Chap 5: Fourier series for periodic functions

Let $f : (-\pi, \pi] \to \mathbb{R}$ be a bounded piecewise continuous function which we continue to be a 2π-periodic function defined on $\mathbb{R}$, i.e.

$$f(x + 2\pi) = f(x), \quad \forall x \in \mathbb{R}.$$

The Fourier series of this function is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x.$$

Sufficient conditions for

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

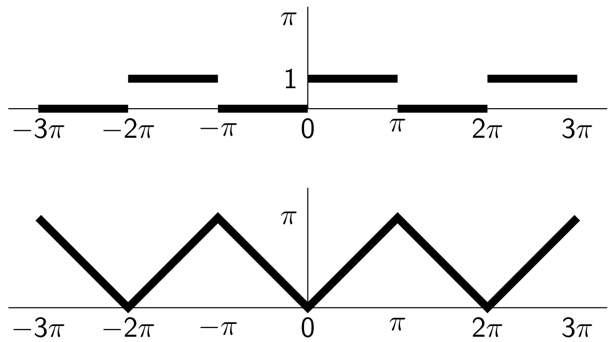
will be considered later.

Examples of 2π periodic functions

Let $f_1(x)$ and $f_2(x)$ be defined on $(-\pi, \pi]$ as follows.

$$f_1(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi, \\ 0, & \text{if } -\pi < x < 0, \end{cases} \quad \text{and} \quad f_2(x) = |x|.$$

Both of these are continued 2π-periodically and a sketch of both is shown below.



Functions which are orthogonal on $(-\pi, \pi)$

Functions f and g defined on $(-\pi, \pi)$ are orthogonal if

$$\int_{-\pi}^{\pi} f(x) g(x)\,\mathrm{d}x = 0.$$

The following functions are orthogonal to each other in this sense.

$$1,\ \cos(x),\ \sin(x)\ \cos(2x),\ \sin(2x), \ldots, \cos(nx),\ \sin(nx), \ldots.$$

These functions are also 2π-periodic with 2π being the least period for $\cos(x)$ and $\sin(x)$.

Addition formulas to help evaluate the integrals

In complex form we have

$$\mathrm{e}^{i(a+b)} = \mathrm{e}^{ia}\mathrm{e}^{ib}, \quad \mathrm{e}^{i(a-b)} = \mathrm{e}^{ia}\mathrm{e}^{-ib}.$$

By adding and subtracting the expanded version of the above and taking real and imaginary parts leads to

$$\begin{aligned} \cos(a)\cos(b) &= \frac{\cos(a+b) + \cos(a-b)}{2}, \\ \sin(a)\cos(b) &= \frac{\sin(a+b) + \sin(a-b)}{2}, \\ \sin(a)\sin(b) &= \frac{\cos(a-b) - \cos(a+b)}{2}, \\ \cos(a)\sin(b) &= \frac{\sin(a+b) - \sin(a-b)}{2}. \end{aligned}$$

If a and b are integers then $a \pm b$ are integers. When $p \ne 0$ is a integer

$$\int_{-\pi}^{\pi} \cos(px)\,\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi} \sin(px)\,\mathrm{d}x = 0.$$

When $f \neq g$ and when $f = g$

If f and g are from the list

$$1, \cos(x), \sin(x) \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots.$$

and $f \neq g$ then

$$\int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x = 0.$$

When $f(x) = g(x)$ we have the following cases.

$$\begin{aligned} \int_{-\pi}^{\pi} \mathrm{d}x &= 2\pi, \\ \int_{-\pi}^{\pi} \cos^2(nx) \, \mathrm{d}x &= \pi, \\ \int_{-\pi}^{\pi} \sin^2(nx) \, \mathrm{d}x &= \pi. \end{aligned}$$

The formula for the Fourier coefficients a_n and b_n

If our starting point are numbers a_n and b_n such that the series converges to define

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

and the convergence is such that all operations such as interchanging infinite sums and integrals are valid then the orthogonality properties imply the following.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x &= a_0 \pi, \\ \int_{-\pi}^{\pi} f(x) \cos(mx) \, \mathrm{d}x &= a_m \pi, \\ \int_{-\pi}^{\pi} f(x) \sin(mx) \, \mathrm{d}x &= b_m \pi. \end{aligned}$$

This gives a justification for the formulas used.

The Heaviside function on $(-\pi, \pi]$

Let

$$f_1(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi, \\ 0, & \text{if } -\pi < x < 0 \end{cases}$$

which we continue in a 2π-periodic way.

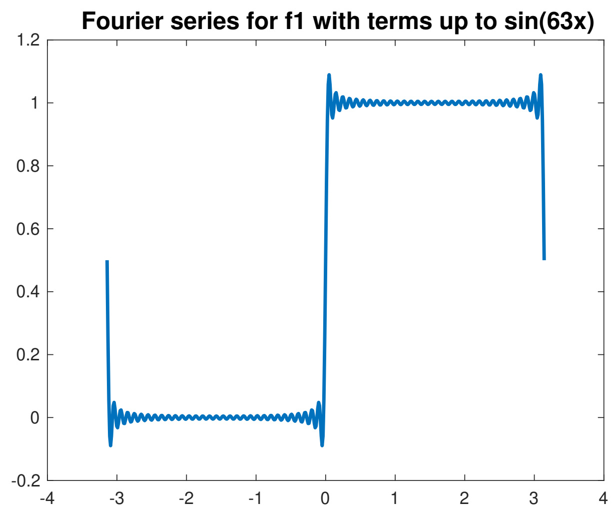
Determining the coefficients gives the following.

$$f_1(x) \sim \frac{1}{2} + \frac{2}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin((2n-1)x}{2n-1} + \cdots \right).$$

We can immediately observe that when $x = k\pi$ the value of the series is $1/2$. The series does not converge to $f_1(x)$ at these points of discontinuity.

A plot of a truncated Fourier series for $f_1(x)$

$$S_{63}(x) = \frac{1}{2} + \frac{2}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin(63x)}{63} \right).$$



The function $f_2(x) = |x|$ on $(-\pi, \pi]$

In this case convergence occurs at all points and we can write for $|x| \le \pi$.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos((2n-1)x)}{(2n-1)^2} + \cdots\right).$$

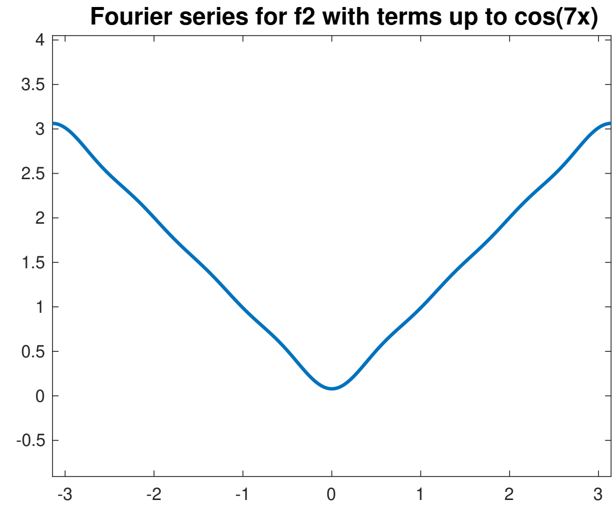
The 2π-periodic function extension is continuous at all points.

Observe that $f_2(x)$ is an even function of x which is why $b_n = 0$ and only cosine terms are involved. Integration by parts is used to determine

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\,\cos(nx)\,\mathrm{d}x.$$

A plot of a truncated Fourier series for $f_2(x)$

$$S_7(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2}\right).$$



The function $f_3(x) = x$ on $(-\pi, \pi]$

In this case for $x \in (-\pi, \pi)$ we have convergence to $f_3(x)$ and we can write

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right).$$

We can immediately observe that when $x = \pi$ the value of the series is 0. The series does converge here but not to $f_3(\pi) = \pi$ or to the limit as we tend to $-\pi$ which is $-\pi$. The 2π-periodic extension is discontinuous at the points $(2k+1)\pi$, $k \in \mathbb{Z}$.

Observe that $f_3(x)$ is an odd function of x which is why $a_n = 0$ and only sine terms are involved. Integration by parts is used to determine

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\,\sin(nx)\,\mathrm{d}x.$$

A plot of a truncated Fourier series for $f_3(x)$

$$S_{64}(x) = 2\sum_{n=1}^{64} \frac{(-1)^{n+1}}{n}\sin(nx).$$