq 0$, the 2×2 principal sub-matrix is non-singular, ..., the $(n-1) \times (n-1)$ principal sub-matrix is non-singular. The 3×3 non-singular matrices in the exercises which do not have a LU factorization have either had $a_{11} = 0$ or the 2×2 principal submatrix has determinant of

My notes from about page 2-10 has further information about this. When the matrix is non-singular there is a re-arrangement of the rows which gives a matrix which does have a LU factorization. This was described in the section on pivoting when we had $PA = LU$. Here P is a permutation matrix.

It can be the case that a singular matrix has a LU factorization although this will involve at least one of the diagonal entries of U being 0. The 3×3 matrix B in question 3 on the exercise sheet is a singular matrix which has a LU factorization with $u_{33} = 0$.

2.6 How do you get the third column of the inverse of A?

The third column $\underline{x}$ of the inverse of A is described by

$$\underline{x} = A^{-1}\underline{e}_3,$$

where $\underline{e}_3$ is the 3rd base vector, i.e the 3rd column of the identity matrix I. If we multiply by A then

$$A\underline{x} = \underline{e}_3.$$

We get $\underline{x}$ by solving a linear with $\underline{e}_3$ as the right hand side vector.

When $A = LU$ the system $A\underline{x} = LU\underline{x}3 = \underline{e}_3$ is solved by defining $\underline{y} = U\underline{x}$ and noting that $L\underline{y} = LU\underline{x} = \underline{e}_3$. We have two triangular systems,

$$\begin{aligned} L\underline{y} &= = \underline{e}_3, \\ U\underline{x} &= = \underline{y}. \end{aligned}$$

In this case we immediately get $\underline{y} = \underline{e}_3$ $\underline{x}$ is the 3rd column of the inverse of U.

If we want instead the 1st or 2nd columns of the inverse of A then the step to get the corresponding vector $\underline{y}$ involves a bit more work.

2.7 2019 Qu 10, do you need to show the first Gauss elimination step for matrix C?

The matrix C in the question is as follows.

$$C = \begin{pmatrix} 3 & 1 & -4 \\ 9 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$

This matric does not have a LU factorization. For the reason it is sufficient here to note that the 2-by-2 principal submatrix of C, i.e.

$$\begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix},$$

is singular or give an equivalent term such as has zero determinant or that the rows or columns are linearly dependent.

In the solution available on Blackboard I have the following.

$$C = \begin{pmatrix} 3 & 1 & -4 \\ 9 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & -4 \\ 0 & 0 & 12 \\ 0 & -2/3 & 2/3 \end{pmatrix}$$

Basic Gauss elimination cannot continue as the $2,2$ entry is 0. The 2×2 principal submatrix is singular.

When you do the basic Gauss elimination operations the determinant of the principal submatrices is the same for each matrix in the sequence of matrices generated.

3 Chap 3, Systems of ODEs questions

4 Chap 4, Taylor expansions and related finite differences questions

4.1 How do you get the expression in the second part of the May 2019 question

The manipulation involved when combining Taylor series when dealing with finite difference approximations is similar that you would probably have done in the module `MA2730`.

In the question all the evaluations in the right hand side are at $x = 0$.

The first step involves the standard Taylor series for $u(h)$.

$$u(h) = u(0) + u'(0)h + \frac{u''(0)}{2}h^2 + \frac{u'''(0)}{6}h^3 + \cdots$$

The series for $u(3h)$ is then obtained by just replacing h by $3h$ and when this is done remember that $(3h)^2 = 9h^2$, $(3h)^3 = 27h^3$ etc.

$$\begin{aligned} u(3h) &= u(0) + u'(0)(3h) + \frac{u''(0)}{2}(3h)^2 + \frac{u'''(0)}{6}(3h)^3 + \cdots \\ &= u(0) + u'(0)(3h) + \frac{u''(0)}{2}9h^2 + \frac{u'''(0)}{6}27h^3 + \cdots \end{aligned}$$

The expression for $9u(h) - u(3h)$ is obtained by multiply the $u(h) =$ series by 9 and subtracted the $u(3h) =$ series. The composite version has terms in $u(0)$, $hu'(0)$, $(h^2/2)u''(0)$ and $(h^3/6)u'''(0)$ and each coefficient is the combination of 2 terms and at this stage we get

$$9 - 1 = 8, \quad 9 - 3 = 6, \quad 9 - 9 = 0 \quad \text{and} \quad 9 - 27 = -18.$$

In full we have

$$\begin{aligned} -u(3h) + 9u(h) &= 8u(0) + u'(0)(6h) + \frac{u'''(0)}{6}(-27 + 9)h^3 + \cdots \\ &= 8u(0) + u'(0)(6h) + u'''(0)(-3h^3) + \cdots . \end{aligned}$$

The last step is just to put things together as in the question and this involves

$$\frac{-u(3h) + 9u(h) - 8u(0)}{6h} = u'(0) - \frac{u'''(0)}{2}h^2 + \cdots$$

Hence $c_1 = 1$, $c_2 = 0$ and $c_3 = -1/2$.

4.2 You sometimes write terms such $\mathcal{O}(h^6)$. When you replace h by $2h$ why do you still write $\mathcal{O}(h^6)$?

To shorten the expressions we let $u_i = u(x_i)$, $u_i' = u'(x_i)$, $u_i'' = u''(x_i)$, etc.. With these abbreviations we have the following Taylor expansions about the point x_i:

$$\begin{aligned} u_{i+1} &= u(x_i+h) = u_i + hu_i' + \frac{h^2}{2!}u_i'' + \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i'''' + \frac{h^5}{5!}u_i^{(5)} + \cdots, \\ u_{i-1} &= u(x_i-h) = u_i - hu_i' + \frac{h^2}{2!}u_i'' - \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i'''' + \frac{h^5}{5!}u_i^{(5)} + \cdots. \end{aligned}$$

Instead of putting $\cdots$ I might put

$$\begin{aligned} u_{i+1} &= u(x_i+h) = u_i + hu_i' + \frac{h^2}{2!}u_i'' + \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i'''' + \frac{h^5}{5!}u_i^{(5)} + \mathcal{O}(h^6), \\ u_{i-1} &= u(x_i-h) = u_i - hu_i' + \frac{h^2}{2!}u_i'' - \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i'''' - \frac{h^5}{5!}u_i^{(5)} + \mathcal{O}(h^6). \end{aligned}$$

The context here is that h is small and when the above is written the last term involves h^6 and we are not concerned with the other detail. If we now replace h by $2h$ then we still write

$$\begin{aligned} u_{i+2} &= u(x_i+2h) = u_i + 2hu_i' + \frac{(2h)^2}{2!}u_i'' + \frac{(2h)^3}{3!}u_i''' + \frac{(2ha)^4}{4!}u_i'''' + \frac{(2h)^5}{5!}u_i^{(5)} + \mathcal{O}(h^6), \\ u_{i-2} &= u(x_i-2h) = u_i - 2hu_i' + \frac{(2h)^2}{2!}u_i'' - \frac{(2h)^3}{3!}u_i''' + \frac{(2h)^4}{4!}u_i'''' - \frac{(2h)^5}{5!}u_i^{(5)} + \mathcal{O}(h^6). \end{aligned}$$

The last term still involves h^6 although the other details are different from the previous case. With this convention we have

$$\begin{aligned} u_{i+1} + u_{i-1} &= 2\left(u_i + \frac{h^2}{2!}u_i'' + \frac{h^4}{4!}u_i'''' + \mathcal{O}(h^6)\right) \\ u_{i+2} + u_{i-2} &= 2\left(u_i + \frac{4h^2}{2!}u_i'' + \frac{16h^4}{4!}u_i'''' + \mathcal{O}(h^6)\right), \\ 16(u_{i+1} + u_{i-1}) - (u_{i+2} + u_{i-2}) &= (32-2)u_i + (16-4)h^2u_i'' + \mathcal{O}(h^6). \end{aligned}$$

We have not attempted to determine the precise details of the last term other than to deduce that the power of h is h^6.

5 Chap 5, Fourier series questions

5.1 Why do you replace $(-\pi, \pi)$ with $(0, \pi)$ sometimes when evaluating integrals?

This is usually done when the integrand is an even function of x. If $g(x)$ denotes any integrand then by properties of the integral

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = \int_{-\pi}^{0} g(x)\,\mathrm{d}x + \int_{0}^{\pi} g(x)\,\mathrm{d}x.$$

If $g(x)$ is even, i.e. if $g(-x) = g(x)$ at all points of continuity, then

$$\int_{-\pi}^{0} g(x)\,\mathrm{d}x = \int_{0}^{\pi} g(x)\,\mathrm{d}x.$$

Hence

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = \int_{-\pi}^{0} g(x)\,\mathrm{d}x + \int_{0}^{\pi} g(x)\,\mathrm{d}x = 2\int_{0}^{\pi} g(x)\,\mathrm{d}x.$$

This was used in the Fourier series examples when $f(x)$ is odd and when $f(x)$ is even. When $f(x)$ is an odd function the integrands of the form

$$g(x) = f(x)\sin(nx)$$

is even. When $f(x)$ is an even function the integrands of the form

$$g(x) = f(x)\cos(nx)$$

is even.

The subsection on half range Fourier series had formula for the Fourier coefficients which made use of the above.

5.2 Why did you replace $(0, \pi)$ by $(\pi/3, \pi)$ in the 2017 question?

In the 2017 question the function $f(x)$ is an odd function and also $f(x) = 0$ when $x \in (-\pi/3, \pi/3)$. Thus

$$\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \int_{0}^{\pi/3} f(x)\sin(nx)\,\mathrm{d}x + \int_{\pi/3}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \int_{\pi/3}^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

As the integrand is 0 in $(0, \pi/3)$ the first part is 0.

When $f(x)$ is piecewise defined in $(-\pi, \pi)$ with say $a_0 = -\pi < a_1 < \cdots a_m = \pi$ denoting the division points then

$$\int_{a_0}^{a_m} g(x)\,\mathrm{d}x = \int_{a_0}^{a_1} g(x)\,\mathrm{d}x + \cdots + \int_{a_{m-1}}^{a_m} g(x)\,\mathrm{d}x.$$

5.3 For 2π-periodic functions can we replace $(-\pi, \pi]$ by $[0, 2\pi)$ throughout?

The answer depends here on what has already been set-up. If the 2π-periodic function $f(z)$ has already been defined then in the formula for the coefficients a_n and b_n we can replace $(-\pi, \pi)$ by $(0, 2\pi)$ or indeed by any interval of length 2π.

If you are considering a function on an interval of length 2π which you then extend in a 2π periodic way to be defined on $\mathbb{R}$ then the extended function will depend on which interval is used. For example, if $f(x) = x$ defined on $(-\pi, \pi)$ is extended in a 2π-periodic way then the 2π-periodic function is an odd function at all points of continuity. However, if $g(x) = x$ defined on on $(0, 2\pi)$ is extended in a 2π-periodic way then the 2π-periodic function is not the same function. There are of course some connections between $f(x)$ and $g(x)$ but, to repeat, they are not exactly the same.

5.4 Does it matter if we write $(-1)^{m-1}$ or $(-1)^{m+1}$?

A term such as $(-1)^n$ is either 1 or -1. In particular, $(-1)^n = 1$ when n is even and $(-1)^n = -1$ when n is odd. If 2 values of n differ by an even number then we have the same number. Thus we can write $(-1)^{m-1}$ or $(-1)^{m+1}$ or $-(-1)^m$ as they all represent the same number.

6 Not specific to a chapter questions

6.1 What study module codes are assessed in the 3-hour MA2815 exam

Since May 2018 it is just MA2712 and MA2715.
MA2712=Multivariable Calculus
MA2715=Advanced Calculus and Numerical Methods

6.2 What topics from MA2715 might be in the exam?

The exam questions from previous years gives an indication of topics. If you take the ones put on Blackboard then the following are relevant topics.

2017 paper

	Topic
Q1	Fourier series. (Somebody else mainly taught this in 2017.)
Q9	Vector and matrix norms.
Q10	$A = LU$ and a column of the inverse.
Q11	Solution of a system of ODEs.
Q12	Taylor expansions and finite difference approximations.

2018 paper

	Topic
Q7	Vector and matrix norms.
Q8	LU factorizations
Q9	Solution of a system of ODEs.
Q10	Taylor expansions and finite difference approximations.
Q11	Fourier series

2019 paper

	Topic
Q8	Vector and matrix norms.
Q9	Forward substitution to get a column of L^{-1}
Q10	LU factorizations
Q11	Solution of a system of ODEs.
Q12	Taylor expansions and finite difference approximations.
Q13	Fourier series

Please note that each year the exam is different. I do update the exercise sheets each year and thus everything that are in these are relevant to this academic year.