Chap 5: Fourier series for periodic functions

Let $f : (-\pi, \pi] \to \mathbb{R}$ be a bounded piecewise continuous function which we continue to be a 2π-periodic function defined on $\mathbb{R}$, i.e.

$$f(x + 2\pi) = f(x), \quad \forall x \in \mathbb{R}.$$

The Fourier series of this function is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x.$$

When certain sufficient conditions about f hold we have that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

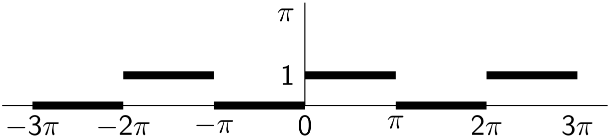
at all the points of continuity of $f(x)$.

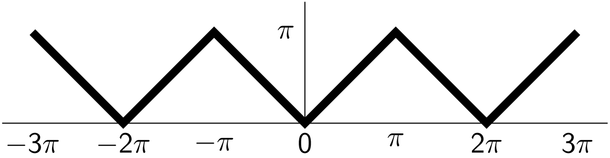
Examples of 2π periodic functions

Let $f_1(x)$ and $f_2(x)$ be defined on $(-\pi, \pi]$ as follows.

$$f_1(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi, \\ 0, & \text{if } -\pi < x < 0, \end{cases} \quad \text{and} \quad f_2(x) = |x|.$$

Both of these are continued 2π-periodically and a sketch of both is shown below.





Orthogonal functions on $(-\pi, \pi)$ and the value of integrals

If f and g are from the list

$$1,\ \cos(x),\ \sin(x)\ \cos(2x),\ \sin(2x), \ldots, \cos(nx),\ \sin(nx), \ldots.$$

and $f \neq g$ then

$$\int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x = 0.$$

When $f(x) = g(x)$ we have the following cases.

$$\begin{aligned} \int_{-\pi}^{\pi} \mathrm{d}x &= 2\pi, \\ \int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x &= \pi, \\ \int_{-\pi}^{\pi} \sin^2(nx)\,\mathrm{d}x &= \pi. \end{aligned}$$

The formula for the Fourier coefficients a_n and b_n

If our starting point are numbers a_n and b_n such that the series converges to define

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

and the convergence is such that all operations such as interchanging infinite sums and integrals are valid then the orthogonality properties imply the following.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x &= a_0\pi, \\
\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x &= a_m\pi, \\
\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x &= b_m\pi.
\end{aligned}$$

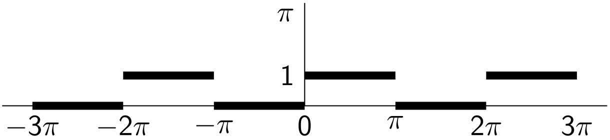
This gives a justification for the formulas used.

The Heaviside function on $(-\pi, \pi]$

Let

$$f_1(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi, \\ 0, & \text{if } -\pi < x < 0 \end{cases}$$

which we continue in a 2π-periodic way.



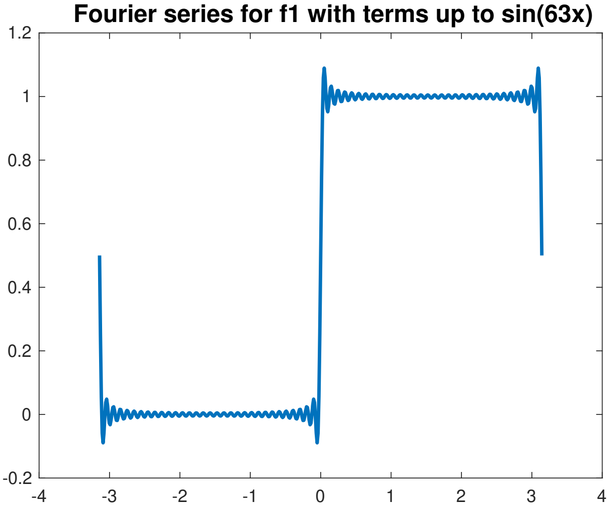
Determining the coefficients gives the following.

$$f_1(x) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin((2n-1)x}{2n-1} + \cdots\right).$$

We can immediately observe that when $x = k\pi$ the value of the series is $1/2$. The series does converge to $f_1(x)$ at these points of discontinuity.

A plot of a truncated Fourier series for $f_1(x)$

$$S_{63}(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin(63x)}{63}\right).$$

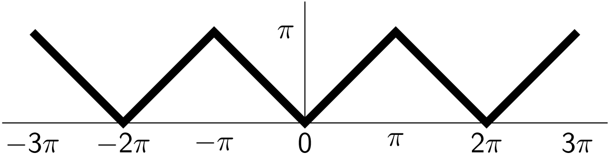


The function $f_2(x) = |x|$ on $(-\pi, \pi]$

In this case convergence occurs at all points and we can write for $|x| \le \pi$.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos((2n-1)x)}{(2n-1)^2} + \cdots\right).$$

The 2π-periodic function extension is continuous at all points.

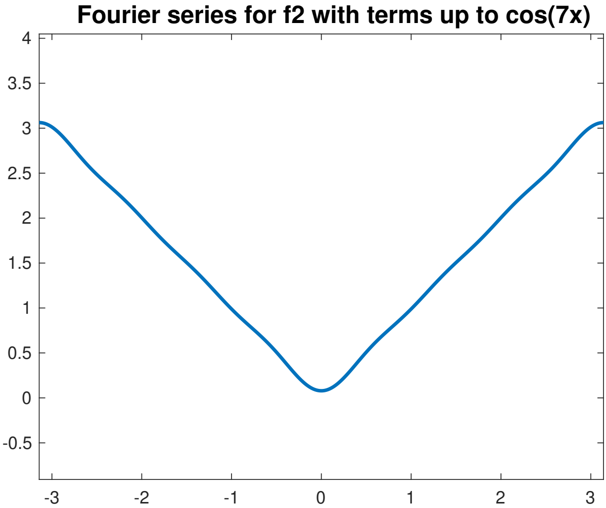


Observe that $f_2(x)$ is an even function of x which is why $b_n = 0$ and only cosine terms are involved. Integration by parts is used to determine

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\,\cos(nx)\,\mathrm{d}x.$$

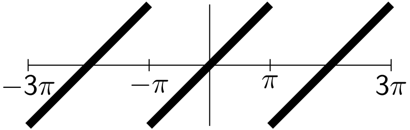
A plot of a truncated Fourier series for $f_2(x)$

$$S_7(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2}\right).$$



The function $f_3(x) = x$ on $(-\pi, \pi]$

A sketch of the 2π-periodic extension is as follows.



When $x \in (-\pi, \pi)$ we have pointwise convergence to $f_3(x)$ and

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = 2 \left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots \right).$$

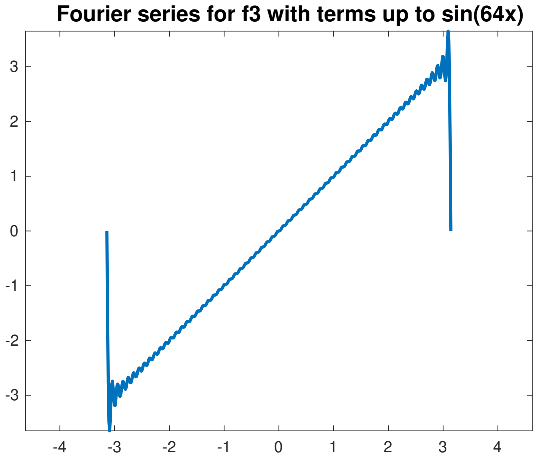
We can immediately observe that when $x = \pi$ the value of the series is 0. The series does converge here but not to $f_3(\pi) = \pi$ or to the limit as we tend to $-\pi$ which is $-\pi$. The 2π-periodic extension is discontinuous at the points $(2k+1)\pi$, $k \in \mathbb{Z}$.

As $f_3(x)$ is an odd function of x we have $a_n = 0$ and only sine terms are involved. Integration by parts is used to determine

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) \, \mathrm{d}x.$$

A plot of a truncated Fourier series for $f_3(x)$

$$S_{64}(x) = 2\sum_{n=1}^{64} \frac{(-1)^{n+1}}{n} \sin(nx).$$



Combining Fourier series

$$f_4(x) = \frac{f_2(x) + f_3(x)}{2} = \frac{|x| + x}{2} = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 \le x \le \pi. \end{cases}$$

Combining the series for $f_2(x)$ and $f_3(x)$ gives

$$\begin{aligned} f_4(x) &= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos((2n-1)x)}{(2n-1)^2} + \cdots\right) \\ &\quad + \left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right) \end{aligned}$$

for $-\pi < x < \pi$.

$x = \pi + 2k\pi$, $k \in \mathbb{Z}$ are points of discontinuity of the 2π-periodic extension.

Sufficient conditions for pointwise convergence

Let $f(x)$ be piecewise continuous on $(-\pi, \pi]$ with the left and right limits $f(x-)$ and $f(x+)$ existing at all points. Suppose also that for some $\delta > 0$ $f(x)$ is differentiable in $(x - \delta, x)$ and in $(x, x + \delta)$ with a left and right derivative at x. These are sufficient conditions for

$$\lim_{m\to\infty} \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) = \frac{f(x-) + f(x+)}{2}.$$

At points of continuity

$$\lim_{m\to\infty} \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx)) = f(x).$$

There are no known conditions which are both necessary and sufficient for the pointwise convergence of Fourier series.

Sufficient conditions continued

The proof is beyond the syllabus of MA2715. If it was done then some steps to show would involve the following.

1. Express a term in the series as an integral.

$$\begin{aligned} & a_n \cos(nx) + b_n \sin(nx) \\ = & \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos(nt)\cos(nx) + \sin(nt)\cos(nx)) f(t)\, \mathrm{d}t \\ = & \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n(x-t)) f(t)\, \mathrm{d}t. \end{aligned}$$

2. When we sum the previous terms we need the Dirichlet kernel which is

$$\frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) = \frac{\sin(m+1/2)t)}{2\sin(t/2)}.$$

Sufficient conditions continued

3. The Riemann Lebesgue lemma. For a suitable function $g(x)$ we have

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} g(x)\sin((m+1/2)x)\,\mathrm{d}x = 0.$$

4. Representation of the partial sums and the convergence.

$$\begin{aligned} S_m(x) &= \frac{a_0}{2} + \sum_{n=1}^{m} (a_n\cos(nx) + b_n\sin(nx)) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\frac{(\sin(m+1/2)t)}{2\sin(t/2)}\,\mathrm{d}t. \end{aligned}$$

The remaining details are to show that

$$S_m(x) - \frac{f(x-)+f(x+)}{2} = \ldots\text{details}\ldots \to 0 \quad \text{as } m\to\infty.$$

Comments about other types of convergence

With, as before,

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx))$$

the orthogonality properties of the functions quickly leads to

$$\int_{-\pi}^{\pi} S_m(x)^2 \,\mathrm{d}x = \pi \left(\frac{a_0^2}{2} + \sum_{n=1}^{m} (a_n^2 + b_n^2) \right)$$

and

$$\int_{-\pi}^{\pi} f(x)^2 \,\mathrm{d}x = \int_{-\pi}^{\pi} S_m(x)^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} (f(x) - S_m(x))^2 \,\mathrm{d}x.$$

It is beyond the syllabus of `MA2715` to cover this but with not too restrictive requirements on $f(x)$ it can be shown that S_m tends to f in the sense that

$$\int_{-\pi}^{\pi} (f(x) - S_m(x))^2 \,\mathrm{d}x \to 0 \quad \text{as } m \to \infty.$$

Half range Fourier series

If $f(x)$ is just given on $(0,\pi)$ then we can continue it as an even function or we can continue as an odd function. The even or odd extensions both have Fourier series.

The half range cosine series for $f(x)$ defined on $(0,\pi)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x.$$

The half range sine series for $f(x)$ defined on $(0,\pi)$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

We hence get two different representations on $(0,\pi)$.