Chap 5: Fourier series for periodic functions

Let $f : (-\pi, \pi] \to \mathbb{R}$ be a bounded piecewise continuous function which we continue to be a 2π-periodic function defined on $\mathbb{R}$, i.e.

$$f(x + 2\pi) = f(x), \quad \forall x \in \mathbb{R}.$$

The Fourier series of this function is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x.$$

When certain sufficient conditions about f hold we have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

at all the points of continuity of $f(x)$.

This was the 1st slide in week 24.

Sufficient conditions for pointwise convergence

Let $f(x)$ be piecewise continuous on $(-\pi, \pi]$ with the left and right limits $f(x-)$ and $f(x+)$ existing at all points. Suppose also that for some $\delta > 0$ $f(x)$ is differentiable in $(x-\delta, x)$ and in $(x, x+\delta)$ with a left and right derivative at x. These are sufficient conditions for

$$\lim_{m\to\infty} \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) = \frac{f(x-) + f(x+)}{2}.$$

At points of continuity

$$\lim_{m\to\infty} \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) = f(x).$$

There are no known conditions which are both necessary and sufficient for the pointwise convergence of Fourier series.

This was the a slide in week 25.

Half range Fourier series

If $f(x)$ is just given on $(0,\pi)$ then we can continue it as an even function or we can continue as an odd function. The even or odd extensions both have Fourier series.

The half range cosine series for $f(x)$ defined on $(0,\pi)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x.$$

The half range sine series for $f(x)$ defined on $(0,\pi)$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

We hence get two different representations on $(0,\pi)$.

This was about the last topic in week 25. The slides that follow were the planned slides with new material for week 26.

Integrating a Fourier series

This is valid but take care with the constant term.

At points of continuity we have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right).$$

If we shift the constant term to the left hand side to give

$$f(t) - \frac{a_0}{2} = \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

and integrate from 0 to x then we get

$$\phi(x) = \int_0^x f(t)\,\mathrm{d}t - \frac{a_0 x}{2} = \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} (\cos(nx) - 1) \right).$$

If $f(x)$ has jump discontinuities then $\phi(x)$ is continuous with jump discontinuities in the first derivative.

Integrating a Fourier series continued

$$\phi(x) = \int_0^x f(t)\,\mathrm{d}t - \frac{a_0 x}{2} = \sum_{n=1}^{\infty} \left(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - 1) \right).$$

When $a_0 \neq 0$ we complete the details by using

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) = \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots$$

which is valid for $|x| < \pi$.

Note that as the 2π-periodic function $\phi(x)$ is continuous on $\mathbb{R}$ the function

$$\int_0^x f(t)\,\mathrm{d}t$$

is only continuous on $\mathbb{R}$ when $a_0 = 0$ when extended as a 2π periodic function.

Example: Integrating x to get the series for $x^2/2$

In previous lectures we obtained the series for the odd function $f_3(x) = x$. By replacing the variable by t we have

$$t = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt) = 2\left(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \cdots\right).$$

Now

$$\frac{x^2}{2} = \int_0^x t\,\mathrm{d}t \quad \text{and} \quad \int_0^x \sin(nt)\,\mathrm{d}t = \frac{1}{n}\left(1 - \cos(nx)\right).$$

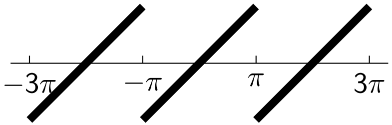
Thus for $|x| \le \pi$

$$\frac{x^2}{2} = 2\sum_1^{\infty} \frac{(-1)^{n+1}}{n^2} + 2\sum_1^{\infty} (-1)^n \frac{\cos(nx)}{n^2}.$$

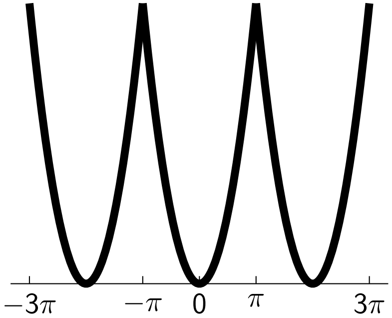
Integrating this expression over $(-\pi, \pi)$ gives for the constant term

$$2\sum_1^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2}{2}\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} x^2\,\mathrm{d}x = \frac{\pi^2}{6}.$$

Sketch of the 2π periodic extension of x and $x^2/2$



This is discontinous at $\pi + 2k\pi$, $k \in \mathbb{Z}$.



This is continuous with discontinous slope at $\pi + 2k\pi$, $k \in \mathbb{Z}$.

Differentiating a Fourier series

Suppose that the 2π-periodic extension of $f(x)$ is continuous so that we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$

In particular this means that

$$f(-\pi+) = f(\pi-).$$

In this case we can differentiate term-by-term to give

$$f'(x) \sim \sum_{n=1}^{\infty} \left(-n a_n \sin(nx) + n b_n \cos(nx) \right).$$

We have equality at all points of continuity of $f'(x)$.

Example

On $(-\pi, \pi]$ let

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x(\pi - x), & 0 \le x \le \pi, \end{cases}$$

and

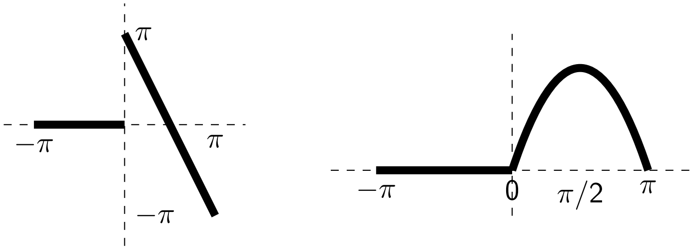
$$g(x) = \begin{cases} 0, & -\pi < x < 0, \\ \pi - 2x, & 0 \le x \le \pi. \end{cases}$$

Note that $g(x) = f'(x)$ in $(-\pi, 0)$ and $(0, \pi)$.

$g(x)$ is discontinuous at $x = 0$ and at $\pm\pi$.

The 2π periodic extension of $f(x)$ is continuous.

Sketch of $g(x)$ (left) and $f(x)$ (right) in $(-\pi, \pi)$



Techniques to get the coefficients of $f(x)$ and $g(x)$

- In $(0, \pi)$ we have polynomials in x and integration by parts is needed. At any stage when integration by parts is used we want to differentiate the polynomial term so that the next integral to consider is simpler.
- We can get the series for $g(x)$ and then use term-by-term integration to get the series for $f(x)$.
- Alternatively we can first get the series for $f(x)$ and then differentiate to get the series for $g(x)$.

Fourier series for $f(x)$ and $g(x)$ in the example

Which ever route its taken the series obtained are as follows.

$$
\begin{aligned}
g(x) &\sim \sum_{n=1}^{\infty} \left(\frac{2}{n^2\pi}(1+(-1)^{n+1})\cos(nx) + \frac{1}{n}(1+(-1)^n)\sin(nx) \right), \\
f(x) &= \frac{\pi^2}{12} + \\
&\quad \sum_{n=1}^{\infty} \frac{2}{n^3\pi}(1+(-1)^{n+1})\sin(nx) - \frac{1}{n^2}(1+(-1)^n)\cos(nx).
\end{aligned}
$$

Note that if we consider either function on $(0, \pi)$ and consider the odd extension of $g(x)$ or the even extension of $f(x)$ then the functions created have period π.

The odd and even extensions have period π

The odd extension of $g(x)$ on $(0, \pi)$ is

$$\sum_{n=1}^{\infty} \frac{1}{n}(1 + (-1)^n) \sin(nx) = \sum_{m=1}^{\infty} \frac{1}{m} \sin(2mx)$$

The even extension of $f(x)$ on $(0, \pi)$ is

$$\begin{aligned} f(x) &= \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{1}{n^2}(1 + (-1)^n) \cos(nx) \\ &= \frac{\pi^2}{12} - \sum_{m=1}^{\infty} \frac{1}{2m^2} \cos(2mx). \end{aligned}$$

In both cases these odd and even extensions have period π which is why only the even terms appear in the representations.