Exercises for revision week

1. Consider the following 3×3 matrices.

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

For each matrix either determine the LU factorization, where L is unit lower triangular and U is upper triangular, or give a reason why the matrix does not have a LU factorization. If a matrix has a LU factorization then you must show intermediate computations to show how you obtained the factors.

Solution

An $n \times n$ matrix $A = (a_{ij})$ has a LU factorization if the principal sub-matrices of size $1, \ldots, n-1$ are all non-singular. We do not actually need A itself to be non-singular to get the factors but if A is singular with the principal sub-matrices being non-singular then the $n.n$ entry of U is 0. Thus when $n = 3$ we have a LU factorization if

$$a_{11} \neq 0 \quad \text{and} \quad \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0.$$

The basic Gauss elimination process gets the factors when it runs to completion.

A_1 does not have a factorization as the $1, 1$ entry is 0.

A_3 does not have a factorization as the 2×2 principal sub-matrix is

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

is singular.

In the case of A_2 the basic Gauss elimination process gives the following.

$$A_2 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = U.$$

The vector of the multipliers at each step are

$$\underline{m}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix},$$

Thus

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the case of A_4 we note that the last 2 rows are the same and thus the matrix is singular but this does not prevent the reduction process running to completion. We get the following.

$$A_4 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = U.$$

The vector of the multipliers at each step are

$$\underline{m}_1 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

Thus

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

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2. In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -7 & -8 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{with } \underline{u}(0) = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

You must explain all the intermediate workings.

Suppose that we replace the initial condition by

$$\underline{u}(0) = \begin{pmatrix} \alpha \\ 2 \end{pmatrix}$$

For what value of α will the solution be such that $\underline{u}(x) \to \underline{0}$ as $x \to \infty$. You need to explain your answer.

Solution

Let A denote the 2×2 matrix, i.e.

$$A = \begin{pmatrix} -7 & -8 \\ 5 & 6 \end{pmatrix}.$$

$$\det(A - \lambda I) = (-7-\lambda)(6-\lambda) - (-40) = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1).$$

The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$.

$$A - \lambda_1 I = \begin{pmatrix} -5 & -8 \\ 5 & 8 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} -8 & -8 \\ 5 & 5 \end{pmatrix},$$

For the eignevectors associated with λ_1 and λ_2 we can take respectively

$$\underline{v}_1 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution of the system of ODEs is

$$\underline{u}(x) = c_1 \mathrm{e}^{-2x} \underline{v}_1 + c_2 \mathrm{e}^{x} \underline{v}_2.$$

To satisfy the initial conditions we need

$$\underline{u}(0) = c_1 \underline{v}_1 + c_2 \underline{v}_2 = \begin{pmatrix} 8 & 1 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

If we add the two equations then we eliminate c_2 and get

$$3c_1 = -3, \quad c_1 = -1.$$

Then from the first equation

$$8(-1) + c_2 = -5, \quad c_2 = 3.$$

The solution is

$$\underline{u}(x) = -\mathrm{e}^{-2x} \begin{pmatrix} 8 \\ -5 \end{pmatrix} + 3\mathrm{e}^{x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Recall that the general solution is

$$\underline{u}(x) = c_1 \mathrm{e}^{-2x} \underline{v}_1 + c_2 \mathrm{e}^{x} \underline{v}_2.$$

We only have $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ when $c_2 = 0$. With the different initial condition we need

$$\begin{pmatrix} \alpha \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -5 \end{pmatrix}.$$

By just considering the direction of the vector we need

$$\frac{\alpha}{2} = \frac{8}{-5}, \quad \alpha = -\frac{16}{5}.$$

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3. Let $f(x)$ be a 2π-periodic function defined on $(-\pi, \pi]$ by

$$f(x) = \begin{cases} x(\pi + x), & -\pi < x \le 0, \\ x(\pi - x), & 0 < x \le \pi. \end{cases}$$

Explain why $f(x)$ is an odd function on $(-\pi, \pi)$.

Let $g(x) = f'(x)$. Sketch $g(x)$ on the interval $(-\pi, 3\pi)$. Indicate whether or not $g(x)$ is a continuous function.

Explain why the Fourier series for $f(x)$ is given by

$$\frac{8}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \cdots + \frac{\sin((2m-1)x)}{(2m-1)^3} + \cdots \right).$$

You need to give intermediate workings to explain why the series has this form.

Solution

Let $a > 0$. From the above $f(a) = a(\pi - a)$ (using the $(0, \pi)$ part).

$f(-a) = (-a)(\pi + (-a)) = -a(\pi - a)$ (using the $(-\pi, 0)$ part).

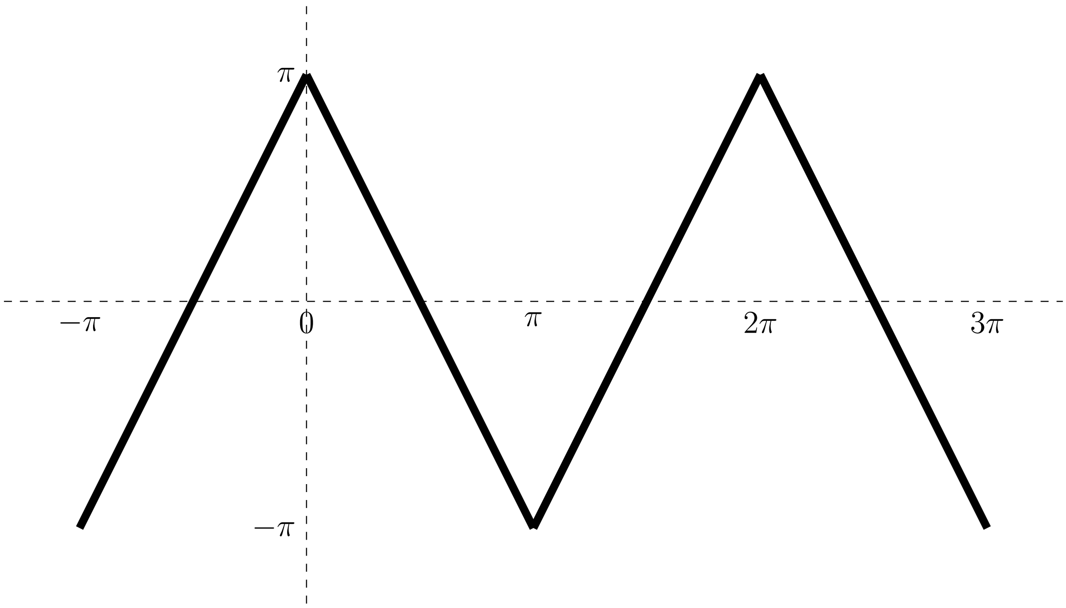
Thus $f(-a) = -f(a)$ and $f(x)$ is an odd function.

Now

$$g(x) = f'(x) = \begin{cases} \pi - 2x, & -\pi < x \le 0, \\ \pi - 2x, & 0 < x \le \pi. \end{cases}$$

This is piecewise linear with continuity at $x = 0$ with $g(0) = \pi$ and the 2π-peroidic extension is also continuous with $g(\pi) = g(-\pi) = -\pi$.

For a sketch of $g(x) = f'(x)$ on $[-\pi, 3\pi]$ we have the following.



As $f(x)$ is an odd function there are no cosine terms and hence $a_n = 0$.

Also, as $f(x)$ is an odd function the integration can be done by just using $(0, \pi)$, i.e.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x. = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

We integration by parts to determine b_n and as $f(x)$ is a degree 2 polynomial on $(0, \pi)$ we need to do this twice. Firstly, we note that $f'(x) = \pi - 2x$ amd $f''(x) = -2$ on this interval. In the integration by parts we always differentiate the polynomial term and a consequence we integrate the trig. term.

Integrating by parts once gives

$$\begin{aligned} b_n &= \frac{2}{\pi} \left(\left[-f(x) \frac{\cos(nx)}{n} \right]_0^{\pi} + \int_0^{\pi} f'(x) \frac{\cos(nx)}{n} \, \mathrm{d}x \right) \\ &= \frac{2}{\pi} \left(\int_0^{\pi} f'(x) \frac{\cos(nx)}{n} \, \mathrm{d}x \right), \quad \text{as } f(0) = f(\pi) = 0. \end{aligned}$$

For the next integration by parts we have

$$
\begin{aligned}
\int_0^{\pi} f'(x)\cos(nx)\,\mathrm{d}x &= \left[f'(x)\frac{\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} f''(x)\frac{\sin(nx)}{n}\,\mathrm{d}x \\
&= -\int_0^{\pi} f''(x)\frac{\sin(nx)}{n}\,\mathrm{d}x, \quad \text{as } \sin(0)=\sin(n\pi)=0 \\
&= 2\int_0^{\pi} \frac{\sin(nx)}{n}\,\mathrm{d}x, \quad \text{as } f''(x)=-2, \\
&= \frac{2}{n^2}\left[-\cos(nx)\right]_0^{\pi} = \frac{2}{n^2}(-\cos(n\pi)+1) = \frac{2}{n^2}(-(-1)^n+1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{4}{n^2} & \text{if } n \text{ is odd}. \end{cases}
\end{aligned}
$$

Putting everything together with the previous $2/(\pi n)$ term gives

$$
b_n = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd}. \end{cases}
$$

4. Consider the following symmetric 3×3 matrix.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Given that

$$A = LU \quad \text{with } L = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

determine the inverse matrices L^{-1} and A^{-1} using forwards and/or backward substitution, as appropriate, and determine $\|A^{-1}\|_\infty$. You need to show intermediate workings.

Solution

Let $\underline{e}_i$ denote the ith base vector, i.e. the ith column of the identity matrix. We can describe the ith column of the inverse matrix as

$$\underline{x} = A^{-1}\underline{e}_i \quad \text{so that } A\underline{x} = LU\underline{x} = \underline{e}_i.$$

Thus we can get A^{-1} column-by-column by solving linear systems and as we already have the LU factorization we just need to use forward and back substitution in each case. In each case we first obtain $\underline{y}$ by solving $L\underline{y} = \underline{e}_i$ and then we get $\underline{x}$ by solving $U\underline{x} = \underline{y}$.

Consider $i = 1$.

$$L\underline{y} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$y_1 = 1, \quad y_2 = 1/2, \quad y_3 = 1/3.$$

$$U\underline{x} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix}.$$

This gives

$$x_3 = 1/4, \quad x_2 = 1/2, \quad x_1 = 3/4.$$

Consider $i = 2$.

$$L\underline{y} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

This gives

$$y_1 = 0, \quad y_2 = 1, \quad y_3 = 2/3.$$

$$U\underline{x} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2/3 \end{pmatrix}.$$

This gives

$$x_3 = 1/2, \quad x_2 = 1, \quad x_1 = 1/2.$$

Consider $i = 3$.

$$L\underline{y} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This gives

$$y_1 = 0, \quad y_2 = 0, \quad y_3 = 1.$$

$$U\underline{x} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This gives

$$x_3 = 3/4, \quad x_2 = 1/2, \quad x_1 = 1/4.$$

We have already solved problems of the form $L\underline{y} = \underline{e}_i$ and thus the vectors $\underline{y}$ are the columns of the inverse of L. To summarize

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 2/3 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}.$$

The row sums of magnitudes of the entries of A are 3, 4 and 3 and hence $\|A\|_\infty = 4$.

The row sums of magnitudes of the entries of A^{-1} are 3/2, 2 and 3/2 and hence $\|A^{-1}\|_\infty = 2$.

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5. Let u be an infinitely differentiable function defined in a region which contains $[-2h, 2h]$, where $h > 0$, and assume that the Maclaurin series expansion are valid at all points in this interval. Assuming that h is small, show that

$$\frac{u(2h) - 4u(h) + 6u(0) - 4u(-h) + u(-2h)}{h^4} = u^{(4)}(0) + c_2 h^2 u^{(6)}(0) + \mathcal{O}(h^4)$$

and determine the constant c_2. You need to show all the Maclaurin series that you use and you ned to show all other intermediate working.

Solution

The Maclaurin expansion of $u(x)$ about 0 evaluated at $x = h$ is

$$u(h) = u(0) + hu'(0) + \frac{h^2}{2}u''(0) + \frac{h^3}{6}u'''(0) + \frac{h^4}{24}u''''(0) + \cdots$$

If we replace h by $-h$ then we have

$$u(-h) = u(0) - hu'(0) + \frac{h^2}{2}u''(0) - \frac{h^3}{6}u'''(0) + \frac{h^4}{24}u''''(0) + \cdots$$

Adding these two expressions cancels the odd powers of h and we get

$$u(h) + u(-h) = 2u(0) + h^2u''(0) + \frac{h^4}{12}u^{(4)}(0) + \frac{h^6}{369}u^{(6)}(0) + \cdots$$

If we replace h by $2h$ in the last expression then we get

$$u(2h) + u(-2h) = 2u(0) + 4h^2u''(0) + \frac{16h^4}{12}u^{(4)}(0) + \frac{64h^6}{360}u^{(6)}(0) + \cdots$$

Thus

$$u(2h) + u(-2h) - 4(u(h) + u(-h)) = -6u(0) + h^4u^{(4)}(0) + \frac{60}{360}h^6u^{(6)}(0) + \cdots$$

Rearranging gives

$$\frac{u(2h) - 4u(h) + 6u(0) - 4u(-h) + u(-2h)}{h^4} = u^{(4)}(0) + \frac{1}{6}h^2u^{(6)}(0) + \mathcal{O}(h^4).$$

Thus $c_2 = 1/6$.