Exercises for revision week

1. Consider the following 3×3 matrices.

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

For each matrix either determine the LU factorization, where L is unit lower triangular and U is upper triangular, or give a reason why the matrix does not have a LU factorization. If a matrix has a LU factorization then you must show intermediate computations to show how you obtained the factors.

---

2. In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -7 & -8 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{with } \underline{u}(0) = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

You must explain all the intermediate workings.

Suppose that we replace the initial condition by

$$\underline{u}(0) = \begin{pmatrix} \alpha \\ 2 \end{pmatrix}$$

For what value of α will the solution be such that $\underline{u}(x) \to \underline{0}$ as $x \to \infty$. You need to explain your answer.

---

3. Let $f(x)$ be a 2π-periodic function defined on $(-\pi, \pi]$ by

$$f(x) = \begin{cases} x(\pi + x), & -\pi < x \le 0, \\ x(\pi - x), & 0 < x \le \pi. \end{cases}$$

Explain why $f(x)$ is an odd function on $(-\pi, \pi)$.

Let $g(x) = f'(x)$. Sketch $g(x)$ on the interval $(-\pi, 3\pi)$. Indicate whether or not $g(x)$ is a continuous function.

Explain why the Fourier series for $f(x)$ is given by

$$\frac{8}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \cdots + \frac{\sin((2m-1)x)}{(2m-1)^3} + \cdots \right).$$

You need to give intermediate workings to explain why the series has this form.

---

4. Consider the following symmetric 3×3 matrix.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Given that

$$A = LU \quad \text{with } L = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

determine the inverse matrices L^{-1} and A^{-1} using forwards and/or backward substitution, as appropriate, and determine $\|A^{-1}\|_\infty$. You need to show intermediate workings.

---

5. Let u be an infinitely differentiable function defined in a region which contains $[-2h, 2h]$, where $h > 0$, and assume that the Maclaurin series expansion are valid at all points in this interval. Assuming that h is small, show that

$$\frac{u(2h) - 4u(h) + 6u(0) - 4u(-h) + u(-2h)}{h^4} = u^{(4)}(0) + c_2 h^2 u^{(6)}(0) + \mathcal{O}(h^4)$$

and determine the constant c_2. You need to show all the Maclaurin series that you use and you ned to show all other intermediate working.
