Chap 3: The problem $\underline{u}' = A\underline{u}$, $\underline{u}(0) = \underline{u}_0$

Here A is $n \times n$ and $\underline{u} = \underline{u}(x)$ is $n \times 1$.

In all cases we express the solution in a "closed form" as

$$\underline{u}(x) = \exp(xA)\underline{u}_0, \quad \exp(xA) \text{ is the exponential matrix of } xA.$$

When A has a complete set of eigenvectors we can do the following.

1. Determine the eigenvalues λ_i and eigenvectors $\underline{v}_i$ of A.
2. Form the matrix $V = (\underline{v}_1, \ldots, \underline{v}_n)$ and solve

   $$V\underline{c} = \underline{u}_0.$$

   This has a unique solution as $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent when A has a complete set of eigenvectors.
3. The solution is given by

   $$\underline{u}(x) = \sum_{i=1}^{n} c_i e^{\lambda_i x} \underline{v}_i.$$

Example: Distinct real eigenvalues of ± 1

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

The eigenvalues of the matrix are ± 1 and the solution is

$$\underline{u}(x) = -\mathrm{e}^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\mathrm{e}^{x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In components we have

$$\begin{aligned} u_1(x) &= -\mathrm{e}^{-x} + 3\mathrm{e}^{x}, \\ u_2(x) &= \mathrm{e}^{-x} + 3\mathrm{e}^{x}, \end{aligned}$$

In this case $u_1'' = u_1$ and $u_2'' = u_2$ and we could have solved two second order linear ODEs.

Example: Complex conjugate pair of eigenvalues

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

The eigenvalues of the matrix are the complex conjugate pair $\pm i$ and using the method the solution is first written as

$$\underline{u}(x) = c_1 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

with

$$c_1 = 1 + 2i, \quad c_2 = 1 - 2i.$$

All the non-real quantities occur in complex conjuage pairs and by using $e^{\pm ix} = \cos x \pm i \sin x$ we can re-express the solution as

$$\begin{aligned} u_1(x) &= 2\cos x + 4\sin x, \\ u_2(x) &= 4\cos x - 2\sin x. \end{aligned}$$

In this case $u_1'' = -u_1$ and $u_2'' = -u_2$ and we could have solved two second order linear ODEs.

An example with distinct real eigenvalues of -6 and 5

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 6 & 6 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 20 \\ -7 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = -6$ and $\lambda_2 = 5$ and the eigenvectors $\underline{v}_1$ and $\underline{v}_2$ are obtained from

$$A - \lambda_1 I = \begin{pmatrix} 12 & 6 \\ -2 & -1 \end{pmatrix}, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

$$A - \lambda_2 I = \begin{pmatrix} 1 & 6 \\ -2 & -12 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$$

The solution is

$$\underline{u}(x) = \mathrm{e}^{-6x} \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \mathrm{e}^{5x} \begin{pmatrix} 18 \\ -3 \end{pmatrix}.$$

Note, if instead

$$\underline{u}(0) = \begin{pmatrix} 20 \\ -40 \end{pmatrix} = 20\underline{v}_1 \quad \text{then } \underline{u}(x) = 20\mathrm{e}^{-6x}\underline{v}_1 \to \underline{0} \quad \text{as } x \to \infty.$$

The complex exponential

When we have more general complex eigenvalues of the form $\lambda = p + iq$, $p, q \in \mathbb{R}$ the complex exponential is defined to mean

$$e^{\lambda x} = e^{(p+iq)x} = e^{px} e^{iqx} = e^{px}(\cos(qx) + i \sin(qx))$$

The behaviour as $x \to \infty$

As $|e^{\lambda x}| = e^{px}$ the solution $\underline{u}(x)$ tends to $\underline{0}$ as $x \to \infty$ for all $\underline{u}_0$ when the real part of all the eigenvalues is negative.

The exponential matrix

For a square matrix B the exponential matrix is defined by

$$\exp(B) = I + B + \frac{1}{2}B^2 + \frac{1}{6}B^3 + \cdots + \frac{1}{n!}B^n + \cdots$$

This series always converges. By taking $B = xA$ the solution of $\underline{u}' = A\underline{u}$ is given by

$$\underline{u}(x) = \exp(xA)\underline{u}_0$$

in all cases.

When the eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent the matrix $V = (\underline{v}_1, \ldots, \underline{v}_n)$ is invertible and we also have

$$\underline{u}(x) = V \exp(xD) V^{-1} \underline{u}_0$$

with $D = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$. In this case

$$\exp(xA) = V \exp(xD) V^{-1}.$$

$\exp(xA)$ in a deficient matrix case

$$A = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 \text{ and } \lambda_2 \text{ are the eigenvalues.}$$

$$A - \lambda_1 I = \begin{pmatrix} 0 & \alpha \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix} \quad \text{and} \quad A - \lambda_2 I = \begin{pmatrix} \lambda_1 - \lambda_2 & \alpha \\ 0 & 0 \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{v}_2 = \begin{pmatrix} \alpha \\ \lambda_2 - \lambda_1 \end{pmatrix} \neq 0.$$

The matrix $V = (\underline{v}_1,\ \underline{v}_2)$ and the inverse V^{-1} are

$$V = \begin{pmatrix} 1 & \alpha \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix} \quad \text{and} \quad V^{-1} = \begin{pmatrix} 1 & \dfrac{-\alpha}{\lambda_2 - \lambda_1} \\ 0 & \dfrac{1}{\lambda_2 - \lambda_1} \end{pmatrix}.$$

If $D = \mathrm{diag}\{\lambda_1, \lambda_2\}$ then

$$V \exp(xD) V^{-1} = \begin{pmatrix} e^{\lambda_1 x} & \alpha \left(\dfrac{e^{\lambda_2 x} - e^{\lambda_1 x}}{\lambda_2 - \lambda_1} \right) \\ 0 & e^{\lambda_2 x} \end{pmatrix} \to e^{\lambda_1 x} \begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}$$

as $\lambda_2 \to \lambda_1$. We will not consider such "more difficult" cases.

Higher order ODEs – a 2nd order case

One higher order ODE can be written as a system of first order ODEs. For example

$$y'' + b_1 y' + b_0 y = 0$$

can be written as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}.$$

We have a closed form solution when b_0 and b_1 are constants. If b_0 and b_1 are replaced by functions then we rarely have “closed form expressions” for the solution.

Higher order systems with constant coefficients

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0, \quad b_i \text{ constants.}$$

$$\begin{array}{rcl} u_1 & = & y, \\ u_2 & = & y' = u_1', \\ u_3 & = & y'' = u_2', \\ \cdots & & \cdots \\ u_n & = & y^{(n-1)} = u_{n-1}' \end{array}$$

From the differential equation

$$u_n' = y^{(n)} = -b_0 u_1 - b_1 u_2 - \cdots - b_{n-1} u_{n-1}.$$

Thus we have

$$\underline{u}' = A\underline{u} \quad \text{with } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & \cdots & -b_{n-1} \end{pmatrix}.$$

Characteristic equation/ auxiliary equation

The characteristic equation of A is the auxiliary equation

$$\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0 = 0.$$

When $n = 4$ expand the determinant about the last row to give

$$\begin{aligned}
\det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ b_0 & b_1 & b_2 & \lambda + b_3 \end{vmatrix} \\
&= (-b_0)\begin{vmatrix} -1 & 0 & 0 \\ \lambda & -1 & 0 \\ 0 & \lambda & -1 \end{vmatrix} + b_1 \begin{vmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & \lambda & -1 \end{vmatrix} \\
&\quad + (-b_2)\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -1 \end{vmatrix} + (\lambda + b_3)\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} \\
&= b_0 + b_1\lambda + b_2\lambda^2 + (b_3 + \lambda)\lambda^3.
\end{aligned}$$

Chap 4: The two-point BVP

$$u''(x) = p(x)u'(x) + q(x)u(x) + r(x), \quad a < x < b,$$
$$u(a) = g_1, \quad u(b) = g_2.$$

The FD approximation – a summary

With a uniform mesh with $h = (b-a)/N$, $x_i = a + ih$, $i = 0, 1, \ldots, N$ and $U_i \approx u(x_i)$ the central difference finite difference approximation involves the following.

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = p_i \left(\frac{U_{i+1} - U_{i-1}}{2h} \right) + q_i U_i + r_i,$$
$$i = 1, 2, \ldots, N-1.$$

$$U_0 = g_1 \quad \text{and} \quad U_N = g_2.$$

The "continuous" problem for $u(x)$, $a \le x \le b$ is approximated by a "discrete" problem involving $U_0, U_1, \ldots, U_N$.

The central difference approximations

Let $u_i = u(x_i)$ and consider Taylor expansions about x_i of $u_{i-1} = u(x_{i-1}) = u(x_i - h)$ and $u_{i+1} = u(x_i + h) = u(x_i + h)$.

$$\begin{aligned} u_{i+1} &= u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' + \cdots \\ u_{i-1} &= u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' + \cdots . \end{aligned}$$

Adding and subtracting gives

$$\begin{aligned} u_{i+1} + u_{i-1} &= 2\left(u_i + \frac{h^2}{2}u_i'' + \frac{h^4}{24}u_i'''' + \cdots\right) \\ u_{i+1} - u_{i-1} &= 2\left(hu_i' + \frac{h^3}{6}u_i''' + \cdots\right). \end{aligned}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + \mathcal{O}(h^2), \quad \frac{u_{i+1} - u_{i-1}}{2h} = u_i' + \mathcal{O}(h^2).$$

The exact values u_i and the discrete values U_i

The exact values satisfy $u_0 = g_1$, $u_N = g_2$ and the following.

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p_i \left(\frac{u_{i+1} - u_{i-1}}{2h} \right) + q_i u_i + r_i + \mathcal{O}(h^2),$$
$$i = 1, 2, \ldots, N-1.$$

The finite approximation satisfy $U_0 = g_1$, $U_N = g_2$ and the following.

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = p_i \left(\frac{U_{i+1} - U_{i-1}}{2h} \right) + q_i U_i + r_i,$$
$$i = 1, 2, \ldots, N-1.$$

Each equation only involves 2 or 3 of the terms U_i.

Let $\underline{U} = (U_1, \ldots, U_{N-1})^T$. $\underline{U}$ is determined by solving a linear system $A\underline{U} = \underline{c}$.

The local truncation error

The local truncation error is concerned with how nearly the exact solution satisfies the difference equations that determine the finite difference approximation. It is defined as follows for $i = 1, \ldots, N-1$.

$$L_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \left(p_i \left(\frac{u_{i+1} - u_{i-1}}{2h} \right) + q_i u_i + r_i \right) = \mathcal{O}(h^2).$$

The linear system $A\underline{U} = \underline{c}$

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{N-2,N-1} \\ 0 & \cdots & 0 & a_{N-1,N-2} & a_{N-1,N-1} \end{pmatrix},$$

$$a_{i,i-1} = -1 - \frac{hp_i}{2}, \quad a_{ii} = 2 + h^2 q_i, \quad a_{i,i+1} = -1 + \frac{hp_i}{2}.$$

$$\begin{aligned} c_1 &= -h^2 r_1 + \left(1 + \frac{hp_1}{2}\right) g_1, \\ c_i &= -h^2 r_i, \quad 2 \le i \le N-2, \\ c_{N-1} &= -h^2 r_{N-1} + \left(1 - \frac{hp_{N-1}}{2}\right) g_2. \end{aligned}$$

It is $\mathcal{O}(N)$ storage and it $\mathcal{O}(N)$ operations to solve for $\underline{U}$.

The system in the special case $u'' = r$

In the "basic" scheme in one of the `MA2895` assignment tasks you have you have $p(x) = 0$ and the central difference version. When we have the further simplification $q(x) = 0$ we have $U_0 = g_1$, $U_N = g_2$ and

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = r_i, \quad i = 1, 2, \ldots, N-1.$$

The tri-diagonal system is as follows.

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-2} \\ U_{N-1} \end{pmatrix} = \begin{pmatrix} -h^2 r_1 + g_1 \\ -h^2 r_2 \\ \vdots \\ -h^2 r_{N-2} \\ -h^2 r_{N-1} + g_2 \end{pmatrix}.$$

It is $\mathcal{O}(N)$ storage and it $\mathcal{O}(N)$ operations to solve for $\underline{U}$.