Organisation: Lecs, Sems and Labs

MA2715_SB=Advanced Calculus and Numerical Methods

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Handouts: http://people.brunel.ac.uk/~icstmkw/ma2715/

Lectures times: Tue 15:00 and Thu 12:00.

The SEMs will be at Mon 11:00 and Tue 16:00 from week 18.

Matlab labs start this week and each person has one of the times Thu 10:00, 14:00 and 15:00. The labs are associated with 10 credit module MA2895_CB=Numerical Analysis Project.

Organisation: The codes MA2715, MA2815, MA2895

Assessment:

MA2715 topics are part of the 20 credit 3-hour MA2815 exam.

MA2895 has a class test on Matlab in week 22 (30%) and a Matlab assignment with a deadline in week 28 (70%).

Labs in WLFB 106 breakdown:

M1: See your individual timetable.

M2: See your individual timetable.

F: FM degree+others, Thu 15:00.

Overview of MA2715
Chapter 1 – revision, norms, ...

Vectors, matrices, **norms of vectors** and **norms of matrices**.

The notation for column vectors and matrices will be as follows.

$$\underline{x} = (x_i) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Eigenvalues and **eigenvectors** are used in chapter 3 and also appear in some expressions and explanations in other parts of MA2715. Recall that $\underline{v} \neq \underline{0}$ is an eigenvector of A if

$$A\underline{v} = \lambda \underline{v}.$$

Norms: The notation will be $\|\underline{x}\|$ and $\|A\|$. Specific norms with be the 2-norm, ∞-norm and 1-norm.

Condition number: $\kappa(A) = \|A\| \, \|A^{-1}\| \geq 1$.

A large condition number means that A is close to a singular matrix and it is difficult to accurately solve $A\underline{x} = \underline{b}$.

Chapter 2 – solving $A\underline{x} = \underline{b}$

This will be about **Gauss elimination** methods to solve

$$A\underline{x} = \underline{b}$$

for a general system of n equations in n unknowns. Here n may be large and thus everything is done on a computer.

Basic Gauss elimination is equivalent to a **factorization**

$$A = LU,$$

where $L =$**unit lower triangular matrix** and $U =$**upper triangular matrix**. Gauss elimination with pivoting of some kind is equivalent to factorizations of the form

$$PA = LU \quad \text{or} \quad PAQ = LU$$

where P and Q are **permutation matrices**. Permutations matrices are obtained from the identity matrix I by re-arranging the rows.

The basic LU factorization when $n = 4$

Let A be a non-singular matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}.$$

This works if and only if all principal sub-matrices are non-singular.
In this case we also have

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}, \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}, \\ a_{11} &= u_{11}. \end{aligned}$$

Also $\det(A) = \det(U) = u_{11} u_{22} u_{33} u_{44}$.

Chapter 3 – $\underline{u}' = A\underline{u}$, $\underline{u}(0) = \underline{u}_0$

This will be about solving the following problem.

$$\underline{u}' = A\underline{u}, \quad \underline{u}(0) = \underline{u}_0,$$

where A is a constant matrix. This involves a **linear system of differential equations**. In full the differential equation part is

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix}.$$

The solution can be given in terms of the eigenvalues and eigenvectors of the matrix A when A is diagonalisable.

In all cases, diagonalisable or not, the solution can be expressed as

$$\underline{u}(x) = \exp(xA)\underline{u}(0)$$

where here $\exp(xA)$ means the **exponential matrix** of xA.

Chapter 4 – 2 point BVP

This will be about the two-point boundary value problem

$$u''(x) = p(x)u'(x) + q(x)u(x) + r(x), \quad a < x < b,$$

with

$$u(a) = g_1, \quad u(b) = g_2.$$

Here p, q and r are suitable functions. Generally we cannot give a "simple closed form expression" for the solution. Instead we approximate the solution by using the **finite difference method**.

The relevant previous study for this is Taylor expansions which you have seen in MA2730.

A Taylor expansion about every mesh point

The "continuous problem" involving $a \le x \le b$ is approximated by a "discrete problem" involving points $a = x_0 < x_1 < \cdots < x_n = b$. Equally spaced points corresponds to $n \ge 1$, $h = (b-a)/n$ and $x_i = a + ih$, $i = 0, 1, \ldots, n$.

The finite difference approximation is derived by considering Taylor expansions about every interior point x_i. With $u_{i-1} = u(x_{i-1}) = u(x_i - h)$ and $u_{i+1} = u(x_i + h) = u(x_i + h)$ we have

$$
\begin{aligned}
u_{i+1} &= u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' + \cdots \\
u_{i-1} &= u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' + \cdots .
\end{aligned}
$$

Using these we get finite difference approximations to the derivatives at all the points x_i.

Chapter 5 – Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ denote a 2π-periodic function which is piecewise continuous. For "most" values of x we can represent this in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x.$$

Among the things that is likely to be considered are the following.

- Determining the coefficients for several functions which will often need integration by parts.
- Stating conditions when the series converges to $f(x)$.
- When it is valid to integrate or differentiate the series to obtain another Fourier series.
- If time permits then some applications of Fourier series may be briefly mentioned.

Key points in the week 17 lectures

Let $\underline{x} = (x_i)$ and $\underline{y} = (y_i)$ denote real column vectors of length n.
Inner product of $\underline{x}$ and $\underline{y}$:

$$\underline{x}^T \underline{y} = x_1 y_1 + \cdots + x_n y_n.$$

Outer product of $\underline{x}$ and $\underline{y}$: $\underline{x}\,\underline{y}^T$, an $n \times n$ matrix.

$\underline{v}_1, \ldots, \underline{v}_n$ are **linearly dependent** if $\exists\ \underline{\alpha} = (\alpha_i) \neq \underline{0}$ such that

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}.$$

Represent a matrix A in terms of its columns as

$$A = (\underline{a}_1,\ \ldots,\ \underline{a}_n).$$

Then

$$A\underline{x} = x_1 \underline{a}_1 + \cdots + x_n \underline{a}_n,$$

a linear combination of the columns of A. For A to be **invertible** (i.e. **non-singular**) we need the columns to be **linearly independent**. When A is invertible the columns of A^T are also linearly independent.

Eigenvalues/Eigenvectors

$\underline{v} \neq \underline{0}$ is an eigenvector of A with eigenvalue λ if $A\underline{v} = \lambda\underline{v}$.

$A - \lambda I$ is a singular matrix when λ is an eigenvalue and λ satisfies the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

$\det(tI - A)$ is called the **characteristic polynomial**.

Suppose $A\underline{v}_i = \lambda\underline{v}_i$, $\underline{v}_i \neq \underline{0}$, $i = 1, \ldots, n$. The **spectrum** of A is the set

$$\{\lambda_1, \ldots, \lambda_n\}.$$

The **spectral radius** of A is

$$\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

Some results about eigenvalues and eigenvectors

1. A is non-singular if and only if $\lambda_i \neq 0$ for $i = 1, \ldots, n$.
2. Let $\underline{v}_1, \ldots, \underline{v}_n$ be eigenvectors of A with $A\underline{v}_i = \lambda_i \underline{v}_i$.

$$A(\underline{v}_1, \ldots, \underline{v}_n) = (A\underline{v}_1, \ldots, A\underline{v}_n) = (\lambda_1 \underline{v}_1, \ldots, \lambda_n \underline{v}_n).$$

The last right hand side expression can be written as VD, i.e.

$$AV = VD, \quad \text{with } V = (\underline{v}_1, \ldots, \underline{v}_n) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

When $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent V is invertible and

$$V^{-1}AV = D, \quad A = VDV^{-1}.$$

A is **diagonalisable** when this is the case. Otherwise the matrix is **deficient**.

Examples with 2×2 matrices

1.

$$A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\det(I_2 - tI_2) = \begin{vmatrix} 1-t & 0 \\ 0 & 1-t \end{vmatrix} = (1-t)^2.$$

We have repeated eigenvalues with $\lambda_1 = \lambda_2 = 1$ and every non-zero vector in $\mathbb{R}^2$ is an eigenvector. $\rho(A) = 1$.

2.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{vmatrix} 2-t & 1 \\ 1 & 2-t \end{vmatrix} = (2-t)^2 - 1 = (1-t)(3-t).$$

Symmetric matrix, distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. $\rho(A) = 3$.

$$A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Examples with 2×2 matrices continued

3.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} = t^2 + 1 = (i - t)(-i - t).$$

A real matrix with complex eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$. The eigenvectors are also complex. $\rho(A) = 1$.

4.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{vmatrix} -t & 1 \\ 0 & -t \end{vmatrix} = t^2.$$

Repeated eigenvalues with $\lambda_1 = \lambda_2 = 0$. $\rho(A) = 0$.

$$A = A - \lambda_1 I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The only direction which is an eigenvector is

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The matrix is deficient.

Norms

We consider these for vectors and square matrices. The value of a norm is usually written using the $\|.\|$ notation rather than the function notation. To be called a norm the following properties must hold.

Vector norm axioms

$\|\underline{x}\| \geq 0 \ \forall \underline{x} \in \mathbb{R}^n$ with $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$.

$\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\| \ \forall \alpha \in \mathbb{R}$ and $\forall \underline{x} \in \mathbb{R}^n$.

$\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\| \ \forall \underline{x}, \underline{y} \in \mathbb{R}^n$.

The common vector norms

$$\begin{aligned}
\|\underline{x}\|_2 &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} = \left(\underline{x}^T \underline{x}\right)^{1/2}, \\
\|\underline{x}\|_\infty &= \max\{|x_1|, \ldots, |x_n|\}, \\
\|\underline{x}\|_1 &= |x_1| + |x_2| + \cdots + |x_n|.
\end{aligned}$$

Matrix norm

The matrix norm induced by a vector norm is

$$\|A\| = \max\{\|A\underline{x}\| : \ \|\underline{x}\| = 1\}.$$

All of the following norm requirements are satisfied.

$\|A\| \geq 0 \ \forall A \in \mathbb{R}^{n,n}$ with $\|A\| = 0$ if and only if $A = 0$.

$\|\alpha A\| = |\alpha| \, \|A\| \ \forall \alpha \in \mathbb{R}$ and $\forall A \in \mathbb{R}^{n,n}$.

$\|A + B\| \leq \|A\| + \|B\| \ \forall A, B \in \mathbb{R}^{n,n}$.

We also have $\|AB\| \leq \|A\| \, \|B\|$.

For any norm of this type $\rho(A) \leq \|A\|$.