Key points in chapter 1

Suppose A is $n \times n$ and suppose $A\underline{v}_i = \lambda_i \underline{v}_i$, $\underline{v}_i \neq \underline{0}$, $i = 1, \ldots, n$.
$\underline{v}_i$ is an **eigenvector**, λ_i is the **eigenvalue**.

$$\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\} = \textbf{spectral radius} \text{ of } A.$$

We have **vector norms** and we use the notation $\|\underline{x}\|$.

The **matrix norm** induced by a vector norm is

$$\|A\| = \max\{\|A\underline{x}\| : \ \|\underline{x}\| = 1\}.$$

For all such matrix norms $\rho(A) \leq \|A\|$.

The **matrix condition number** is defined by

$$\kappa(A) = \|A\| \, \|A^{-1}\|, \quad 1 \leq \kappa(A) \leq \infty.$$

We say $\kappa(A) = \infty$ when A does not have an inverse.

$\kappa(A)$ is large when A is near to a matrix which has no inverse.
When this is the case the solution $\underline{x}$ to $A\underline{x} = \underline{b}$ is very sensitive to changes in A and/or $\underline{b}$.

The common vector norms

$$\begin{aligned} \|\underline{x}\|_2 &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} = \left(\underline{x}^T \underline{x}\right)^{1/2}, \\ \|\underline{x}\|_\infty &= \max\{|x_1|, \ldots, |x_n|\}, \\ \|\underline{x}\|_1 &= |x_1| + |x_2| + \cdots + |x_n|. \end{aligned}$$

Expressions for the common vector norms

$$\begin{aligned} \|A\|_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|, \\ \|A\|_1 &= \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|, \\ \|A\|_2 &= \left(\rho(A^T A)\right)^{1/2}. \end{aligned}$$

$\|A\|_\infty$ and $\|A\|_1$ are “easy to calculate”.

$\|A\|_2$ is “expensive to calculate”.

Chap 2: Direct methods for solving $A\underline{x} = \underline{b}$

Upper triangular systems

$$
\begin{array}{ccccccccc}
u_{11}x_1 & + & u_{12}x_2 & + & \cdots & + & u_{1n}x_n & = & b_1 \\
 & + & u_{22}x_2 & + & \cdots & + & u_{2n}x_n & = & b_2 \\
 & & \ddots & & & & & & \vdots \\
 & & & & u_{n-1,n-1}x_{n-1} & + & u_{n-1,n}x_n & = & b_{n-1} \\
 & & & & & & u_{nn}x_n & = & b_n
\end{array}
$$

Backward substitution:

$$
\begin{aligned}
x_n &= b_n/u_{nn} \\
x_i &= \left(b_i - \sum_{k=i+1}^{n} u_{ik}x_k \right) /u_{ii}, \quad i = n-1, \ldots, 1 \ .
\end{aligned}
$$

$\mathcal{O}(n^2/2)$ entries in U and about n^2 operations to get $\underline{x}$.

With lower triangular systems we use forward substitution which has the same number of operations.

Solving $A\underline{x} = \underline{b}$ when $A = LU$

Here L is lower triangular and U is upper triangular.

$$A\underline{x} = LU\underline{x} = \underline{b}.$$

Algorithm:

Solve $L\underline{y} = \underline{b}$ by forward substitution.

Solve $U\underline{x} = \underline{y}$ by backward substitution.

The number of operations is about the same as computing

$$A^{-1}\underline{b}$$

if the inverse matrix A^{-1} is available.

It is rare to need to have A^{-1}. When we write $\underline{x} = A^{-1}\underline{b}$ it should be considered as a way of describing the solution and not a preferred method to compute the solution.

Reduction to triangular form

At each stage in the basic reduction process we create zeros below the diagonal in a column and we have a vector of the **multipliers**.

$$
A = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \quad \to \quad \begin{pmatrix} \times & \times & \times & \times \\ 0 & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{pmatrix} = A^{(1)}, \quad \underline{m}_1 = \begin{pmatrix} 0 \\ m_{21} \\ m_{31} \\ m_{41} \end{pmatrix},
$$

$$
\to \quad \begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{pmatrix} = A^{(2)}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ m_{32} \\ m_{42} \end{pmatrix},
$$

$$
\to \quad \begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & X \end{pmatrix} = U, \quad \underline{m}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_{43} \end{pmatrix}.
$$

x and X are potentially non-zero.

3×3 example

Solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 13 \end{pmatrix}.$$

Basic Gauss elimination involves the following sequence.

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 5 \\ 3 & 2 & 1 & 13 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -7 \\ 0 & -1 & -2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

At the 1st stage we subtract multiples of row 1 from rows 2 and 3.
At the 2nd stage we subtract multiples of row 2 from row3.

Back substiution then gives $x_3 = 2$, $x_2 = 1$ and $x_1 = 3$. Here collecting the multipliers together gives $A = LU$ with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The $A = LU$ factorization

When the basic reduction is possible we have the factorization

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

$$L = M_1^{-1} M_2^{-1} M_3^{-1} = I + \underline{m}_1 \underline{e}_1^T + \underline{m}_2 \underline{e}_2^T + \underline{m}_3 \underline{e}_3^T,$$

where each $M_k = I - \underline{m}_k \underline{e}_k^T$ is a **Gauss transformation matrix** with the inverse being $M_k^{-1} = I + \underline{m}_k \underline{e}_k^T$.

Later we sometimes write $l_{ij} = m_{ij}$ for the entries of the lower triangular matrix L. As the diagonal entries of L are all equal to 1 the matrix is said to be **unit lower triangular**.

Factorization of the principal sub-matrices

$$a_{11} = u_{11}.$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

In general

$$\det(A_k) = u_{11} \cdots u_{kk}.$$

This factorization is possible if all the principle sub-matrices are invertible, i.e. the entries $u_{kk} \neq 0$, $k = 1, \ldots, n-1$. We also need $u_{nn} \neq 0$ to solve a system.