Chap 3: The problem $\underline{u}' = A\underline{u}$, $\underline{u}(0) = \underline{u}_0$

Here A is $n \times n$ and $\underline{u} = \underline{u}(x)$ is $n \times 1$.

The scalar case ($n = 1$)

A first order ODE with a constant coefficient

When $u = u(x)$ and

$$u' = au, \quad u(0) = u_0,$$

with a being a constant, the solution is

$$u(x) = u_0 \mathrm{e}^{ax}.$$

Later the exponential matrix $\exp(xA)$ will be introduced and we show that the solution can always be written in the form

$$\underline{u}(x) = \exp(xA)\underline{u}_0.$$

A first order ODE system with a constant matrix

Suppose now that $\underline{u} = \underline{u}(x)$ is a vector of length n and

$$\underline{u}' = A\underline{u}, \quad \underline{u}(0) = \underline{u}_0,$$

where A is $n \times n$ matrix not depending on x. In full this is

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix}.$$

Let $\underline{v}_i \neq \underline{0}$ be an eigenvector of A with eigenvalue λ_i for $i = 1, \ldots, n$. The vector valued functions

$$\mathrm{e}^{\lambda_i x}\underline{v}_i, \quad i = 1, \ldots, n$$

all satisfy $\underline{u}' = A\underline{u}$.

The general solution and the specific solution

$$\underline{u}' - A\underline{u} = \underline{0}$$

is a linear differential equation and the general solution is

$$\underline{u}(x) = \sum_{i=1}^{n} c_i e^{\lambda_i x} \underline{v}_i$$

for constants $c_1, \ldots, c_n$ when $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent.

$$\underline{u}(0) = c_1 \underline{v}_1 + \cdots + c_n \underline{v}_n = \left(\underline{v}_1, \ldots, \underline{v}_n\right) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = V\underline{c}$$

where

$$V = \left(\underline{v}_1, \ldots, \underline{v}_n\right).$$

The particular solution satisfying $\underline{u}(0) = \underline{u}_0$ requires that

$$V\underline{c} = \underline{u}_0.$$

Summary of the method for solving $\underline{u}' = A\underline{u}, \qquad \underline{u}(0) = \underline{u}_0$

1. Determine the eigenvalues λ_i and eigenvectors $\underline{v}_i$ of A.
2. Form the matrix $V = (\underline{v}_1, \ldots, \underline{v}_n)$. If V is non-singular then solve

$$V\underline{c} = \underline{u}_0.$$

   This requires that $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent.
3. The solution is given by

$$\begin{aligned}
\underline{u}(x) &= \sum_{i=1}^{n} c_i e^{\lambda_i x} \underline{v}_i \\
&= V \begin{pmatrix} e^{\lambda_1 x} & & \\ & \ddots & \\ & & e^{\lambda_n x} \end{pmatrix} \underline{c} \\
&= V \exp(xD) V^{-1} \underline{u}_0.
\end{aligned}$$

Example: Distinct real eigenvalues cases

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

The eigenvalues of the matrix are ± 1 and the solution is

$$\underline{u}(x) = -\mathrm{e}^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\mathrm{e}^{x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 6 & 6 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 20 \\ -7 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = -6$ and $\lambda_2 = 5$ with eigenvectors given respectively by

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$$

The solution is

$$\underline{u}(x) = \mathrm{e}^{-6x} \begin{pmatrix} 2 \\ -4 \end{pmatrix} + \mathrm{e}^{5x} \begin{pmatrix} 18 \\ -3 \end{pmatrix}.$$

Example: Complex conjugate pair of eigenvalues

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \underline{u}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

The eigenvalues of the matrix are the complex conjugate pair $\pm i$ and using the method the solution is first written as

$$\underline{u}(x) = c_1 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

with

$$c_1 = 1 + 2i, \quad c_2 = 1 - 2i.$$

All the non-real quantities occur in complex conjuage pairs and by using $e^{\pm ix} = \cos x \pm i \sin x$ we can re-express the solution as

$$\begin{aligned} u_1(x) &= 2\cos x + 4\sin x, \\ u_2(x) &= 4\cos x - 2\sin x. \end{aligned}$$

In this case $u_1'' = -u_1$ and $u_2'' = -u_2$ and we could have solved two second order linear ODEs.

The complex exponential

When we have more general complex eigenvalues of the form $\lambda = p + iq$, $p, q \in \mathbb{R}$ the complex exponential is defined to mean

$$e^{\lambda x} = e^{(p+iq)x} = e^{px}e^{iqx} = e^{px}(\cos(qx) + i\sin(qx))$$

The behaviour as $x \to \infty$

As $|e^{\lambda x}| = e^{px}$ the solution $\underline{u}(x)$ tends to $\underline{0}$ as $x \to \infty$ for all $\underline{u}_0$ when the real part of all the eigenvalues is negative.

The exponential matrix

For a square matrix B the exponential matrix is defined by

$$\exp(B) = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{m!}B^m + \cdots$$

This series always converges. By taking $B = xA$ the solution of $\underline{u}' = A\underline{u}$ is given by

$$\underline{u}(x) = \exp(xA)\underline{u}_0$$

in all cases.

When the eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent the matrix $V = (\underline{v}_1, \ldots, \underline{v}_n)$ is invertible and we also have

$$\underline{u}(x) = V \exp(xD) V^{-1} \underline{u}_0$$

with $D = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$. In this case

$$\exp(xA) = V \exp(xD) V^{-1}.$$