2019 Fourier series question, a piecewise defined function, sketching, odd-even and points of discontinuity

Let f_1 and f_2 be 2π-periodic function defined on $(-\pi, \pi]$ as follows.

$$
\begin{aligned}
f_1(x) &= \begin{cases} 1, & \text{if } |x| \le \pi/2, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi, \end{cases} \\
f_2(x) &= \begin{cases} 1, & \text{if } 0 \le x \le \pi/2, \\ -1, & \text{if } -\pi/2 \le x < 0, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi. \end{cases}
\end{aligned}
$$

In both cases the expressions giving the value of the function at a point x depends on which part of $(-\pi, \pi]$ the point lies.

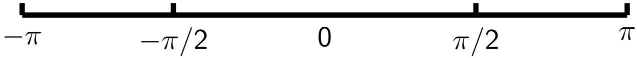
For $f_1(x)$ the ranges x are $(-\pi, -\pi/2)$, $[-\pi/2, \pi/2]$ and $(\pi/2, \pi/2]$.

For $f_2(x)$ the ranges x are $(-\pi, -\pi/2]$, $(-\pi/2, 0]$ $(0, -\pi/2)$ and $[\pi/2, \pi)$.

The function $f_1(x)$ on $(-\pi, \pi]$

For $f_1(x)$ the ranges x are $(-\pi, -\pi/2)$, $[-\pi/2, \pi/2]$ and $(\pi/2, \pi/2]$.

Planning: The x-axis.



The "join points" in $(-\pi, \pi)$ are $x = -\pi/2$ and $x = \pi/2$.

The function values near the join points?

As x increases $f_1(x)$ abruptly changes from 0 to 1 as we move through $-\pi/2$.

As x increases $f_1(x)$ abruptly changes from 1 to 0 as we move through $\pi/2$.

The points $x = \pm\pi/2$ are points of discontinuity of $f_1(x)$.

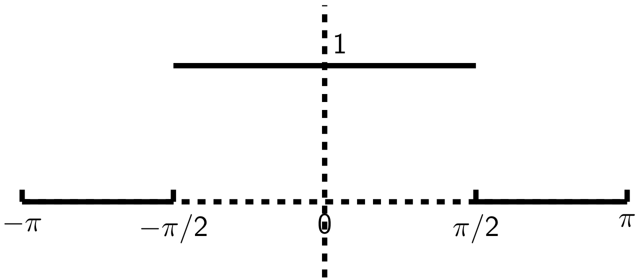
When we restrict to $(-\pi, \pi)$ the Fourier series for $f_1(x)$ is the same as $f_1(x)$ at all points of continuity of $f_1(x)$ which is all points in $(-\pi, \pi)$ except the two points $x = \pm\pi/2$.

$f_1(x)$ on $(-\pi, \pi]$, a sketch

Recall again the definition of f_1.

$$f_1(x) = \begin{cases} 1, & \text{if } |x| \le \pi/2, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi, \end{cases}$$

There are only two different values of the function and it is described as a piecewise constant function. We have enough information to give a sketch.

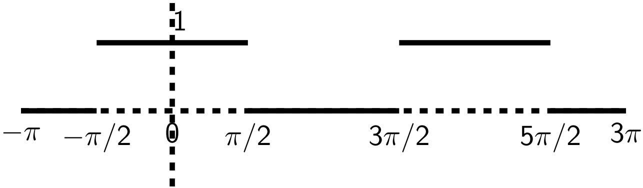


$f_1(x)$ extending the sketch to $(-\pi, 3\pi)$

Before this it is given note that $f_1(\pi) = 0$ and the limit of $f_1(x) as x$ tends to $-\pi$ is also 0. The 2π-periodic version is continuous at such points.

The points of discontinuity in $(-\pi, \pi)$ are $x = \pm\pi/2$ and $x = -\pi/2 + 2\pi$ and $\pi/2 + 2\pi$.

The sketch on $(-\pi, 3\pi)$ is given next.



Is f_1, odd, even or neither?

A function $g(x)$ is even if $g(-x) = g(x)$.

A function $g(x)$ is odd if $g(-x) = -g(x)$.

In the Fourier series context where the functions are piecewise defined we can restrict this to be just being satisfied at the points of continuity.

If neither property holds then the function is not an even function and it is not an odd function.

In the case of $f_1(x)$ we have an even function and we confirm this by noting that it holds in $(-\pi/2, \pi/2)$ and it holds in the "outer parts", i.e. in $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$.

How does the even property of f_1 affect things?

As $f_1(x)$ is even and as $\cos(nx)$ is even the product $g(x) = f_1(x)\cos(nx)$ is even and we have the following two results. For any integrand we have

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = \int_{-\pi}^{0} g(x)\,\mathrm{d}x + \int_{0}^{\pi} g(x)\,\mathrm{d}x$$

and as $g(x)$ is even we have

$$\int_{-\pi}^{0} g(x)\,\mathrm{d}x = \int_{0}^{\pi} g(x)\,\mathrm{d}x.$$

Thus

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = 2\int_{0}^{\pi} g(x)\,\mathrm{d}x$$

and

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f_1(x)\cos(nx)\mathrm{d}x.$$

The even property of f_1 and $b_n = 0$

As $f_1(x)$ is even and as $\sin(nx)$ is odd the product $g(x) = f_1(x)\sin(nx)$ is odd and we have the following two results. For any integrand we have

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = \int_{-\pi}^{0} g(x)\,\mathrm{d}x + \int_{0}^{\pi} g(x)\,\mathrm{d}x$$

and as $g(x)$ is odd we have

$$\int_{-\pi}^{0} g(x)\,\mathrm{d}x = -\int_{0}^{\pi} g(x)\,\mathrm{d}x.$$

Thus

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = 0.$$

and

$$b_n = 0.$$

An even function has a Fourier series which only has cosine terms.

Determining a_n with $f_1(x)$ being piecewise defined

We still need to compute the following.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \cos(nx) \mathrm{d}x.$$

As $f_1(x)$ has one value in $(0, \pi/2)$ and a different value in $(\pi/2, \pi)$ we have

$$\begin{aligned} \int_0^{\pi} f_1(x) \cos(nx) \mathrm{d}x &= \int_0^{\pi/2} f_1(x) \cos(nx) \mathrm{d}x + \int_{\pi/2}^{\pi} f_1(x) \cos(nx) \mathrm{d}x, \\ &= \int_0^{\pi/2} \cos(nx) \mathrm{d}x. \end{aligned}$$

At the last step we have used the property that the integrand is 0 in $(\pi/2, \pi)$ and $f_1(x) = 1$ in $(0, \pi/2)$.

Finishing the computation for $f_1(x)$

When $n \geq 1$ we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi/2} \cos(nx)\mathrm{d}x \\ &= = \frac{2}{\pi}\frac{\sin(n\pi/2)}{n}. \end{aligned}$$

When n is even $\sin(n\pi/2) = 0$ and hence $a_n = 0$ when $n \geq 1$ is even.

An odd number is a number of the form $n = 2m - 1$, $m = 1, 2, \ldots$.

$$n\pi/2 = m\pi - \pi/2$$

and

$$\sin(m\pi - \pi/2) = -\cos(m\pi) = -(-1)^m = (-1)^{m+1} = (-1)^{m-1}.$$

Thus

$$a_{2m-1} = \frac{2}{\pi}\frac{(-1)^{m+1}}{2m-1}.$$

The coefficient a_0 for $f_1(x)$ and the series

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}x = \frac{2}{\pi}\frac{\pi}{2} = 1.$$

To summarise, and this was in the question, the Fourier series for $f_1(x)$

$$\begin{aligned} &\frac{a_0}{2} + \sum_{m=1}^{\infty} a_{2m-1}\cos((2m-1)x) \\ = \quad & \frac{1}{2} + \frac{2}{\pi}\left(\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} \right. \\ & \left. \quad + \cdots + (-1)^{m+1}\frac{\cos((2m-1)x)}{2m-1} + \cdots \right). \end{aligned}$$

It was not part of the question but you can note from this expression that at the points of discontinuity the value of the series is $a_0/2 = 1/2$ which is the average of the values for x either side of the points $\pm\pi/2$.

The function $f_2(x)$ is an odd function

$$f_2(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi/2, \\ -1, & \text{if } -\pi/2 \le x < 0, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi. \end{cases}$$

The value in $(-\pi/2, 0)$ is -1 which is -1 times the value in $(0, \pi/)$. Thus the odd property holds here.

The odd property also holds in the outer part as -1 times 0 is 0.

Hence the odd property holds at all points of continuity.

The implication of this for the Fourier series is that it only involves sine terms.

The coefficients in the series for $f_2(x)$

As the function is an odd function we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f_2(x)\sin(nx)\,\mathrm{d}x.$$

As $f_2(x) = 1$ in $(0, \pi/2)$ and it is 0 in $(\pi/2, \pi)$ we have

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} \sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi/2} = \frac{2}{n\pi}\left(-\cos(n\pi/2) + 1\right).$$

$\cos(n\pi/2)$ takes values 0, -1, 0, 1 and 0 as $n = 1, \ldots, 5$. Thus

$$b_1 = \frac{2}{\pi}, \quad b_2 = \frac{4}{2\pi} = b_1, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{2}{5\pi}.$$

2018 Fourier series question, expanded explanation of the solution

In the question we have the following.

Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ denote the 2π-periodic functions given on $(-\pi, \pi]$ by

$$f_1(x) = \begin{cases} 1\,, & -\pi < x \le -\pi/2\,, \\ 0\,, & -\pi/2 < x < \pi/2\,, \\ 1\,, & \pi/2 \le x \le \pi\,, \end{cases} \quad \text{and} \quad f_2(x) = -\frac{x}{2} + \int_0^x f_1(t)\,\mathrm{d}t\,.$$

The first thing to note about the piecewise constant function $f_1(x)$ is that it is even.

This is the case in $(-\pi/2, \pi/2)$ where $f(x) = 0$.

In the "outer" parts $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$ this is true with $f(x) = 1$.

As $f_1(x)$ is even the Fourier series only has cosine terms.

The Fourier coefficients of $f_1(x)$

For the constant term

$$\pi a_0 = 2\int_0^{\pi} \, \mathrm{d}x = 2\int_{\pi/2}^{\pi} \mathrm{d}x = \pi, \quad a_0 = 1.$$

For $n \ge 1$,

$$\pi a_n = 2\int_{\pi/2}^{\pi} \cos(nx)\,\mathrm{d}x = \frac{2}{n}\left[\sin(nx)\right]_{\pi/2}^{\pi} = -\frac{2}{n}\sin(n\pi/2).$$

The values of $\sin(n\pi/2)$ are respectively 1, 0, -1, 0, 1 etc.

When n is even we hence have $a_n = 0$.

To represent a general odd number let $n = 2m - 1$.

$$\frac{n\pi}{2} = m\pi - \frac{\pi}{2}, \quad \sin(n\pi/2) = -\cos(m\pi) = -(-1)^m.$$

Hence

$$a_{2m-1} = \frac{2(-1)^m}{\pi(2m-1)}.$$

The piecewise linear function $f_2(x)$

Firstly when $|x| < \pi/2$ we have $f_1(x) = 0$ and thus

$$\int_0^x f_1(t)\,\mathrm{d}t = 0.$$

Hence when $|x| < \pi/2$, $f_2(x) = -x/2$.

For $x \geq \pi/2$,

$$\begin{aligned}\int_0^x f_1(t)\,\mathrm{d}t &= \int_0^{\pi/2} f_1(t)\,\mathrm{d}t + \int_{\pi/2}^x f_1(t)\,\mathrm{d}t \\ &= \int_{\pi/2}^x f_1(t)\,\mathrm{d}t = \int_{\pi/2}^x \mathrm{d}t = x - \pi/2.\end{aligned}$$

Thus in $(\pi/2, \pi)$ we have

$$f_2(x) = \frac{x - \pi}{2}.$$

The piecewise linear function $f_2(x)$ continued

For $x \le -\pi/2$,

$$\begin{aligned} \int_0^x f_1(t)\,\mathrm{d}t &= \int_0^{-\pi/2} f_1(t)\,\mathrm{d}t + \int_{-\pi/2}^x f_1(t)\,\mathrm{d}t \\ &= \int_{-\pi/2}^x f_1(t)\,\mathrm{d}t = \int_{-\pi/2}^x \mathrm{d}t = x + \pi/2. \end{aligned}$$

Thus in $(-\pi, -\pi/2)$ we have

$$f_2(x) = \frac{x+\pi}{2}.$$

A sketch of the piecewise linear function $f_2(x)$

To help understand what a sketch of $f_2(x)$ looks like you might note that the derivative (where it exists) is the value of $f_1(x)$ minus $1/2$, i.e.

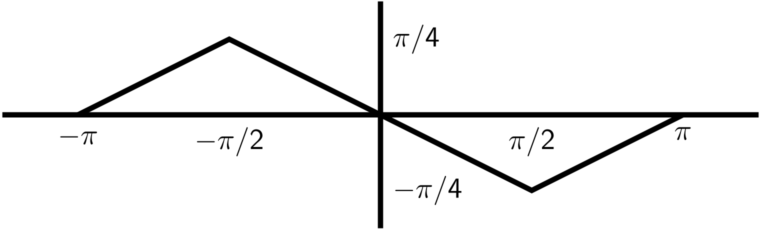
$$f_2'(x) = \begin{cases} 1/2\,, & -\pi < x \le -\pi/2\,, \\ -1/2\,, & -\pi/2 < x < \pi/2\,, \\ 1/2\,, & \pi/2 \le x \le \pi. \end{cases}$$

The function $f_2(x)$ has constant slope in $(-\pi, -\pi/2)$, $(-\pi/2, \pi/2)$, and $(\pi/2, \pi)$. It is a straight line segment in each part and at the join points it is continuous. The sketch just involves 3 straight line segments.

The values at the join points are

$$f_2(-\pi) = f_2(\pi) = 0, \quad f_2(-\pi/2) = \pi/4, \quad f_2(\pi/2) = -\pi/4$$

Finally the sketch of $f_2(x)$



This function is an odd function.

The Fourier series of $f_2'(x)$

As $f_2(x)$ is an odd function the Fourier series only involves sine terms.

We have already noted that

$$f_2'(x) = -\frac{1}{2} + f_1(x).$$

The Fourier series for this function is the series for $f_1(x)$ without the constant term, i.e.

$$a_1 \cos(x) + a_3 \cos(3x) + a_5 \cos(5x) + \cdots$$

Term-by-term integration gives us our series just having sine terms and if we also not that $f_2(0) = 0$ the answer is

$$f_2(x) = a_1 \sin(x) + \frac{a_3}{3} \sin(3x) + \frac{a_5}{5} \sin(5x) + \cdots$$

We can put $=$ here as $f_2(x)$ is continuous for all x and thus the Fourier series of $f_2(x)$ is the same as $f_2(x)$ at all points.

The use of Taylor series and finite difference approximations

Let $u(x)$ be sufficiently smooth near to 0 and let h be sufficiently small such that all Taylor expansions about 0 are valid. Consider the following.

$$
\begin{aligned}
I(h) &= \frac{-u(3h)+9u(h)-8u(0)}{6h}, \\
J(h) &= \frac{-u(2h)+8u(h)-8u(-h)+u(-2h)}{12h}.
\end{aligned}
$$

We want Taylor expansions of $I(h)$ and $J(h)$ about 0.

The points $3h$ and h in $I(h)$ are both on the same side of 0.

The points $\pm 2h$ and $\pm h$ are symmetrical about 0.

Both cases need the following Taylor expansion

$$
u(h) = u(0) + u'(0)h + \frac{u''(0)}{2}h^2 + \frac{u'''(0)}{6}h^3 + \cdots
$$

The expression for $I(h)$

$$u(h) = u(0) + u'(0)h + \frac{u''(0)}{2}h^2 + \frac{u'''(0)}{6}h^3 + \cdots$$

If we replace h by $3h$ then we get

$$u(3h) = u(0) + u'(0)(3h) + \frac{u''(0)}{2}(3h)^2 + \frac{u'''(0)}{6}(3h)^3 + \cdots$$

If we multiply the $u(h)$ version by 9 then

$$9u(h) = 9u(0) + 9u'(0)h + \frac{9u''(0)}{2}h^2 + \frac{9u'''(0)}{6}h^3 + \cdots$$

Thus

$$\begin{aligned} 9u(h) - u(3h) &= 8u(0) + 6u'(0)h + \frac{(-18)u'''(0)}{6}h^3 + \cdots \\ &= 8u(0) + 6u'(0)h - 3u'''(0)h^3 + \cdots . \end{aligned}$$

Hence

$$I(h) = \frac{9u(h) - u(3h) - 8u(0)}{6h} = u'(0) - \frac{u'''(0)}{2}h^2 + \cdots$$

The expression for $J(h)$

$$u(h) = u(0) + u'(0)h + \frac{u''(0)}{2}h^2 + \frac{u'''(0)}{6}h^3 + \cdots$$

If we replace h by $-h$ then we get

$$u(-h) = u(0) - u'(0)h + \frac{u''(0)}{2}h^2 - \frac{u'''(0)}{6}h^3 + \cdots$$

and subtracting from $u(h)$ gives

$$u(h) - u(-h) = 2u_0'h + \frac{u_0'''}{3}h^3 + \frac{u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7).$$

There are no even index terms. Replacing h by $2h$ in the above gives

$$u(2h) - u(-2h) = 4u_0'h + \frac{8u_0'''}{3}h^3 + \frac{32u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7).$$

The expression for $J(h)$ continued

A key point is to note that

$$-u(2h)+8u(h)-8u(-h)+u(-2h) = 8(u(h)-u(-h))-(u(2h)-u(-2h)).$$

$$\begin{aligned}
u(h)-u(-h) &= 2u_0'h + \frac{u_0'''}{3}h^3 + \frac{u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7), \\
8(u(h)-u(-h)) &= 16u_0'h + \frac{8u_0'''}{3}h^3 + \frac{8u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7), \\
u(2h)-u(-2h) &= 4u_0'h + \frac{8u_0'''}{3}h^3 + \frac{32u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7).
\end{aligned}$$

Thus

$$8(u(h)-u(-h))-(u(2h)-u(-2h)) = 12u_0'h - \frac{24u_0^{(5)}}{60}h^5 + \mathcal{O}(h^7)$$

and

$$J(h) = u_0' - \frac{u_0^{(5)}}{30}h^4 + \mathcal{O}(h^7).$$

The use of Taylor series — some comments

$$
\begin{aligned}
I(h) &= \frac{-u(3h) + 9u(h) - 8u(0)}{6h}, \\
J(h) &= \frac{-u(2h) + 8u(h) - 8u(-h) + u(-2h)}{12h}.
\end{aligned}
$$

- When giving the details attempt to line up the corresponding powers of h as this makes it easier when combining terms.
- Do not try and do too many steps in one go. You are more likely to make mistakes.
- When there is symmetry, e.g. in the $J(h)$ case, make use of it so that less has to be written down.